Principal Killing Strings

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GC2018, Yukawa Institute, Kyoto,
February 16, 2018

Based on:

" Principal Killing strings in higher-dimensional Kerr-NUT-(A)dS spacetimes", Jens Boos and V.F, e-Print: arXiv:1801.00122 (2018);

"Stationary black holes with stringy hair", Jens Boos and V.F., e-Print: arXiv:1711.06357 (2017), (to appear in PRD);

"Stationary strings and branes in the higher-dimensional Kerr-NUT-(A)dS spacetimes", David Kubiznak and V.F., JHEP 0802 (2008) 007; e-Print: arXiv:0711.2300.

"Black holes, hidden symmetries, and complete integrability",

V.F., Pavel Krtous and David Kubiznak,

Living Rev.Rel. 20 (2017) no.1, 6;

e-Print: arXiv:1705.05482.

Solving stationary string equations in the Kerr-NUT-(A)dS background

Killing-Yano objects

Conformal KY tensor (CKY) of rank p in D dims:

$$\nabla_X \omega = \frac{1}{p+1} X \bullet (\nabla \wedge \omega) + \frac{1}{D-p+1} X \wedge (\nabla \bullet \omega) + 0,$$

Killing tensor (KYT): $\nabla \bullet \omega \sim \delta\omega = 0;$

Closed conformal KY tensor (CCKY): $\nabla \wedge \omega \sim d\omega = 0.$

$$\nabla_X (*\omega) = \frac{1}{p_* + 1} X \bullet (\nabla \wedge *\omega) + \frac{1}{D - p_* + 1} X \wedge (\nabla \bullet * \omega),$$

$$p_* = D - p$$

Properties:

- $*$(KYT)=CCKY
- $*$(CCKY)=KYT
- CCKY $\wedge$ CCKY=CCKY

Principal tensor = non-degenerate rank 2 CCKY tensor

$$D = 2n + \varepsilon$$

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_c, \quad \xi_a = \frac{1}{D-1}\nabla^b h_{ba}.$$

$$\mathbf{h} = r\,\mathbf{l}_+ \wedge \mathbf{l}_- + \sum_{\mu=1}^{n-1} x_\mu \mathbf{e}^\mu \wedge \hat{\mathbf{e}}^\mu,$$

$$\mathbf{g} = -\mathbf{l}_+\mathbf{l}_- - \mathbf{l}_-\mathbf{l}_+ + \sum_{\mu=1}^{n-1} (\mathbf{e}^\mu \mathbf{e}^\mu + \hat{\mathbf{e}}^\mu \hat{\mathbf{e}}^\mu) + \varepsilon \hat{\mathbf{e}}^0 \hat{\mathbf{e}}^0.$$

$$(\mathbf{l}_+, \mathbf{l}_-) = -1, \; (\mathbf{e}^\mu, \mathbf{e}^\mu) = (\hat{\mathbf{e}}^\mu, \hat{\mathbf{e}}^\mu) = (\hat{\mathbf{e}}^0, \hat{\mathbf{e}}^0) = 1.$$

Darboux basis: $(\mathbf{l}_+, \mathbf{l}_-, \mathbf{e}^\mu, \hat{\mathbf{e}}^\mu, \hat{\mathbf{e}}^0)$.

Non-degenerate: There are exactly n non-vanishing "eigenvalues" (r, x_μ) that are functionally independent in some domain. In this domain none of the gradients of them is a null vector.

A metric which admits a principal tensor
is off-shell Kerr-NUT-(A)dS metric.
It contains *n* arbitrary functions of 1 variable.

On-shell: Einstein equations are satisfied $\Rightarrow$
Kerr-NUT-(A)dS solution.

- ξ is a primary Killing vector: $L_\xi \mathbf{g} = L_\xi \mathbf{h} = 0$;
- $\mathbf{h}^{(j)} = \frac{1}{j!}\mathbf{h}^{\wedge j}$ is a CCKY $2j$ – form;
- $\mathbf{f}^{(j)} = *\,\mathbf{h}^{(j)}$ is a KY (D-2j) form
- $k^{ab}_{(j)} = \frac{1}{(D-2j-1)!} f^{(j)a}{}_{c_1 \ldots c_{D-2j-1}} f^{(j)bc_1 \ldots c_{D-2j-1}}$ is a rank 2 Killing tensor;
- $\zeta_{(j)} = \mathbf{k}_{(j)} \cdot \xi, \quad (j = 0, \ldots, n-1+\varepsilon \equiv m)$ are commuting (secondary) Killing vectors;
- $\mathbf{k}_{(0)} = g$;
- Frobenius theorem: $\xi = \partial_\tau, \quad \zeta_{(j)} = \partial_{\psi_j}$;
- (r, x_μ, τ, ψ_j) are canonical coordinates;
- $l_\pm$ are principal null directions; their integral lines are geodesics.

Geodesic equations are completely integrable:

There exist D integrals of motion for a free particle,

$(n+\varepsilon)$ first order $\zeta_{(j)} \bullet p$ and n second order $p \bullet \mathbf{k}_{(j)} \bullet p$.

Q: If instead of a particle one has a string:

Are Nambu-Goto string equations completely integrable in the Kerr-NUT-(A)dS geometry?

A: In a general case - No.

If a string is stationary - Yes.

Stationary strings in Kerr-NUT-(A)dS

Killing vector $\xi = \partial_t$, coordinates - (t, y^i)

$$g_{ab} = p_{ab} + \frac{\xi_a \xi_b}{\xi^2}, \quad F = -\xi^2, \quad A_i = \frac{\xi_i}{\xi^2},$$

$$ds^2 = -F(dt + A_i dy^i)^2 + p_{ij} dy^i dy^j .$$

Nambu-Goto action for a string

$$I = -\Delta t E, \quad E = \mu \int \sqrt{F} dl = \mu \int d\sigma \sqrt{F p_{ij} \frac{dy^i}{d\sigma} \frac{dy^j}{d\sigma}} .$$

String configuration $y^i(\sigma)$ is a geodesic in $(D-1)-$ dimensional space with metric $\tilde{p}_{ij} = F p_{ij}$.

If metric g_{ab} admits the principal tensor it has $n+\varepsilon$ Killing vectors and n rank 2 Killing tensors. This gives $D=2n+\varepsilon$ integrals of motion for a free particle.

The reduced ref-shifted metric $\tilde{p}_{ij}$ does not admit a principal tensor, However, when ξ is a primary Killing vector, it has $n-1+\varepsilon$ Killing vectors and n rank 2 Killing tensors. This gives $D=2n-1+\varepsilon$ integrals of motion. Thus, the stationary string equations are completely integrable.

[V.F. and David Kubiznak (2008)]

Principal Killing Strings

$\ell_{\pm} = \beta\, l_{\pm}$ is a tangent vector to a principal null geodesic in the affine parametrization:

$\nabla_{\ell_{\pm}} \ell_{\pm} = 0, \quad [\ell_{\pm}, \xi] = 0.$

Frobenius theorem implies that a ST is foliated by 2-D (Killing) surfaces Σ. Coordinates $Y^a = (z^A, y^i)$.

Equations of a given Σ are $y^i = const$.

$z^A = (v, \lambda_{\pm})$ are coordinates on Σ, such that

$\xi = \partial_v, \quad \ell_{\pm} = \partial_{\lambda_{\pm}}.$

The Killing surface in the off - shell Kerr - NUT - (A)dS metric is minimal.

Induced 2-D metric: $d\gamma^2 = \gamma_{AB}\mathrm{d}z^A\mathrm{d}z^B = \xi^2\mathrm{d}\lambda_\pm^2 - 2\mathrm{d}v\mathrm{d}\lambda_\pm$.

$n^\mu_{(i)}$ are (D-2) mutually orthogonal unit vectors normal to $\Sigma_\pm$. The extrinsic curvature is:

$$\Omega_{(i)AB} = g_{ab}n^a_{(i)}Y^c{}_{,A}\nabla_c Y^b{}_{,B}\,, \quad \Omega_{(i)} = g_{ab}n^a_{(i)}Z^b = 0.$$

$$Z^b = \gamma^{AB}Y^c{}_{,A}\nabla_c Y^b{}_{,B} = -\left(\xi^a\nabla_a\ell^b_\pm + \ell^a_\pm\nabla_a\xi^b + \xi^2\ell^a_\pm\nabla_a\ell^b_\pm\right).$$

$$Z^b = 2\ell^a_\pm\nabla_a\xi^b = -F^b{}_a\ell^a_\pm = -\kappa_\pm\ell^b_\pm\,.$$

$$F^a{}_b\ell^b_\pm = \kappa_\pm\ell^a_\pm\,, \quad \kappa_\pm = \frac{1}{2}\ell^a_\pm(\xi^2)_{;a}\,.$$

Incoming principal Killing string:

$$\lambda = -r, \quad \psi_j = -P_n^{(j)}(r), \quad x_\mu = \text{const},$$

$$P_n^{(j)} = \int \frac{r^{2(n-1-j)} dr}{X_n(r)}.$$

$$(\tau, r, x_\mu, \psi_k) \to (v, r, x_\mu, \hat{\varphi}_k)$$

$$\mathrm{d}\tau = \mathrm{d}v - \sum_{j=1}^{m} a_j \mathrm{d}\hat{\varphi}_j - \frac{r^{2(n-1)}}{X_n} \mathrm{d}r,$$

$$\mathrm{d}\psi_j = a_j^{-1} \mathrm{d}\hat{\varphi}_j - \frac{1}{X_n} r^{2(n-1-j)} \mathrm{d}r.$$

String equation in the null incoming coordinates

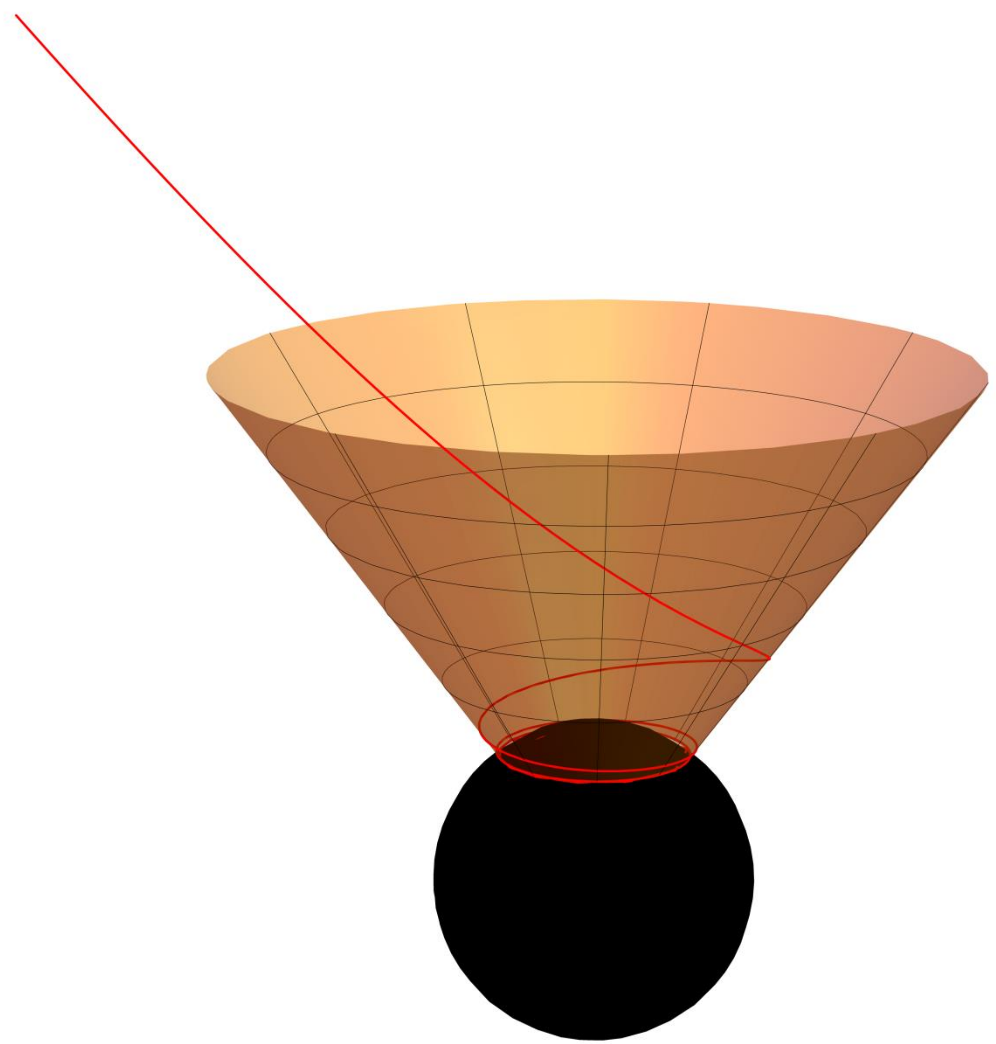
$$\hat{\varphi}_j = \hat{\varphi}_j^0 = \text{const}, \quad x_\mu = x_\mu^0 = \text{const}.$$

String's stress-energy tensor in in-coming coordinates

$$T^{ab} = -\frac{\mu_s}{\sqrt{-g}}\left(2\xi^{(a}\ell^{b)} + \xi^2 \ell^a \ell^b\right) q,$$

$$q = q(x_\mu, \hat{\varphi}_j \mid x^0_\mu, \hat{\varphi}^0_j) \equiv$$

$$\prod_{\mu=1}^{n-1} \delta(x_\mu - x^0_\mu) \prod_{j=1}^{m} \delta(\hat{\varphi}_j - \hat{\varphi}^0_j).$$

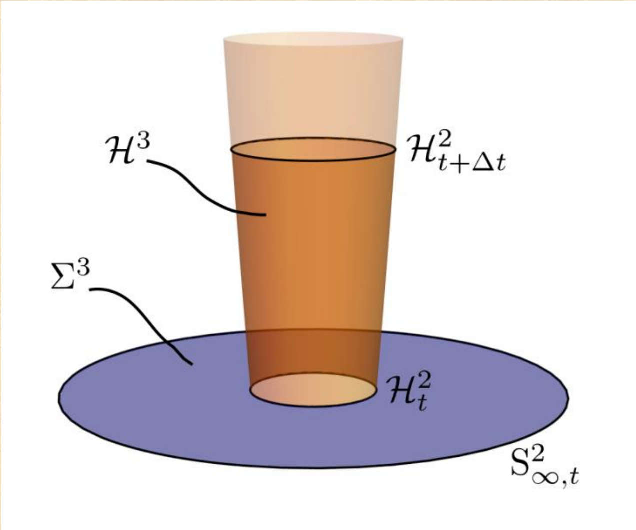


Applications to Myers-Perry ST

$$-16\pi\frac{D-3}{D-2}M=\oint_B \nabla^a \xi^b \mathrm{d}S_{ab},$$

$$-16\pi J_{(i)}=\oint_B \nabla^a \zeta^b_{(i)} \mathrm{d}S_{ab}.$$

$$J_{(i)}=\frac{2}{D-2}Ma_i.$$

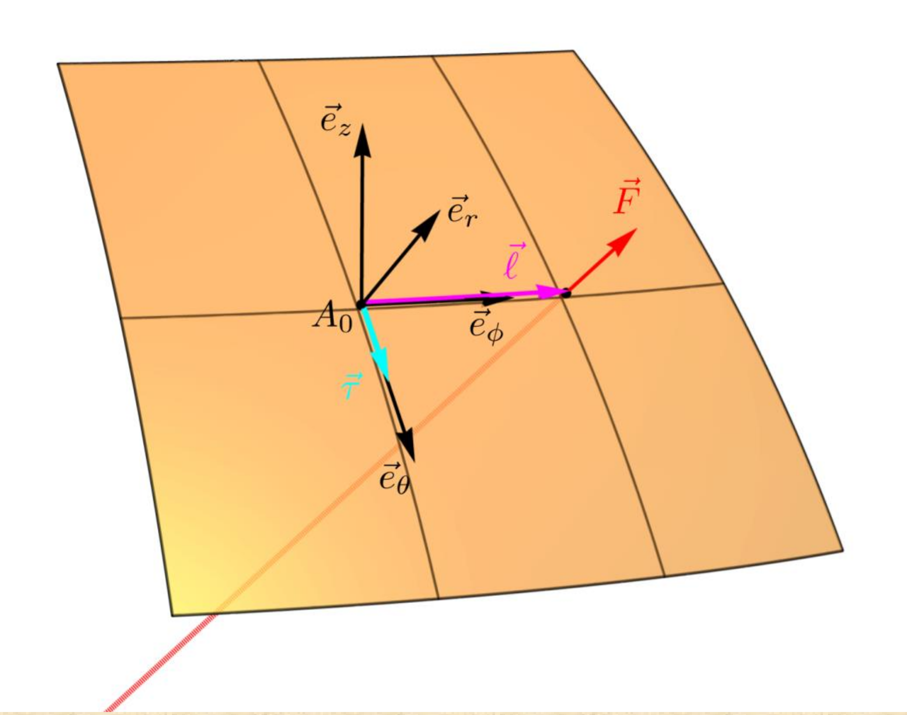


$$\Delta E=\int_{H^{D-1}} T^a{}_b \xi^b \mathrm{d}\Sigma_a, \quad \Delta J_i=\int_{H^{D-1}} T^a{}_b \zeta^b_i \mathrm{d}\Sigma_a$$

$$\dot{M}=0, \quad \dot{J}_{(i)}=-\mu_s a_i\left(\mu_i^0\right)^2.$$

$$J_{(i)}=J_{(i)}^0\exp(-v/v_i), \quad v_i=\frac{2M}{D-2}\frac{1}{\mu_s(\mu_i^0)^2}.$$

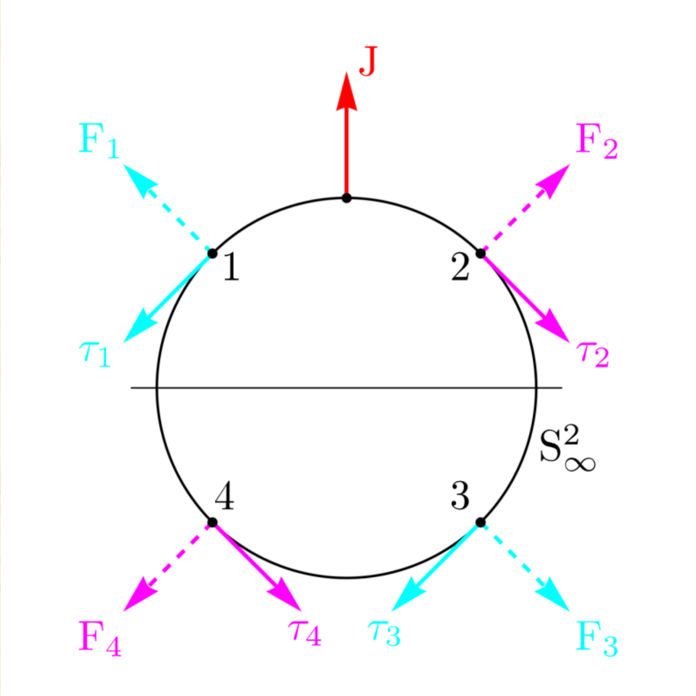
$\mathbf{j} = \text{Projection}(\boldsymbol{\tau})$



e_i and $\hat{e}_i$ are unit vectors along x_i and y_i; ω^i and $\hat{\omega}^i$ are dual unit forms. $\mathbf{J} = \sum_{i=1}^{m} \frac{2Ma_i}{D-2} \omega^i \wedge \hat{\omega}^i$; $x_i^0 = 0 \quad \Rightarrow \boldsymbol{\delta} = \sum_{i=1}^{m} a_i \mu_i^0 e_i$,

$$\mathbf{F} = \frac{\mu_s}{r}[\sum_{i=1}^{m}\left(x_i^0 e_i + y_i^0 \hat{e}_i\right) + (1-\varepsilon) z^0 \hat{e}_z].$$

$$\boldsymbol{\tau} \equiv \boldsymbol{\delta} \wedge \mathbf{F} \equiv \mu_s[\sum_{i,j=1}^{m} a_i \mu_i^0 \mu_i^0 \omega^i \wedge \hat{\omega}^j + \sum_{i=1}^{m} a_i \mu_i^0 \mu_0^0 (1-\varepsilon) z^0 \omega^i \wedge \omega^0].$$



2^n string segments $= 2^{n-1}$ "infinite captured strings"

Instead of Summary…

Happy Birthday