Mass Ladder Operators

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Introduction

In quantum mechanics, ladder operators are very powerful tools.
We can derive physical properties without a detailed knowledge of solutions.

Today, we show ladder operators for massive Klein-Gordon equations on curved spacetime.
We expect this will be also powerful tool.

Introduction

Purpose of this project:

- construct ladder operator for KG eq
- reproduce known results from different point of view
- find new applications

Introduction

My personal motivation:

A phenomena around an extremal black hole is effectively described by a massive KG eq in AdS_2.

There exists a “conserved quantity” if the mass takes special values.

I guessed that there should be mathematically deeper understanding.

mass ladder operators

In n Dim spacetime (or space), if there exists a closed conformal Killing vector ζ_ν

$$\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = Q g_{\mu\nu} \quad (Q = n^{-1} \nabla_\mu \zeta^\mu)$$

$$\nabla_\mu \zeta_\nu - \nabla_\nu \zeta_\mu = 0$$

and ζ_ν is an eigen vector of Ricci tensor

$$R^\mu{}_\nu \zeta^\nu = \chi (n-1) \zeta^\mu \quad (\chi : \text{const.})$$

then, $D_k := \mathcal{L}_{\zeta^\mu} - kQ$ satisfies

$$[\Box, D_k]\Phi = \chi(2k+n-2) D_k \Phi$$
$$+2Q(\Box + \chi k(k+n-1))\Phi$$

$$\begin{cases} m^2 := -\chi k(k+n-1) \\ m'^2 := -\chi(k-1)(k+n-2) \end{cases}$$

Eq. becomes

$$(\Box - m'^2)D_k\Phi = (D_k + 2Q)(\Box - m^2)\Phi$$

If Φ is a sol. of KG eq with m^2

$D_k\Phi$ becomes a sol. of KG eq with m'^2

D_k is mass ladder operator for KG eq

Both m^2, m'^2 are real $\iff$ k is real

$$m^2 = -\chi k(k + n - 1)$$

$$\Longrightarrow k = k_\pm = \frac{1 - n \pm \sqrt{(n-1)^2 - 4m^2/\chi}}{2}$$

$$\frac{\chi}{4}(n-1)^2 \leq m^2, \quad \chi < 0$$ (e.g. AdS)

$$m^2 \leq \frac{\chi}{4}(n-1)^2, \quad \chi > 0$$ (e.g. dS)

Comment

D_k is surjective (onto) map

We can construct all solutions for m'^2

from the solutions for m^2

(proof is straightforward, but need hard calculation)

In this sense,
two different mass systems are "same"

S^2 and Spherical harmonics

$$(\triangle_{S^2} + \ell(\ell+1))Y_{\ell,m} = 0$$

$$L_{\pm}Y_{\ell,m} = \sqrt{(\ell \mp m)(\ell \pm m + 1)}Y_{\ell,m\pm 1}$$

$D_k = \sin\theta\partial_\theta - k\cos\theta$ can shift ℓ

$$D_\ell Y_{\ell,m} = -\sqrt{\frac{(2\ell+1)(\ell^2 - m^2)}{2\ell - 1}}Y_{\ell-1,m}$$

$$D_{-\ell} Y_{\ell-1,m} = \sqrt{\frac{(2\ell-1)(\ell^2 - m^2)}{2\ell + 1}}Y_{\ell,m}$$

KK mode in $AdS_5 \times S^5$

$$\Box_{AdS_5 \times S^5} \Phi = 0 \qquad \Phi = Y_\ell \tilde{\Phi}$$

$$\Longrightarrow (\Box_{AdS_5} - \Lambda \ell(\ell + 4))\tilde{\Phi} = 0 \quad (\ell = 0, 1, 2, \cdots)$$

mass spectrum corresponds to the masses which can be mapped from massless scalar fields in AdS_5

there is a duality among the zero mode and Kaluza-Klein modes on massless scalar fields in $AdS_5 \times S^5$

Aretakis const.

Aretakis showed the “instability” of test scalar field on 4Dim extremal RN BH

[Aretakis 2011]

It is useful to use the Aretakis const.

$$\partial_r^{\ell+1}\Phi|_{\mathcal{H}} = \mathbf{const.}$$

Relation with Newman Penrose const.?

[Bizon, Friedrich, 2013]

We can derive Aretakis const from ladder operator D_k

Aretakis const in AdS_2

$$ds^2 = -r^2 dv^2 + 2dvdr$$

KG eq: $2\partial_v\partial_r\Phi + \partial_r(r^2\partial_r\Phi) = m^2\Phi$

If we assume $m^2 = \ell(\ell+1), (\ell = 0, 1, 2, \cdots)$

$$\partial_v\partial_r^{\ell+1}\Phi\Big|_{r=0} = 0$$

AdS_2 is maximally sym, we can find a quantity which takes const. on every outgoing null hypersurface

$$\underbrace{\left(\partial_v + \frac{r^2}{2}\partial_r\right)}_{\text{outgoing null}} \underbrace{\left[\left(\frac{vr}{2}+1\right)^{2(\ell+1)} \partial_r^{\ell+1}\Phi\right]}_{A_k} = 0$$

outgoing null

A_k

Ladder operators in AdS_2

$$ds^2 = -\frac{4|\Lambda|}{(x^+ - x^-)^2} dx^+ dx^-$$

closed conformal Killing vector :

$$\zeta_{-1} = \partial_+ - \partial_-$$

$$\zeta_0 = x^+ \partial_+ - x^- \partial_-$$

$$\zeta_1 = (x^+)^2 \partial_+ - (x^-)^2 \partial_-$$

$$D_{i,k} = \mathcal{L}_{\zeta_i} - kQ_i \quad (i = -1, 0, 1)$$

KG eq: $(\Box - \ell(\ell+1))\Phi = 0 \qquad (\ell = 0, 1, 2, \cdots)$

Acting D_k ℓ times, $D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi$ becomes massless

$\Box(D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi) = 0$

$$D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi = F(x^+) + G(x^-)$$

$$\frac{\partial}{\partial x^-} D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi = G'(x^-)$$

This coincides with Aretakis const

4D extremal RN black hole

$$ds^2 = -\left(1 - \frac{1}{\rho}\right)^2 dv^2 + 2dvd\rho + \rho^2 d\Omega_{S^2}$$

We can also derive Aretakis const in 4Dim extremal Reissner–Nordström black hole

$$\partial_\rho (D_1 D_2 \cdots D_\ell (e^{(\rho-1)/2}\Phi))\Big|_{\mathcal{H}} = \text{const.}$$

Ladder operator is useful for less symmetric spacetimes which have approximate conformal symmetry

Summary, future works

- We can construct mass ladder operator D_k from closed conformal Killing vector
- D_k is a powerful tool to understand
  - ladder operator for $Y_{\ell,m}$
  - duality in KK mode in $AdS_5 \times S^5$
  - Aretakis constant
  - susy quantum mechanics

  I hope that many other topics will be explained by D_k.
- **Higher derivative operator →Tsuyoshi's talk**
- vector, tensor, spinor
- AdS/CFT
- relation with AdS instability