PBH abundance from the random Gaussian curvature perturbation and a local density threshold

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Primordial BHs

[Zeldovich and Novikov(1967),Hawking(1971)]

◎Remnant of primordial non-linear inhomogeneity

◎Trace the inhomogeneity in the early universe

◎May provide a fraction of dark matter and BH binaries

◎Several aspects

-Inflationary models which provide a large number of PBHs

-Threshold of PBH formation

-Observational constraints on PBH abundance

-Spin distribution of PBHs

Estimation of Abundance

◎Simplest conventional estimation(Press-Schecheter)

- Assumption 1:threshold is given by the amplitude of ζ or δ

- Assumption 2:Gaussian distribution of ζ at each peak of ζ or δ

- Production probability(PBH fraction to the total density) β_0

$$\beta_0 = 2\left(2\pi\sigma^2\right)^{1/2}\int_{|\delta_{\rm th}|}^{\infty} \exp\left[-\frac{\delta^2}{2\sigma^2}\right] d\delta = \mathrm{erfc}\left(\frac{|\delta_{\rm th}|}{\sqrt{2}\sigma}\right)$$

◎Questions

- Is Gaussian distribution of δ valid?

- Is giving the threshold by ζ appropriate?

Gaussian δ?

[Kopp et. al.(2011)]

◎Flat background

$$\bar{H}^2 = \frac{8\pi}{3}\bar{\rho}$$

◎Closed FLRW model as an overdense region

- Metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(\mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega^2) = -\mathrm{d}t^2 + a^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega^2\right)$$

- Hubble eq.

$$H^2 = \frac{8\pi}{3}\rho - \frac{1}{a^2}$$

- Density perturbation on uniform Hubble slice

$$\delta^{\mathrm{UH}} = \frac{\rho}{\bar{\rho}} - 1 = \frac{H^2 + \frac{1}{a^2}}{H^2} - 1 = \frac{1}{a^2H^2} > 0$$

- δ^{UH} at horizon entry($H^{-1} \sim ar$)

$$\delta^{\mathrm{UH}}_H = \frac{1}{a^2H^2} \sim \frac{a^2r^2}{a^2} = r^2 = \sin^2\chi < 1$$

$\Rightarrow \delta$ cannot have Gaussian pdf

Threshold of ζ?

[Young et. al.(2014),Harada et. al(2015)]

◎$\zeta \sim \phi$:**Newton potential,** $\delta \sim \rho$: **density**

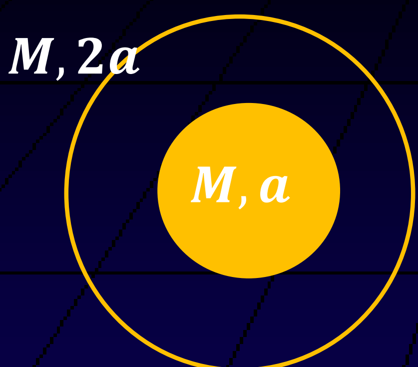
◎**Case 1: homogeneous sphere with radius** a

$$\phi(r) = -\frac{GM}{r} \quad \text{for } r \geq a$$

$$\phi(r) = -\frac{3GM}{2a} + \frac{GM}{2a^3} r^2 \quad \text{for } r < a$$

$$\Rightarrow \phi(0) = -\frac{3GM}{2a}$$

◎**Case 2: homogeneous sphere + shell**



$$\phi(r) = -\frac{2GM}{r} \quad \text{for } r \geq 2a$$

$$\phi(r) = -\frac{2GM}{a} + \frac{GM}{2a^3} r^2 \quad \text{for } r < a$$

$$\Rightarrow \phi(0) = -\frac{2GM}{a}$$

◎**The potential($\phi \sim \zeta$) depends on environments**

$\delta_{\rm th}$ and Statistics of ζ

◎Threshold should be set by δ

◎Statistical properties are well known for ζ

◎What we have to do

- Statistics of ζ ⇒ probability of δ ⇒ PBH formation prob.
- w/ long-wavelength approx. and w/o linear approx.

◎Relation between ζ and δ w/ long-wavelength approx.

$$\delta = -\frac{4(1+w)}{3w+5}\frac{1}{a^2H^2}\ e^{5/2\zeta}\Delta \mathrm{e}^{-\zeta/2}$$

comoving slicing, $p = w\rho$

Expansion around Extremum

◎**Spatial metric**

$$\mathrm{d}l^2 = a^2 \mathrm{e}^{-2\zeta} \tilde{\gamma}_{ij} \mathrm{d}x^i \mathrm{d}x^j$$

◎**Taylor expansion of ζ**

$$\zeta = \zeta_0 + \zeta_1^i x_i + \frac{1}{2} \zeta_2^{ij} x_i x_j + O(x^3)$$

◎**Density perturbation at an extremum($\zeta_1^i = 0$)**

$$\delta_{\mathrm{ext}} = \frac{2(1+w)}{3w+5} \frac{1}{a^2 H^2} \mathrm{e}^{2\zeta_0} \zeta_2 \quad \textbf{with } \zeta_2 = \zeta_2^{11} + \zeta_2^{22} + \zeta_2^{33}$$

◎**Renormalized scale factor** $\bar{a} := a\mathrm{e}^{-\zeta_0}$

$$\delta_{\mathrm{ext}} = \frac{2(1+w)}{3w+5} \frac{1}{\bar{a}^2 H^2} \zeta_2$$

Horizon Entry

◎**Scale of the perturbation:** $1/k_*$

$$k_*^2 := -\zeta_2/\zeta_0$$

- cf. single Fourier mode $\zeta_0 \cos(k_* x) \simeq \zeta_0 \left(1 - \frac{1}{2} k_*^2 x^2 + \cdots\right)$
- cf. Gaussian $\zeta_0 \exp(-\frac{1}{2} k_*^2 x^2) \simeq \zeta_0 \left(1 - \frac{1}{2} k_*^2 x^2 + \cdots\right)$

◎**Horizon entry condition**

$q k_* = \bar{a} H$ **with** $q = O(1)$**: uncertainty of horizon entry**

◎**Density perturbation at horizon entry**

$$\delta_{\rm ext} = \frac{2(1+w)}{3w+5} \frac{1}{\bar{a}^2 H^2} \zeta_2 \Rightarrow \quad \delta_{\rm H} = \frac{2(1+w)}{3w+5} \frac{\mu}{q^2} \quad \text{with} \quad \mu := -\zeta_0$$

◎**Condition for PBH formation:** $\delta_{\rm H} < \delta_{\rm th} \Rightarrow \mu_{\rm th} := \frac{3w+5}{2(1+w)} q^2 \delta_{\rm th}$

Gaussian Dist. of ζ

[Bardeen et. al(1986)]

◎Probability distribution of linear combinations of $\zeta(x^i)$

$$\mathcal{P}(V_I)\mathrm{d}^nV = (2\pi)^{-n/2}|\mathrm{det}\mathcal{M}|^{-1/2}\exp\left[-\frac{1}{2}V_I(\mathcal{M}^{-1})^{IJ}V_J\right]\mathrm{d}^nV$$

correlation matrix: $\mathcal{M}_{IJ} = \int \frac{d\vec{k}}{(2\pi)^3}\frac{d\vec{k}'}{(2\pi)^3} < \tilde{V}_I^*\left(\vec{k}\right)\tilde{V}_J\left(\vec{k}'\right) >$

$$\tilde{V}_I\left(\vec{k}\right) = \int d^3x V_I(\vec{x})e^{i\vec{k}\vec{x}}$$

◎Non-zero correlations in pairs of ζ_0, ζ_1^i, ζ_2^{ij}

$$\sigma_0^2 := \int \frac{dk}{k}P(k) =< \zeta_0\zeta_0 >$$

$$\sigma_1^2 := \int \frac{dk}{k}k^2P(k) = -3 < \zeta_0\zeta_2^{ii} > = 3 < \zeta_1^i\zeta_1^i >$$

$$\sigma_2^2 := \int \frac{dk}{k}k^4P(k) = 5 < \zeta_2^{ii}\zeta_2^{ii} > = 15 < \zeta_2^{ii}\zeta_2^{jj} > = 15 < \zeta_2^{ij}\zeta_2^{ij} > \text{ with } i \neq j$$

◎Power spectrum $P(k)$ fixes everything

Variable Transformation

◎All 10 variables: $V_I = (\zeta_0, \zeta_1^1, \zeta_1^2, \zeta_1^3, \zeta_2^{11}, \zeta_2^{22}, \zeta_2^{33}, \zeta_2^{12}, \zeta_2^{23}, \zeta_2^{31})$

◎$(\zeta_2^{11}, \zeta_2^{22}, \zeta_2^{33}, \zeta_2^{12}, \zeta_2^{23}, \zeta_2^{31}) \to (\lambda_1, \lambda_2, \lambda_3, \theta_1, \theta_2, \theta_3)$

eigen values of the matrix ζ_2^{ij} with $\lambda_1 \geq \lambda_2 \geq \lambda_3$

Euler angles to take the principal direction

$$\mathrm{d}^6\zeta_2 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)\mathrm{d}^3\lambda \sin\theta_1 \mathrm{d}^3\theta$$

Integration w.r.t. θ_i→factor 2π

◎7 variables: $(\zeta_0, \zeta_1^i, \lambda_i) \to (\nu, \eta_i, \xi_i)$ [Bardeen et. al(1986)]

$$\nu = -\zeta_0/\sigma_0 \qquad \xi_1 = (\lambda_1 + \lambda_2 + \lambda_3)/\sigma_2$$

$$\eta_i = \zeta_1^i/\sigma_1 \qquad \xi_2 = \frac{1}{2}(\lambda_1 - \lambda_3)/\sigma_2 \qquad \xi_2 = \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3)/\sigma_2$$

Extremum Number Density

[Bardeen et. al(1986)]

◎$(\nu,\xi_1)\to(\chi,\xi_1)$ **with** $\chi:=\xi_1/\nu\sim k_*^2$

◎**7 variables:** $\mathcal{P}(\nu,\chi,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3)\mathrm{d}\nu\mathrm{d}\chi\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\vec{\eta}$

◎**Number density distribution of extrema in** $(\vec{x},\nu,\chi)$

$n_{\rm ext}(\vec{x},\nu,\chi)\Delta\vec{\mathrm{x}}\Delta\nu\Delta\chi$:=**number of extrema in** $\Delta\vec{\mathrm{x}}\Delta\nu\Delta\chi$

$$\Rightarrow n_{\rm ext}(\vec{x},\nu,\chi)\mathrm{d}\vec{x}\mathrm{d}\nu\mathrm{d}\chi=\sum_p\delta(\vec{x}-\vec{x}_p)\delta(\nu-\nu_p)\delta(\chi-\chi_p)\mathrm{d}\vec{x}\mathrm{d}\nu\mathrm{d}\chi$$

$\vec{x}_p$: **extremum position** ν_p,χ_p:**value at the extremum**

◎**Extremum** $\zeta_1^i=0\Rightarrow\eta_i=0$

$$\Rightarrow\delta(\vec{x}-\vec{x}_p)=\sigma_1^{-3}|\lambda_1\lambda_2\lambda_3|\delta(\vec{\eta})$$ **with** $\lambda_1\lambda_2\lambda_3=\frac{1}{27}\left((\xi_1+\xi_3)^2-9\xi_2^2\right)(\xi_1-2\xi_3)\sigma_2^3$

Peak Number Density(1)

[Bardeen et. al(1986)]

◎**Number density distribution of extrema in** $(\vec{x},\nu,\chi)$

$$n_{\rm ext}(\vec{x},\nu,\chi)\mathrm{d}\vec{x}\mathrm{d}\nu\mathrm{d}\chi=\sum_p \sigma_1^{-3}|\lambda_1\lambda_2\lambda_3|\delta(\vec{\eta})\delta(\nu-\nu_p)\delta(\chi-\chi_p)\mathrm{d}\vec{x}\mathrm{d}\nu\mathrm{d}\chi$$

◎**Averaged peak number density** $n_{\rm pk}(\nu,\chi)=\langle n_{\rm ext}\Theta(\lambda_3)\rangle$

$$n_{\rm pk}(\nu,\chi)\mathrm{d}\nu\mathrm{d}\chi=\langle n_{\rm ext}\Theta(\lambda_3)\rangle\,\mathrm{d}\nu\mathrm{d}\chi$$

$$=\sigma_1^{-3}\,\mathrm{d}\nu\mathrm{d}\chi\int \mathrm{d}\nu_p\mathrm{d}\chi_p\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\vec{\eta}$$

$$\times\left[\mathcal{P}\left(\nu_p,\chi_p,\vec{\xi}_2,\vec{\xi}_3,\vec{\eta}\right)|\lambda_1\lambda_2\lambda_3|\delta(\vec{\eta})\delta(\nu-\nu_p)\delta(\chi-\chi_p)\Theta(\lambda_3)\right]$$

$$=\frac{3^{1/2}}{(2\pi)^{3/2}}\left(\frac{\sigma_2}{\sigma_1}\right)^3 f(\nu\chi)\mathcal{P}_1(\nu,\chi)\,\mathrm{d}\nu\mathrm{d}\chi$$

$$f(u)=\frac{1}{2}u(u^2-3)\left[\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{5}{2}}u\right)+\mathrm{erf}\left(\sqrt{\frac{5}{2}}u\right)\right]+\sqrt{\frac{2}{5\pi}}\left[\left(\frac{8}{5}+\frac{31}{4}u^2\right)\exp\left(-\frac{5}{8}u^2\right)+\left(-\frac{8}{5}+\frac{1}{2}u^2\right)\exp\left(-\frac{5}{2}u^2\right)\right]$$

$$\mathcal{P}_1(\nu,\chi)\mathrm{d}\nu\mathrm{d}\chi=\frac{1}{2\pi}\frac{1}{\sqrt{1-\gamma^2}}|\nu|\exp\left(-\frac{1}{2}\frac{\nu^2(\chi-\gamma)^2}{1-\gamma^2}-\frac{1}{2}\nu^2\right)\mathrm{d}\nu\mathrm{d}\chi$$

$$\gamma=\sigma_0\sigma_2/\sigma_1^2$$

Peak Number Density(2)

◎Averaged peak number density $n_{\mathrm{pk}}(\nu,\chi)\mathrm{d}\nu\mathrm{d}\chi$

◎Transformation of variables(1)

$$(\nu,\chi) \to (\mu,k_*) \;\Rightarrow\; n_{\mathrm{pk}}(\nu,\chi)\mathrm{d}\nu\mathrm{d}\chi \to n_{\mathrm{pk}}^{(k_*)}(\mu,k_*)\mathrm{d}\mu\mathrm{d}k_*$$

$$\mu = -\zeta_0 = \nu\sigma_0,\; k_* = -\zeta_2/\zeta_0 = \sigma_2\chi/\sigma_0$$

◎Consider each moment σ_i as a function of k_*

Window functions

$$P(k) \to P(k)\big(W(k/k_*)\big)^2 \qquad \text{e.g. } W_{\mathrm{G}}(k/k_*) = \exp\left(-\frac{1}{2}\frac{k^2}{k_*^2}\right)$$

Moments are k_* dependent

$$\text{e.g.} \quad \sigma_0^2 = \int_0^\infty \frac{\mathrm{d}k}{k}\big(W(k/k_*)\big)^2 P(k)$$

PBH Number Density

◎**Transformation of variables(2)**

$$(\mu, k_*) \to (\mu, M) \;\Rightarrow\; n_{\rm pk}^{(k_*)}(\mu, k_*)\mathrm{d}\mu\mathrm{d}k_* \to n_{\rm pk}^{(M)}(\mu, M)\mathrm{d}\mu\mathrm{d}M$$

$$M(k_*, \mu) = \frac{1}{2}\alpha H^{-1} = \frac{1}{2}\alpha\frac{\bar{a}}{qk_*} = \frac{1}{2}\alpha\frac{a}{qk_*}\mathrm{e}^{\mu} = \frac{1}{2}\alpha\frac{ak_*}{qk_*^2}\mathrm{e}^{\mu} = M_{\rm eq}\frac{k_{\rm eq}^2}{k_*^2}\mathrm{e}^{2\mu}$$

where $H^{1/2} \propto k_* \propto 1/a$, $qak_* = \mathrm{e}^{\mu}H_{\rm eq}a_{\rm eq}^2$ **and** $M_{\rm eq} = \alpha H_{\rm eq}^{-1}/2$

◎**PBH number density** $n_{\rm BH}(M)\mathrm{d}\ln M$

$$n_{BH}\mathrm{d}\ln M := \left(\int_{\mu_{th}}^{\infty} n_{\rm pk}^{M}\mathrm{d}\mu\right) M\mathrm{d}\ln M$$

◎**PBH fraction**

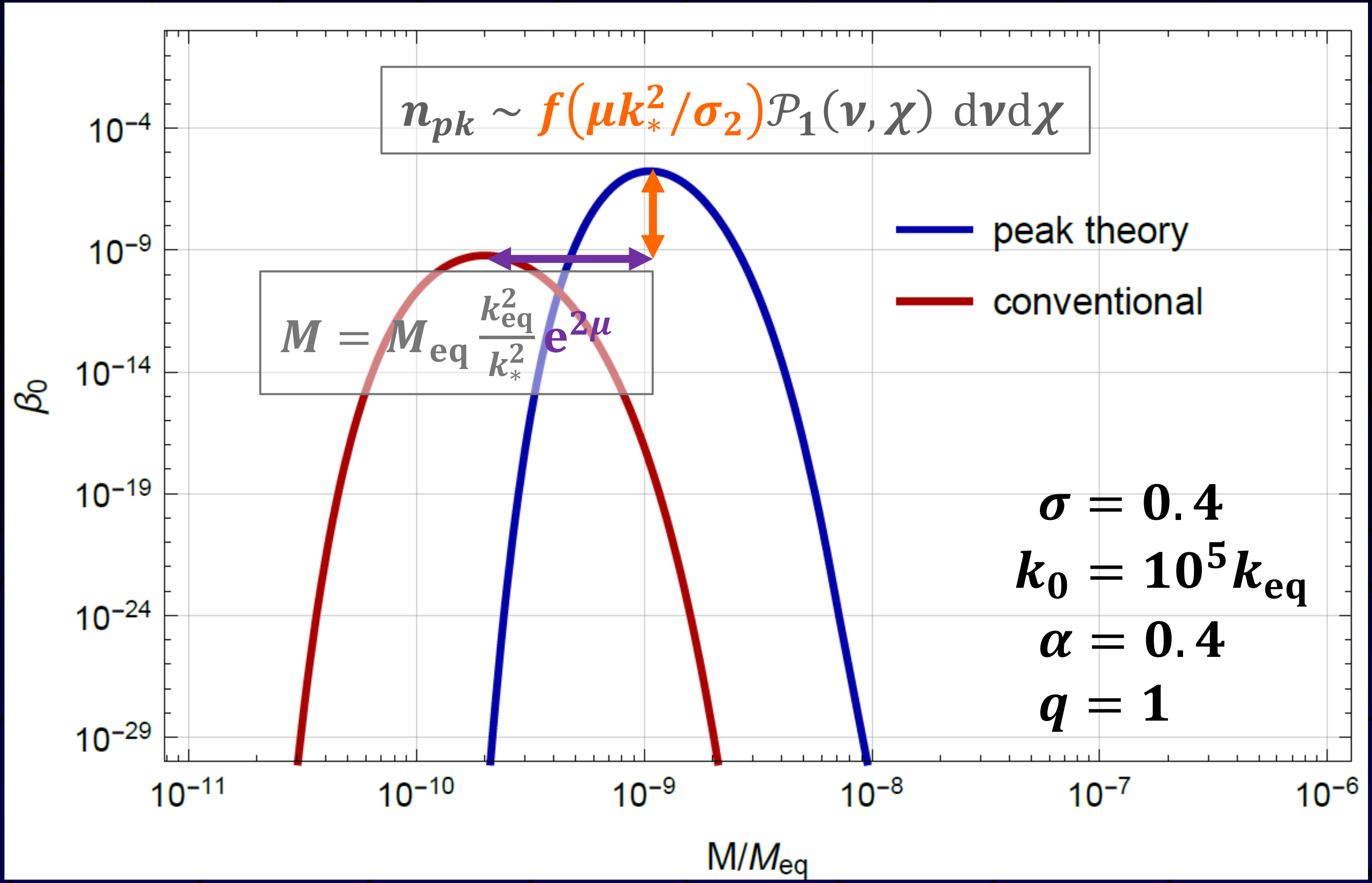
$$\beta_0\mathrm{d}\ln M := \frac{Mn_{\rm BH}}{\rho a^3}\mathrm{d}\ln M$$

$$\beta_0 = 2\cdot\frac{3^{1/2}\alpha}{(2\pi)^{3/2}q^3}\left(\frac{M}{M_{\rm eq}}\right)^{1/2}k_{\rm eq}^{-1}$$

$$\times\int_{\mu_{\rm th}}^{\infty}\frac{\sigma_2^2\mathrm{e}^{2\mu}\mu f(\mu k_*^2/\sigma_2)}{\sigma_0\sigma_1^3\sqrt{1-\gamma^2}}\exp\left[-\frac{\mu^2}{2\sigma_2^2}\left(\frac{\left(k_*^2-\gamma\sigma_2/\sigma_0\right)^2}{1-\gamma^2}\right)\right]\exp\left(-\frac{\mu^2}{2\sigma_0^2}\right)\mathrm{d}\mu\,\mathrm{d}\ln M$$

An Extended $P(k)$

◎ $P(k) = \sigma^2 \left(\frac{k}{k_0}\right)^2 \exp\left(-\frac{k^2}{k_0^2}\right)$ **with Gaussian window function**



Summary

◎PBH abundance from $\delta_{\rm th}$ and Gaussian prob. dis. of ζ

◎Renormalization of zero mode of ζ

◎PBH number density from the peak theory

◎The mass spectrum is shifted to larger mass scales

due to the non-linear relation $M = M_{\rm eq}\frac{k_{\rm eq}^2}{k_*^2}\mathrm{e}^{2\mu}$

◎The max value of the spectrum becomes larger

due to the factor $f\left(\mu k_*^2/\sigma_2\right)$ which
originates from the relation $\delta\left(\vec{x}-\vec{x}_p\right)=\sigma_1^{-3}|\lambda_1\lambda_2\lambda_3|\delta(\vec{\eta})$
the measure difference between the param. and real spaces

Thank you
for your attention!