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Hidden conformal symmetries of spacetime and higher-order ladder operators for Klein-Gordon equation

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(in progress)

The structure of ladder operators for Klein-Gordon equation

The structure of ladder operators for Klein-Gordon equation

- $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$: Laplacian on (M, $g_{\mu\nu}$)
- Klein-Gordon equation

$$(\Box - m^2)\psi = 0$$

m^2: constant

- $\boldsymbol{D}$: Ladder operator

$$[\Box, \boldsymbol{D}] = \delta m^2\, \boldsymbol{D} + \boldsymbol{Q}(\Box - m^2)$$

$\delta m^2, m^2$: constants, $\boldsymbol{Q}$: operator

For a solution $\bar{\psi}$ to $(\Box - m^2)\psi = 0$,

$$\Box(\boldsymbol{D}\bar{\psi}) = [\Box, \boldsymbol{D}]\bar{\psi} + \boldsymbol{D}(\Box\bar{\psi})$$

$$= \delta m^2\, \boldsymbol{D}\bar{\psi} + \underbrace{\boldsymbol{Q}(\Box - m^2)\bar{\psi}}_{=0} + \underbrace{\boldsymbol{D}(\Box\bar{\psi})}_{=\boldsymbol{D}(m^2\bar{\psi})}$$

$$= (m^2 + \delta m^2)\, \boldsymbol{D}\bar{\psi}\,.$$

$\Rightarrow$ $\boldsymbol{D}\bar{\psi}$ is a solution to $\left(\Box - (m^2 + \delta m^2)\right)\psi = 0$.

- $\boldsymbol{D}$: Ladder operator

mass squared shifted

$$[\Box, \boldsymbol{D}] = \delta m^2\, \boldsymbol{D} + \boldsymbol{Q}(\Box - m^2)$$

difference

$\delta m^2, m^2$: constants, $\boldsymbol{Q}$: operator

- $\boldsymbol{D}$: Ladder operator

mass squared shifted

$$[\Box, \boldsymbol{D}] = \delta m^2 \boldsymbol{D} + \boldsymbol{Q}(\Box - m^2)$$

difference

$\delta m^2, m^2$: constants, $\boldsymbol{Q}$: operator

- Differential operator of order *p*

$$\boldsymbol{D} = K_{(p)}^{\mu_1\mu_2\dots\mu_p} \nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_p} + K_{(p-1)}^{\mu_1\dots\mu_{p-1}} \nabla_{\mu_1}\dots\nabla_{\mu_{(p-1)}}$$

$$+ \dots + K_{(2)}^{\mu\nu}\nabla_\mu\nabla_\nu + K_{(1)}^{\mu}\nabla_\mu + K_{(0)}$$

First-order ladder operators for Klein-Gordon equation (review)

cf. Kimura's talk

Symmetries of spacetime

$$\nabla_\mu \xi_\nu = Q g_{\mu\nu}$$

Closed conformal Killing (CCKV)

$$\nabla_{[\mu}\xi_{\nu]} = 0$$

$\subset$

$$\nabla_{(\mu}\xi_{\nu)} = Q g_{\mu\nu}$$

Conformal Killing (CKV)

$\cup$

$$\nabla_\mu Q = 0$$

$$\lambda = 0$$

Killing (KV) $\subset$ Homothetic (HV)

$$\nabla_{(\mu}\xi_{\nu)} = 0$$

$$\nabla_{(\mu}\xi_{\nu)} = \lambda g_{\mu\nu}$$

Our result ① (First-order ladder operators)

If a spacetime (M, $g_{\mu\nu}$) admits a closed conformal Killing vector ξ^μ,

$$\nabla_\mu \xi_\nu = Q g_{\mu\nu}\,, \quad Q = \frac{1}{n}\nabla_\mu \xi^\mu$$

and ξ^μ is an eigenvector of the Ricci tensor,

$$R^\mu{}_\nu \xi^\nu = (n-1)\chi\xi^\mu$$

then the differential operator (with $k \in \mathbb{R}$)

$$\boldsymbol{D}_k \equiv \xi^\mu \nabla_\mu - kQ$$

satisfies the commutation relation

mass squared shifted

$$[\square, \boldsymbol{D}_k] = \chi(2k+n-2)\boldsymbol{D}_k + 2\boldsymbol{Q}(\square + \chi k(k+n-1))$$

difference

mass squared shifted

$$\boldsymbol{D}_k : \; -\chi k(k+n-1) \;\rightarrow\; -\chi(k-1)(k+n-2)$$

$$\chi(2k+n-2)$$

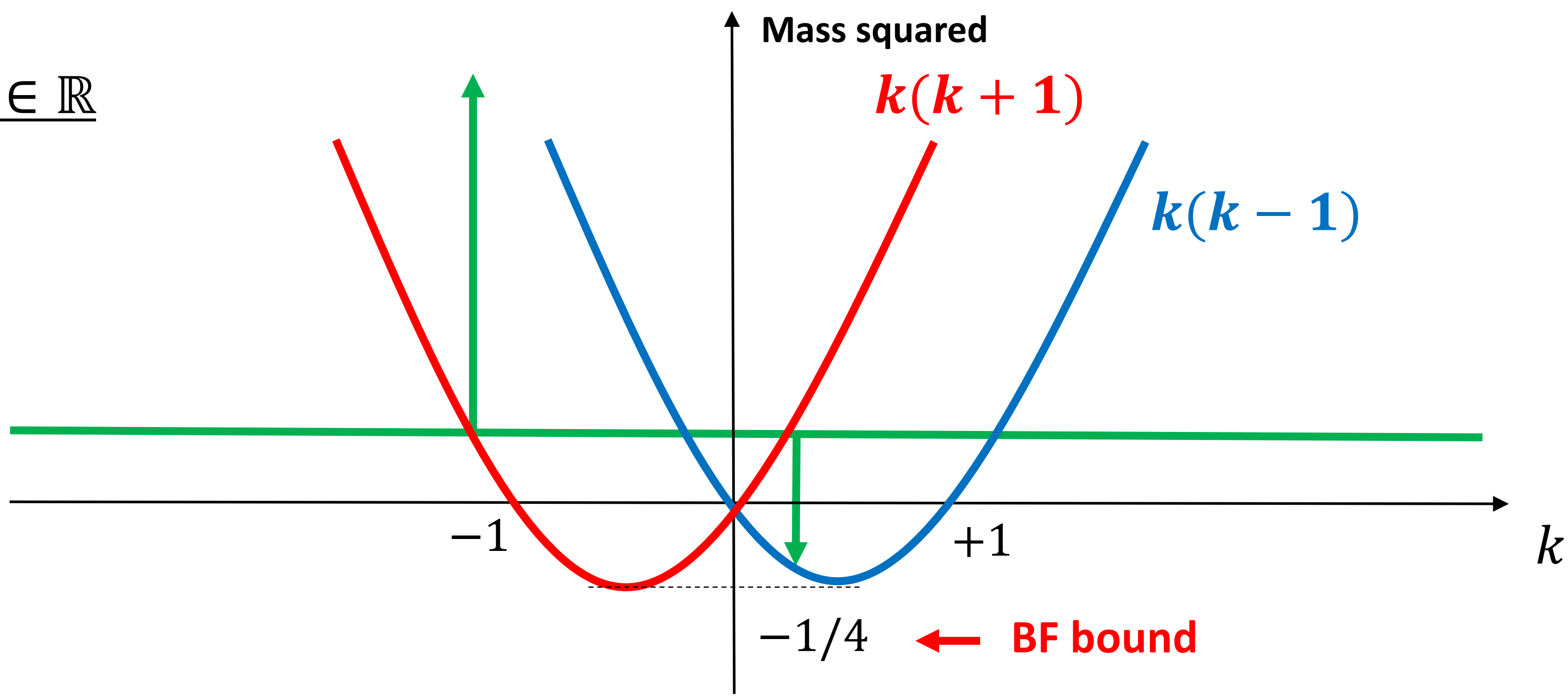
difference

$\boldsymbol{k \rightarrow k-1}$

<u>Example:</u> AdS_2 spacetime ($n=2, \chi=-1$)

$$\boldsymbol{D}_k : \; k(k+1) \;\rightarrow\; k(k-1)$$

$\boldsymbol{k} \in \mathbb{R}$



mass squared shifted

$$\boldsymbol{D}_k : \; -\chi k(k+n-1) \;\rightarrow\; -\chi(k-1)(k+n-2)$$

$$\chi(2k+n-2)$$

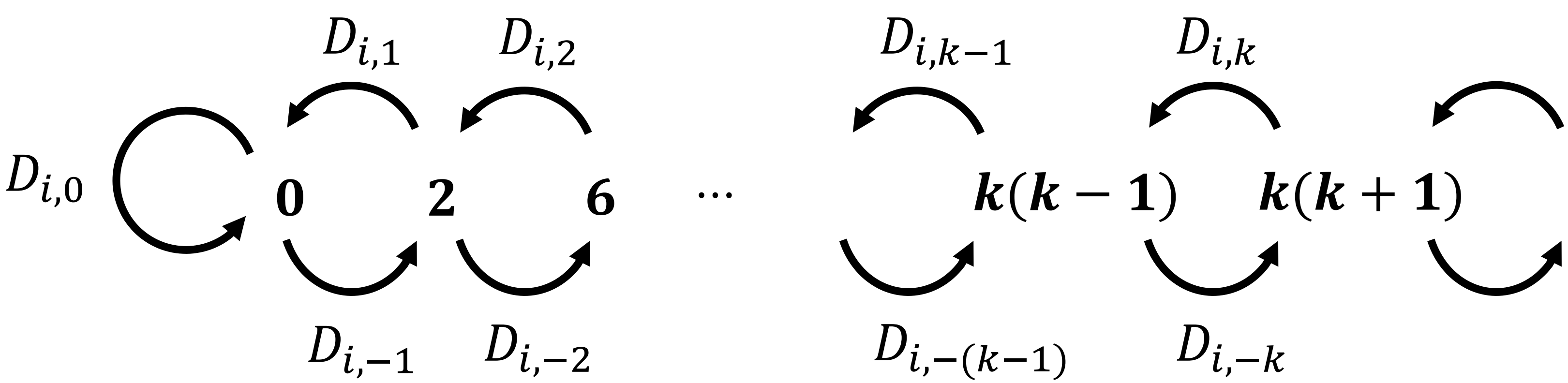
difference

$\boldsymbol{k \rightarrow k-1}$

Example: AdS_2 spacetime ($n = 2, \chi = -1$)

$$\boldsymbol{D}_k : \; k(k+1) \;\rightarrow\; k(k-1)$$

$\boldsymbol{k \in \mathbb{Z}}$



Questions:

- Can we obtain higher-order ladder operators that cannot be multiple of first-order ladders?
- If they exist, what kind of symmetry is related to higher-order ladder operators?
- How do they shift mass squared?

cf. First-order

$$\boldsymbol{D}_k : \quad -\chi k(k+n-1) \;\; \rightarrow \;\; -\chi(k-1)(k+n-2)$$

- Which mass squared is connecting to zero mass squared?

cf. First-order $\quad m^2 = -\chi(k-1)(k+n-2) \quad (\boldsymbol{k} \in \mathbb{Z})$

- Beyond BF bound?

Higher-order ladder operators for Klein-Gordon equation

In previous work

- First-order ladder operator

$$\boldsymbol{D}_k \equiv \underline{\xi^\mu} \nabla_\mu - k\underline{Q}$$

closed CKV $\qquad \sim \nabla_\mu \xi^\mu$

In this work

- Higher-order ladder operator

$$\boldsymbol{D} = \underline{K_{(p)}^{\mu_1\mu_2\dots\mu_p}} \nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_p} + \underline{K_{(p-1)}^{\mu_1\dots\mu_{p-1}}} \nabla_{\mu_1}\dots\nabla_{\mu_{p-1}}$$

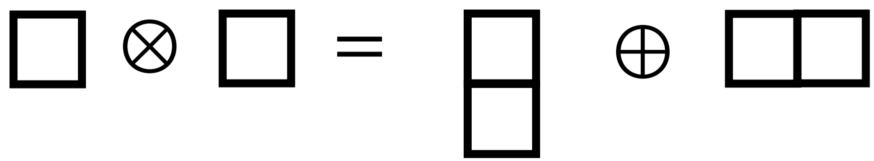
"closed" CKT ? $\qquad \sim \nabla_\nu K_{(p)}^{\nu\mu_1\mu_2\dots\mu_{p-1}}$?

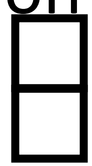
$$+\underline{K_{(p-2)}^{\mu_1\dots\mu_{p-2}}} \nabla_{\mu_1}\dots\nabla_{\mu_{p-2}} + \cdots$$

$\sim \nabla_\nu \nabla_\rho K_{(p)}^{\nu\rho\mu_1\mu_2\dots\mu_{p-2}}$?

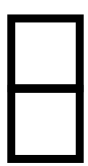
With respect to irreducible representations of $GL(n)$, we decompose the action of the covariant derivative on a 1-form ξ_μ as

$$\boldsymbol{\nabla_\mu \xi_\nu = \nabla_{[\mu} \xi_{\nu]} + \nabla_{(\mu} \xi_{\nu)}} \,.$$



The usual closed condition is equivalent to vanishing the antisymmetric part  of the irreducible representations,

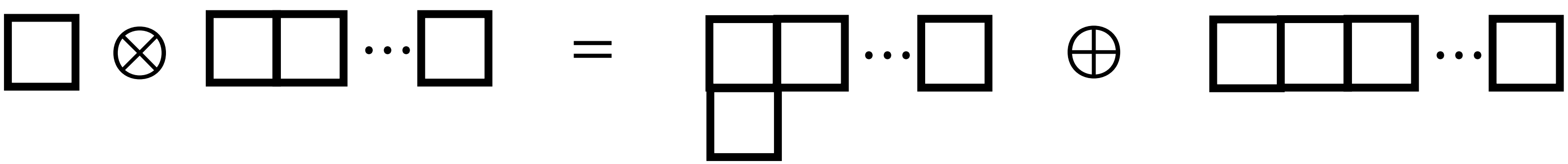
$$\boldsymbol{\nabla_{[\mu} \xi_{\nu]} = 0} \quad \Leftrightarrow \quad \boldsymbol{\nabla_\mu \xi_\nu = \nabla_{(\mu} \xi_{\nu)}}$$



$$= \mathbf{0}$$

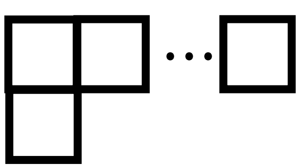
With respect to irr. rep. of $GL(n)$, we are able to decompose the action of the covariant derivative on a symmetric (0,p)-tensor $K_{\mu_1 \dots \mu_p}$ as

$$\boldsymbol{\nabla}_{\nu} \boldsymbol{K}_{\mu_1 \dots \mu_p} = \frac{2}{p+1} \sum_i \boldsymbol{\nabla}_{[\nu} \boldsymbol{K}_{\mu_i] \mu_1 \dots \widehat{\mu_i} \dots \mu_p} + \boldsymbol{\nabla}_{(\nu} \boldsymbol{K}_{\mu_1 \dots \mu_p)}$$



We define “closed condition” for a symmetric (0,p)-tensor $K_{\mu_1 \dots \mu_p}$ by vanishing the first part,

$$\boldsymbol{\nabla}_{[\nu} \boldsymbol{K}_{\mu_1] \mu_2 \dots \mu_p} = 0$$



$= 0$

$$\Leftrightarrow \quad \boldsymbol{\nabla}_{\nu} \boldsymbol{K}_{\mu_1 \dots \mu_p} = \boldsymbol{\nabla}_{(\nu} \boldsymbol{K}_{\mu_1 \dots \mu_p)}$$

rank-p conformal Killing tensor

$$\nabla_{(\mu}K_{\nu_1\ldots\nu_p)} = g_{(\mu\nu_1}L_{\nu_2\ldots\nu_p)}$$

+

"closed condition" for a rank-p sym. tensor

$$\nabla_{\mu}K_{\nu_1\ldots\nu_p} = \nabla_{(\mu}K_{\nu_1\ldots\nu_p)}$$

$$\Downarrow$$

rank-p "closed" conformal Killing tensor

$$\nabla_{\mu}K_{\nu_1\ldots\nu_p} = g_{(\mu\nu_1}L_{\nu_2\ldots\nu_p)}$$

Our result ② (Second-order ladder operators)

If a spacetime (M, $g_{\mu\nu}$) admits a rank-2 "closed" conformal Killing tensor $K^{\mu\nu}$,

$$\nabla_\mu K_{\nu\rho} = g_{(\mu\nu}L_{\rho)}\,, \quad L_\mu = \frac{1}{n}\nabla^\nu \widehat{K}_{\nu\mu}\,, \quad S = \nabla^\mu L_\mu$$

and if the traceless part $\widehat{K}^{\mu\nu}$ of $K^{\mu\nu}$ satisfies

$$R_\mu{}^\rho \widehat{K}_{\nu\rho} - R^\rho{}_{\mu\nu}{}^\sigma \widehat{K}_{\rho\sigma} = \alpha n \widehat{K}_{\mu\nu}\,, \quad R_\mu{}^\nu L_\nu = \beta(n-1)L_\mu\,,$$

then the differential operator (with $k \in \mathbb{R}$)

$$\boldsymbol{D}_k \equiv \widehat{K}^{\mu\nu}\,\nabla_\mu\nabla_\nu + \frac{2k}{3}L^\mu\nabla_\mu + \frac{\alpha k(4\alpha k - \alpha(n+2) + \beta(n-2))}{3n(3\alpha+\beta)}S$$

satisfies the commutation relation

difference

mass squared shifted

$$[\Box, \boldsymbol{D}_k] = 2\alpha(n-1-2k)\boldsymbol{D}_k + 2\boldsymbol{Q}\left(\Box + \frac{(4\alpha k - (3\alpha+\beta)n)\big(4\alpha k - \alpha(n+2) + \beta(n-2)\big)}{4(3\alpha+\beta)}\right)$$

Shift of mass squared

$$\boldsymbol{D}_k : \quad -\frac{(4\alpha \boldsymbol{k} - (3\alpha+\beta)n)\big(4\alpha \boldsymbol{k} - \alpha(n+2) + \beta(n-2)\big)}{4(3\alpha+\beta)}$$

$$\downarrow \qquad \boldsymbol{k} \to \boldsymbol{k} + \frac{\boldsymbol{3\alpha+\beta}}{\boldsymbol{2\alpha}}$$

$$-\frac{\left(4\alpha\left(\boldsymbol{k}+\frac{\boldsymbol{3\alpha+\beta}}{\boldsymbol{2\alpha}}\right) - (3\alpha+\beta)n\right)\left(4\alpha\left(\boldsymbol{k}+\frac{\boldsymbol{3\alpha+\beta}}{\boldsymbol{2\alpha}}\right) - \alpha(n+2) + \beta(n-2)\right)}{4(3\alpha+\beta)}$$

- In the case $\alpha = \beta \equiv \chi$ (including a maximally symmetric spacetime)

$$\boldsymbol{D}_k : \quad -\chi(k-n)(k-1) \;\to\; -\chi(k+2-n)(k+1)$$

$$\boldsymbol{k} \to \boldsymbol{k}+\boldsymbol{2}$$

- For arbitrary α and β, the minimum (or maximum) of mass squared is given by

BF bound

$$m^2 = -\frac{(\alpha+\beta)^2(n-1)^2}{4(3\alpha+\beta)} \overset{\mathrm{AdS}_n}{\Rightarrow} -\frac{(n-1)^2}{4}$$

Summary

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- We have constructed a second-order ladder operator for KG equation,

$$\boldsymbol{D}_k \equiv \widehat{K}^{\mu\nu}\,\nabla_\mu\nabla_\nu + \frac{2k}{3}L^\mu\nabla_\mu + \frac{k(4\alpha k - \alpha(n+2) + \beta(n-2))}{3n(3\alpha+\beta)}S$$

by using the traceless part $\widehat{K}_{\mu\nu}$ of a rank-2 “closed” conformal Killing tensor $K_{\mu\nu}$,

$$\nabla_\mu K_{\nu\rho} = g_{(\mu\nu}L_{\rho)}\,, \quad L_\mu = \frac{1}{n}\nabla^\nu \widehat{K}_{\nu\mu}\,, \quad S = \nabla^\mu L_\mu$$

together with two additional conditions,

$$R_\mu{}^\rho \widehat{K}_{\nu\rho} - R^\rho{}_{\mu\nu}{}^\sigma \widehat{K}_{\rho\sigma} = \alpha n \widehat{K}_{\mu\nu}\,, \quad R_\mu{}^\nu L_\nu = \beta(n-1)L_\mu\,.$$

- In the case $\alpha \neq \beta$, our ladder operators shift mass squared in a different way from those of first order; whereas, in the case $\alpha = \beta$ (including a maximally symmetric spacetime) they reduce to the squares of first-order ladders.