The problem of motion in gravity theories

history, and

new perspectives :

The example of binary black-holes

in Einstein-Maxwell-Dilaton (EMD) theory

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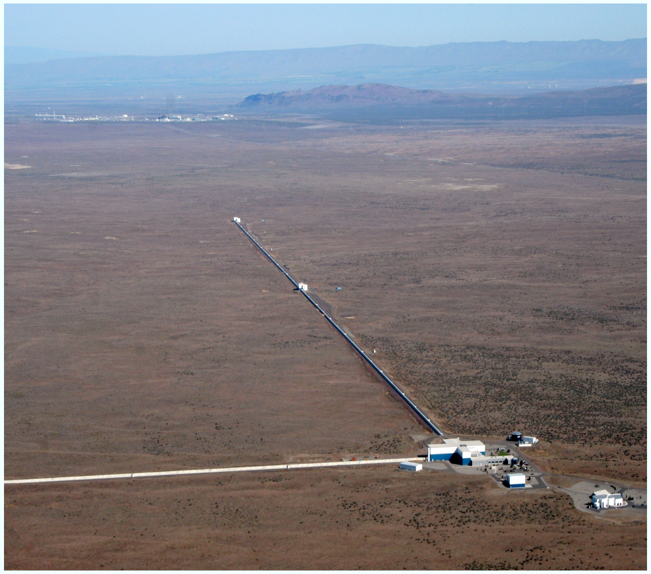
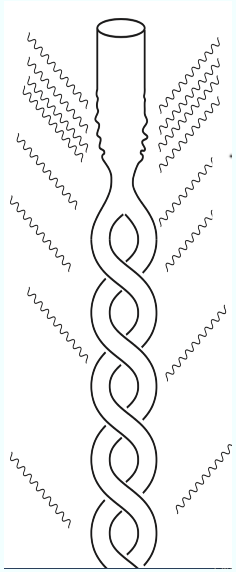
with Félix-Louis Julié and Marcela Cárdenas
CNRS, APC-Paris Diderot

Kyoto, 23 February 2018

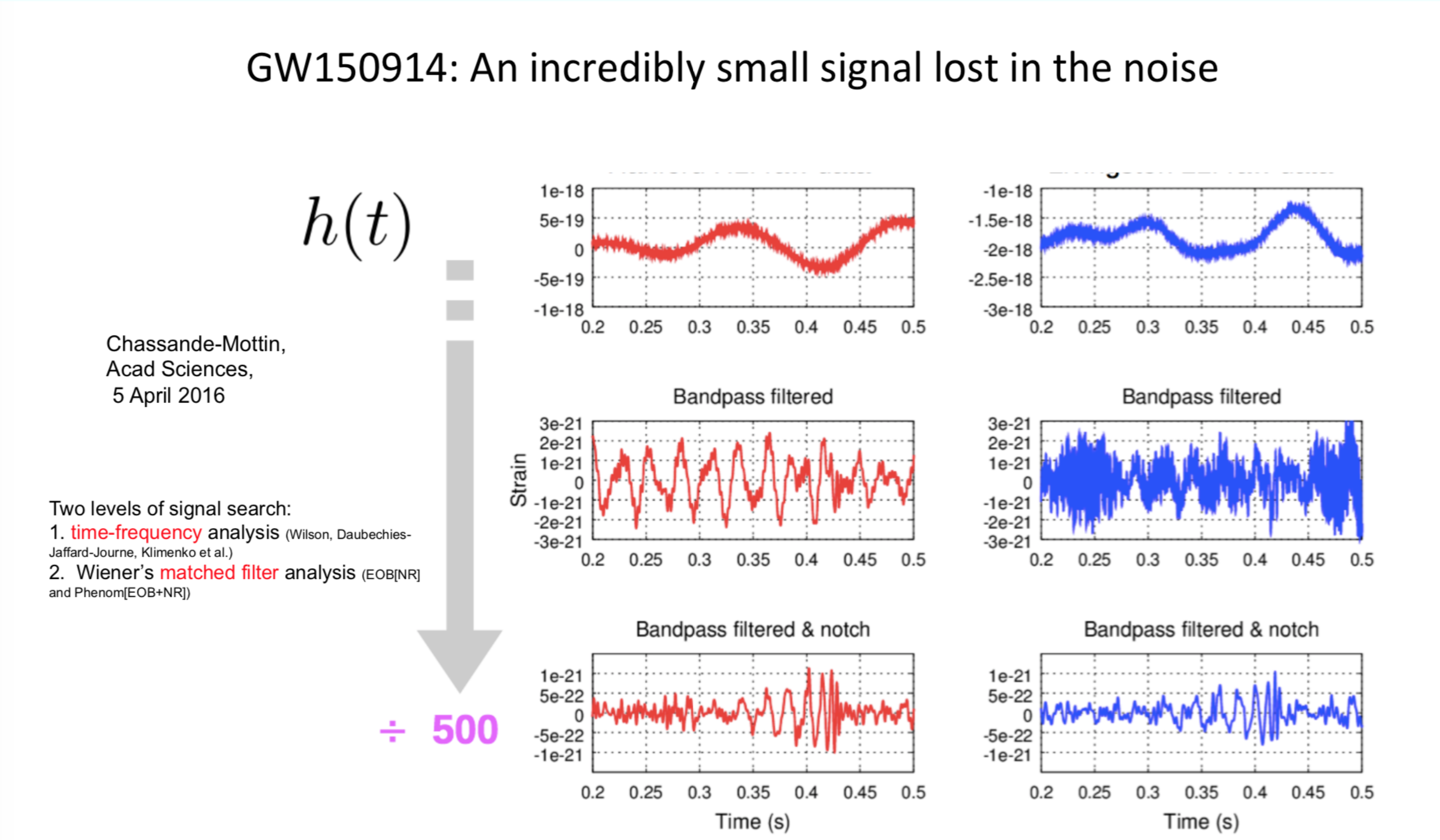
The new era in astronomy

GW150914 : first observation of a BBH coalescence by LIGO
GW170817: first observation of a BNS coalescence by LIGO/Virgo with EM counterparts

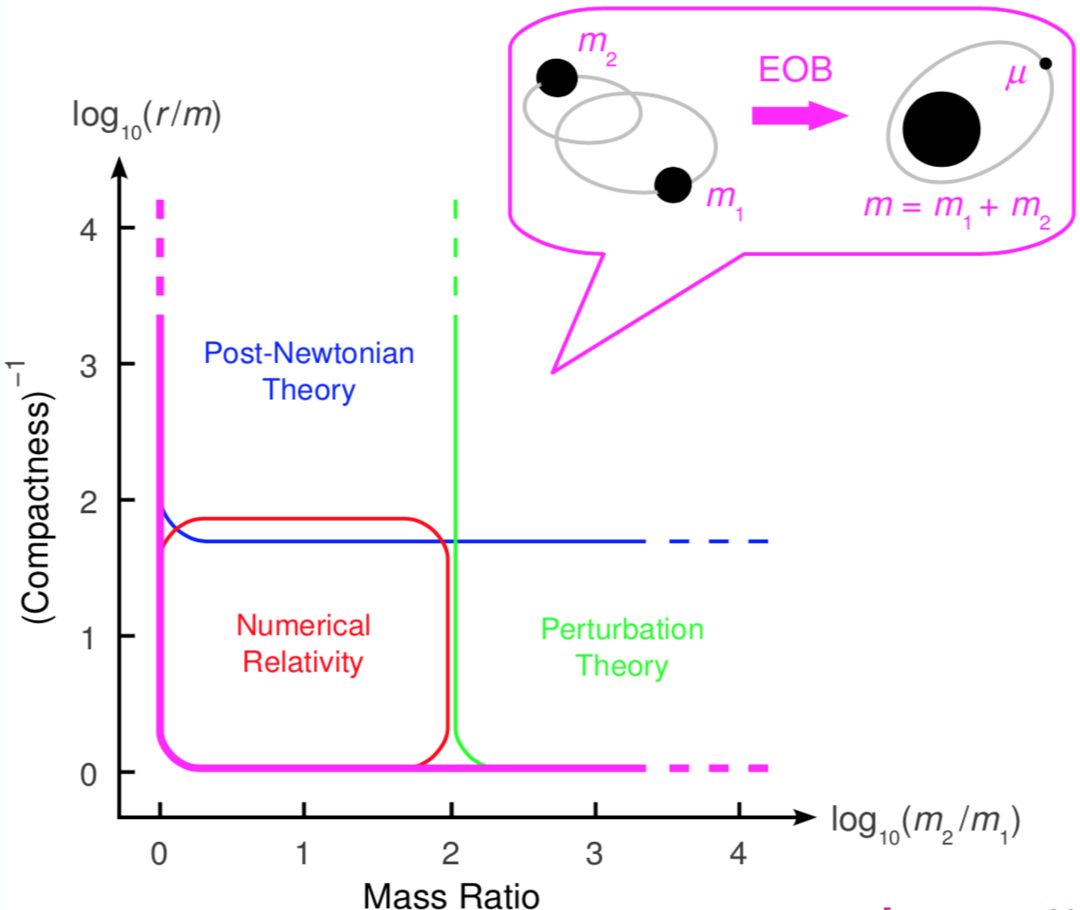

Will allow to probe **modified theories of gravity**, in the strong-field regime near merger, an "important and doable problem, which is still in infancy" (to paraphrase Takashi Nakamura).

Needles in a haystack



(from T. Damour conference, Hannover 2016)

“Knowing the chirp to hear it”...



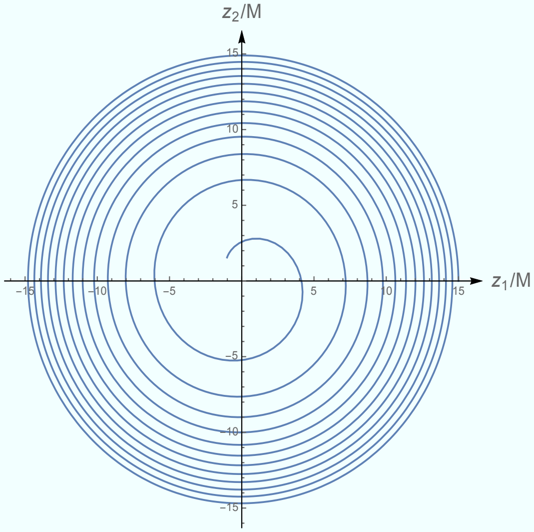
[courtesy Alexandre Le Tiec]

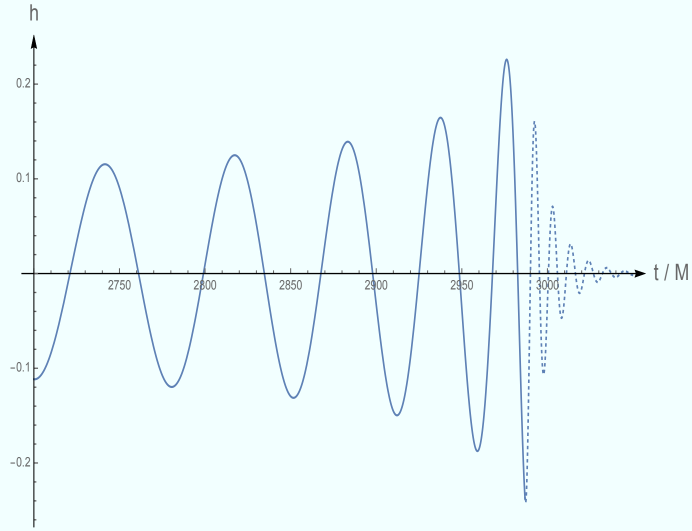
from L. Blanchet conference, Hannover 2016

The "effective-one-body" (EOB) approach

A. Buonanno and T. Damour, 1998

- maps the two-body general relativistic Post-Newtonian (PN) dynamics to the motion of a test particle in an effective SSS metric
- defines a resummation of the PN dynamics to describe analytically the coalescence of 2 compact objects from inspiral to merger
- is instrumental to build libraries of waveform templates for LIGO/Virgo





Aim : extend the EOB approach to modified gravities

Outline of the talk

1. The Einstein-Maxwell-Dilaton (EMD) black hole
   as a simple example of a "hairy" black hole

2. The action for a binary EMD black hole system
   or, how to "skeletonize" hairy black holes

3. The (conservative) dynamics of an EMD black hole binary
   vs "state-of-the-art" in scalar-tensor theories and GR

   - Lagrangian and Hamiltonian for the relative motion
   - Mapping to an effective-one-body (EOB) hamiltonian
   - A first flavour of possible tests

References (all in arXiv)

Thermodynamics sheds light on black hole dynamics
Marcela Cárdenas, Félix-Louis Julié, Nathalie Deruelle, arXiv:1712.02672

On the motion of hairy black holes in EMD theories
Félix-Louis Julié, JCAP 1801 (2018)

Reducing the 2-body problem in ST theories to the motion of a test particle : a ST-EOB approach
Félix-Louis Julié Phys.Rev. D97 (2018) no.2, 024047

Two body pb in ST theories as a deformation of GR : an EOB approach
Félix-Louis Julié, Nathalie Deruelle Phys.Rev. D95 (2017) 12, 124054

On conserved charges and thermodynamics of AdS4 dyonic BHs
Marcela Cárdenas, Oscar Fuentealba, Javier Matulich, JHEP 1605 (2016)

Einstein-Katz action, variational principle, Noether charges and the thermodynamics of AdS-BHs
Andrés Anabalón, Nathalie Deruelle, Félix-Louis Julié, JHEP 1608 (2016)

The Einstein-Maxwell-Dilaton (EMD) black hole

Isolated EMD black holes

G. W. Gibbons 1982, GWG and K. i. Maeda 1988, GWG 1996
D. Garfinkle, G. T. Horowitz and A. Strominger 1991

Vacuum Einstein-Maxwell-dilaton action of gravity

$$16\pi\, I_{\rm vac}[g_{\mu\nu}, A_\mu, \varphi] = \int d^4x \sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - e^{-2a\varphi}F^2\right)$$

Field equations :

$$R_{\mu\nu} = 2\partial_\mu\varphi\,\partial_\nu\varphi + 2e^{-2a\varphi}\left(F_\mu{}^\lambda F_{\nu\lambda} - \tfrac{1}{4}g_{\mu\nu}F^2\right)$$
$$D_\mu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 0 \quad , \quad \Box\,\varphi = -\tfrac{1}{2}e^{-2a\varphi}F^2$$

Static, spherically symmetric, solutions depend a priori on 5 integration constants. "Electric" black hole solutions depend on only 3. For $a = 1$:

$$ds^2 = -\left(1 - \tfrac{r_+}{r}\right)dt^2 + \left(1 - \tfrac{r_+}{r}\right)^{-1}dr^2 + r^2\left(1 - \tfrac{r_-}{r}\right)d\Omega^2$$

$$A_t = -\sqrt{\tfrac{r_+ r_-}{2}}\,\frac{e^{\varphi_\infty}}{r}\,, \qquad A_i = 0\,, \qquad \varphi = \varphi_\infty + \tfrac{1}{2}\ln\left(1 - \tfrac{r_-}{r}\right)$$

EMD black hole thermodynamics
(case $a = 1$)

Temperature : $T = \frac{1}{4\pi r_+}$ (or surface gravity $\kappa = 2\pi T$)

Electric potential : $\Phi = A_t(r \to \infty) - A_t(r_+) = \sqrt{\frac{r_+ r_-}{2}}\, \frac{e^{\varphi_\infty}}{r_+}$

Entropy : $S = \pi r_+^2 \left(1 - \frac{r_-}{r_+}\right)$ (or area : $\mathcal{A} = 4S$; or $M_{\rm irr} = \sqrt{\frac{\mathcal{A}}{4\pi}}$)

Associated global charges :

$$Q = \sqrt{\frac{r_+ r_-}{2}}\, e^{-\varphi_\infty} \quad , \quad M = \frac{1}{2} r_+ - \frac{1}{2} \int r_- d\varphi_\infty$$

(see M. Henneaux et al 2002,..., Cárdenas et al 2016, Julié et al 2016)

The variations of S, Q, and M wrt r_+, r_- and φ_∞, are such that

$$T\delta S = \delta M - \Phi \delta Q$$

The action for a binary EMD black hole system

“Skeletonizing” an EMD black hole

in GR : Mathisson 1931, Infeld 1950,...

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - e^{-2a\varphi}F^2\right) + I_{\rm bh}\left[\Psi, g_{\mu\nu}, \varphi, A^\mu\right]$$

$$I_{\rm bh} = -\int m(\varphi)\,ds + q\int A_\mu\,dx^\mu$$

Linear coupling to A^μ, and q constant, to preserve $U(1)$ symmetry ;

$m_A(\varphi)$: $m \neq const$ because φ cannot be “gauged away”

(Eardley 1975, Damour Esposito-Farese 1992)

Question : how are q and $m(\varphi)$ related to the parameters characterizing the black hole, that is, r_+, r_- and φ_∞ ?

Answer : by identifying the EMD black hole solution to that of the field equations for the skeletonized body above.

Félix-Louis Julié, 2017

The "sensitivity" of an EMD black hole

- Field equations (with $T^{\mu\nu} = \int ds\, m(\varphi)\frac{\delta^{(4)}(x-z)}{\sqrt{-g}} u^\mu u^\nu$)

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + e^{-2a\varphi}\left(2F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{2}g_{\mu\nu}F^2\right) + 8\pi\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T\right)$$

$$D_\nu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 4\pi q \int ds\, \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} u^\mu$$

$$\Box\varphi = -\tfrac{a}{2}e^{-2a\varphi}F^2 + 4\pi \int ds\, \frac{\delta^{(4)}(x-z)}{\sqrt{-g}}\frac{dm}{d\varphi}$$

- Lowest order asymptotic solution in the body rest-frame :

$$g^{\text{asym}}_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{2m_\infty}{r}\right),\ A^{\text{asym}}_t = -\frac{q\, e^{2\varphi_\infty}}{r},\ \varphi^{\text{asym}} = \varphi_\infty - \frac{1}{r}\frac{dm}{d\varphi}|_\infty$$

to be identified with the EMD black hole solution (case $a = 1$) :

$$g^{\text{asym}}_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{r_+}{r}\right),\ A^{\text{asym}}_t = -\sqrt{\frac{r_+r_-}{2}}\frac{e^{\varphi_\infty}}{r},\ \varphi^{\text{asym}} = \varphi_\infty - \frac{r_-}{2r}$$

Hence a differential equation, with a unique solution

$r_+ = 2m_\infty$, $r_- = 2\frac{dm}{d\varphi}$, $q = \sqrt{\frac{r_+r_-}{2}}e^{-\varphi_\infty}|_\infty$ so that $q^2 = 2m\frac{dm}{d\varphi}e^{2\varphi}|_\infty$

$$m(\varphi) = \sqrt{\mu^2 + q^2\frac{e^{2\varphi}}{2}}$$

Félix-Louis Julié, 2017

The parameters of a skeletonized $(a = 1)$ EMD black hole

$q = \sqrt{\frac{r_+ r_-}{2}} e^{-\varphi_\infty}$, $r_+ = 2m_\infty$, $r_- = 2\frac{dm}{d\varphi}|_\infty$, and $m(\varphi) = \sqrt{\mu^2 + q^2 \frac{e^{2\varphi}}{2}}$

Recall : the global charges and entropy of an EMD black hole are

$$Q = \sqrt{\frac{r_+ r_-}{2}} e^{-\varphi_\infty} \ , \ M = \frac{1}{2} r_+ - \frac{1}{2} \int r_- d\varphi_\infty, \text{ and } S = \pi r_+^2 \left(1 - \frac{r_-}{r_+}\right)$$

Hence $Q = q$ is a constant : $\delta Q = 0$. Also : $\delta M = \delta m_\infty - \frac{dm}{d\varphi} \delta\varphi|_\infty = 0$

Our skeletonized BHs exchange no charge nor energy with their environment.

Now, since $T\delta S = \delta M - \Phi \delta Q$,

the black hole entropy is also a constant.

Therefore μ can be identified to a function of the BH entropy. Indeed :

$$\mu = \sqrt{\frac{S}{4\pi}} \quad \Longrightarrow \quad m(\varphi) = \sqrt{\frac{S}{4\pi} + \frac{e^{2\varphi}}{2} Q^2}$$

with (for an Einstein-Hilbert action) $S = \frac{A}{4}$ and $M_{\rm irr}^2 = \frac{S}{4\pi}$

Cárdenas, Julié, ND, 2018

Hence, all in all,

Skeletonized action for a binary EMD black hole system :

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - e^{-2a\varphi}F^2\right) + I_{\rm bbh}\left[g_{\mu\nu}, \varphi, A^\mu\right]$$

$$I_{\rm bbh} = -\sum_A \int m_A(\varphi) ds_A + \sum_A q_A \int A_\mu\, dx^\mu_A$$

with $q_A = Q_A$ and $m_A(\varphi) = \sqrt{\frac{S_A}{4\pi} + \frac{e^{2\varphi}}{2}Q_A^2}$ (for $a = 1$)

where the charges Q_A remain constant (true until coalescence)

where the entropies S_A *also* remain constant (*not* true at coalescence).

*

The action I is the starting point
to study the relative motion of the two black holes.

The (conservative) dynamics of an EMD black hole binary

Lagrangian and Hamiltonian for the relative motion

The 1st Post-Newtonian (1PN) Lagrangian of an EMD BH binary

- Field equations (with $T^{\mu\nu}_A = \int ds_A \, m_A(\varphi) \frac{\delta^{(4)}(x-z_A)}{\sqrt{-g}} u^\mu_A u^\nu_A$)

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + e^{-2a\varphi}\left(2F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{2}g_{\mu\nu}F^2\right) + 8\pi\sum_A \left(T^A_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^A\right)$$

$$D_\nu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 4\pi q_A \sum_A \int ds_A \frac{\delta^{(4)}(x-z_A)}{\sqrt{-g}} u^\mu_A$$

$$\Box\varphi = -\tfrac{a}{2}e^{-2a\varphi}F^2 + 4\pi\sum_A \int ds_A \frac{\delta^{(4)}(x-z_A)}{\sqrt{-g}} \frac{dm_A}{d\varphi}$$

- Work in harmonic and Lorenz gauges

  Write : $g_{00} = -e^{-2U}$, $g_{0i} = -4g_i$, $g_{ij} = \delta_{ij}e^{2V}$

  $A_t = \delta A_t$, $A_i = \delta A_i$, $\varphi = \varphi_\infty + \delta\varphi$

  Weak field $\mathcal{O}(v^2) \sim \mathcal{O}(m/r)$ iteration.

- Solve and obtain

$$V = U + \mathcal{O}\left(v^6\right),\; g_i = \sum_A \frac{m^\infty_A v^i_A}{r_A} + \mathcal{O}\left(v^5\right),\; \varphi = \varphi_\infty + \sum_A \frac{m'^\infty_A}{r_A} + \cdots, \text{ etc}$$

The fields being known at 1st PN order,
plug their expressions in the Lagrangian for body A in the field of B :

$$I_{\rm A} = \int dt L_A \quad \text{with} \quad L_A = -m_A(\varphi)\frac{ds_A}{dt} + q_A A_\mu \frac{dx_A^\mu}{dt}$$

Symmetrize, regularize and obtain (FL Julié) :

$$L_{1PN}^{\rm EMD} = -(m_A + m_B) + \left[\tfrac{1}{2}(m_A v_A^2 + m_B v_B^2) + \frac{G_{AB}\, m_A m_B}{R}\right]$$
$$+\tfrac{1}{8}(m_A v_A^4 + m_B v_B^4)$$
$$+\frac{G_{AB}\, m_A m_B}{R}\left[\tfrac{3}{2}(v_A^2 + v_B^2) - \tfrac{7}{2}(v_A.v_B) - \tfrac{1}{2}(N.v_A)(N.v_B) + \bar\gamma_{AB}(\vec v_A - \vec v_B)^2\right]$$
$$-\frac{{G_{AB}}^2\, m_A m_B}{2R^2}\left[m_A(1+2\bar\beta_B) + m_B(1+2\bar\beta_A)\right]$$

where $G_{AB} = 1 + \alpha_A\alpha_B - e_A e_B$ with $e_A = (q_A/m_A)\, e^{\varphi_\infty}$

$$m_A = m_A|_{\varphi_\infty},\ \alpha_A = (m'_A/m_A)|_\infty,\ \beta_A = \alpha'_A|_{\varphi_\infty}$$

$$\bar\gamma_{AB} = \frac{-4\alpha_A\alpha_B + 3e_A e_B}{2(1+\alpha_A\alpha_B - e_A e_B)} \quad \bar\beta_A = \frac{1}{2}\frac{\beta_A {\alpha_B}^2 - 2e_A e_B(a\alpha_B - \alpha_A\alpha_B) + e_B^2(1 + a\alpha_A - e_A^2)}{1+\alpha_A\alpha_B - e_A e_B}$$

Deviations from GR to be expected ?

$$G_{AB} = 1 + \alpha_A \alpha_B - e_A e_B, \quad e_A = (q_A/m_A)\, e^{\varphi_\infty}$$

$$m_A = m_A|_{\varphi_\infty},\ \alpha_A = (m'_A/m_A)|_\infty,\ \beta_A = \alpha'_A|_{\varphi_\infty}$$

$$\bar\gamma_{AB} = \frac{-4\alpha_A\alpha_B + 3e_Ae_B}{2(1+\alpha_A\alpha_B - e_Ae_B)} \quad \bar\beta_A = \frac{1}{2}\frac{\beta_A{\alpha_B}^2 - 2e_Ae_B(a\alpha_B - \alpha_A\alpha_B) + e_B^2(1+a\alpha_A - e_A^2)}{1+\alpha_A\alpha_B - e_Ae_B}$$

In scalar tensor theories (where $q_A = q_B = 0$),

the deviations to GR are driven by α_A^2, α_B^2 or $\alpha_A\alpha_B$. Now,

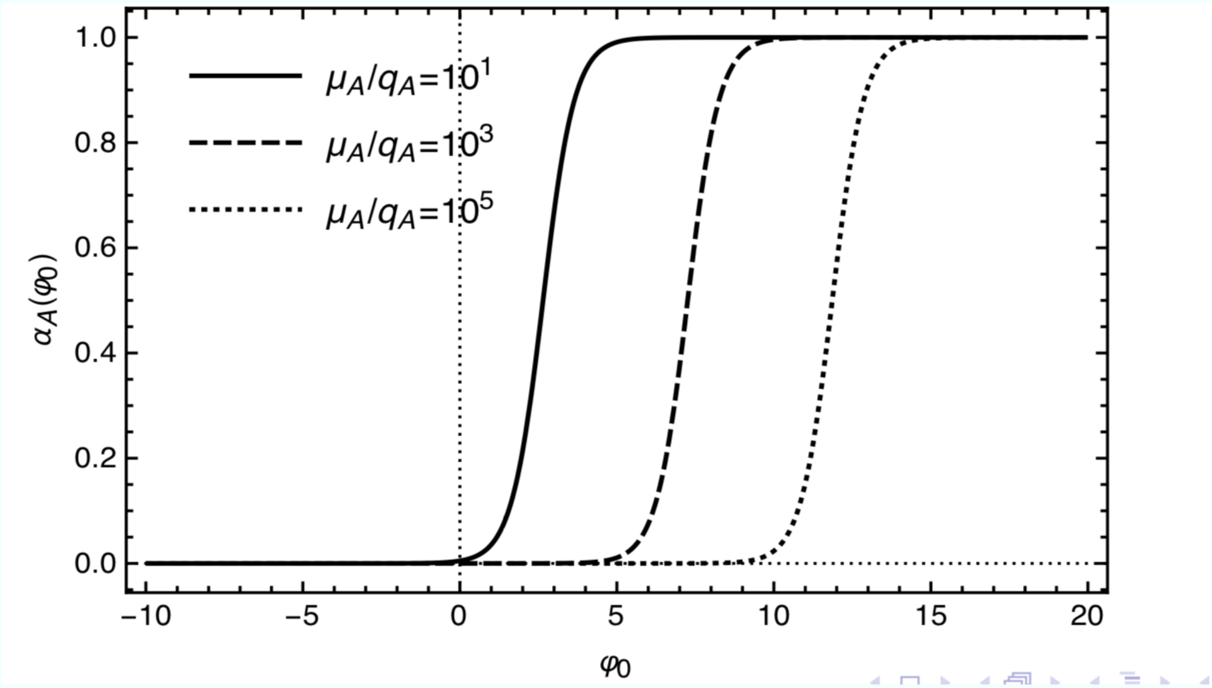
Black holes have no scalar (primary) hair (m_A and m_B are constant) : no deviations from GR,

In EMD theories, BH do have hair, $m_A(\varphi) = \sqrt{\frac{S_A}{4\pi} + \frac{e^{2\varphi}}{2}Q_A^2}$ (for $a = 1$)

In EMD theories (F Julié, 2017)

$m_A(\varphi) = \sqrt{\mu^2 + e^{2\varphi} q_A^2/2}$ $(a = 1)$, with $q_A = Q_A$ and $\mu_A = \sqrt{S_A/4\pi}$

hence : $\alpha_A \equiv (m'_A/m_A)|_\infty = \frac{1}{1+\exp 2\left[\ln\left(\frac{\mu_A\sqrt{2}}{q_A}\right)-\varphi_\infty\right]}$



See also E.W. Hirschmann, L. Lehner, et al. arXiv:1706.09875

Studying the dynamics of hairy (EMD) BH is perhaps worth the effort...

L^{EMD}_{1PN} at 1PN and the state-of-the-art

Scalar-tensor theories 2-body lagrangians ($q_A = q_B = 0$) :

1PN : T. Damour and G. Esposito-Farèse, 1992 (25 years before L^{EMD}_{1PN})

2PN : S. Mirshekari, C. Will, 2013 :
In Einstein frame (FL Julié, ND 2017), see FLJ poster

"Conjecture" : Its extension to describe the dynamics in EMD theories at 2PN requires the calculation of only a few new coefficients.

3PN : L. Bernard, 2018 Talk, March 1st

The 2-body lagrangian in general relativity

1PN Lorentz- Droste (1917) ; Fichtenholz (1950) (100 years before L^{EMD}_{1PN})

4PN L. Bernard, L.Blanchet, G.Faye, and T. Marchand, 2017
(plus A. Bohé and S. Marsat)

The 2PN 2-body lagrangian in scalar-tensor theories (harmonic coordinates)

$$
\begin{aligned}
L_{\rm 2PK} =\ & \frac{1}{16} m_A^0 V_A^6 \\
&+ \frac{G_{AB}\, m_A^0 m_B^0}{R}\left[\frac{1}{8}(7+4\bar\gamma_{AB})\left(V_A^4 - V_A^2(\vec N\cdot\vec V_B)^2\right) - (2+\bar\gamma_{AB})V_A^2(\vec V_A\cdot\vec V_B) + \frac{1}{8}(\vec V_A\cdot\vec V_B)^2\right. \\
&\qquad \left. + \frac{1}{16}(15+8\bar\gamma_{AB})V_A^2V_B^2 + \frac{3}{16}(\vec N\cdot\vec V_A)^2(\vec N\cdot\vec V_B)^2 + \frac{1}{4}(3+2\bar\gamma_{AB})\vec V_A\cdot\vec V_B(\vec N\cdot\vec V_A)(\vec N\cdot\vec V_B)\right] \\
&+ \frac{G_{AB}^2\, m_B^0 (m_A^0)^2}{R^2}\left[\frac{1}{8}\left(2+12\bar\gamma_{AB}+7\bar\gamma_{AB}^2+8\bar\beta_B-4\delta_A\right)V_A^2 + \frac{1}{8}\left(14+20\bar\gamma_{AB}+7\bar\gamma_{AB}^2+4\bar\beta_B-4\delta_A\right)V_B^2\right. \\
&\qquad - \frac{1}{4}\left(7+16\bar\gamma_{AB}+7\bar\gamma_{AB}^2+4\bar\beta_B-4\delta_A\right)\vec V_A\cdot\vec V_B - \frac{1}{4}\left(14+12\bar\gamma_{AB}+\bar\gamma_{AB}^2-8\bar\beta_B+4\delta_A\right)(\vec V_A\cdot\vec N)(\vec V_B\cdot\vec N) \\
&\qquad \left. + \frac{1}{8}\left(28+20\bar\gamma_{AB}+\bar\gamma_{AB}^2-8\bar\beta_B+4\delta_A\right)(\vec N\cdot\vec V_A)^2 + \frac{1}{8}\left(4+4\bar\gamma_{AB}+\bar\gamma_{AB}^2+4\delta_A\right)(\vec N\cdot\vec V_B)^2\right] \\
&+ \frac{G_{AB}^3 (m_A^0)^3 m_B^0}{2R^3}\left[1+\frac{2}{3}\bar\gamma_{AB}+\frac{1}{6}\bar\gamma_{AB}^2+2\bar\beta_B+\frac{2}{3}\delta_A+\frac{1}{3}\epsilon_B\right] + \frac{G_{AB}^3 (m_A^0)^2 (m_B^0)^2}{8R^3}\left[19+8\bar\gamma_{AB}+8(\bar\beta_A+\bar\beta_B)+4\zeta\right] \\
&- \frac{1}{8}G_{AB}\, m_A^0 m_B^0\left(2(7+4\bar\gamma_{AB})\vec A_A\cdot\vec V_B(\vec N\cdot\vec V_B) + \vec N\cdot\vec A_A(\vec N\cdot\vec V_B)^2 - (7+4\bar\gamma_{AB})\vec N\cdot\vec A_A V_B^2\right) \\
&\qquad + (A\leftrightarrow B)
\end{aligned}
$$

where $\delta_A \equiv \dfrac{(\alpha_A^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}$ $\quad\epsilon_A \equiv \dfrac{(\beta'_A\alpha_B^3)^0}{(1+\alpha_A^0\alpha_B^0)^3}$ $\quad\zeta \equiv \dfrac{\beta_A^0\alpha_A^0\alpha_B^0\beta_B^0}{(1+\alpha_A^0\alpha_B^0)^3}$ $(A\leftrightarrow B)$

S. Mirshekari, C. Will, 2013 ; (Félix-Louis Julié, ND, 2017)

The 2 PN Hamiltonian

$$L^{\rm EMD}_{1PN} = -(m_A + m_B) + \left[\tfrac{1}{2}(m_A v_A^2 + m_B v_B^2) + \frac{G_{AB}\, m_A m_B}{R}\right]$$

$$+\tfrac{1}{8}(m_A v_A^4 + m_B v_B^4)$$

$$+\frac{G_{AB}\, m_A m_B}{R}\left[\tfrac{3}{2}(v_A^2 + v_B^2) - \tfrac{7}{2}(v_A.v_B) - \tfrac{1}{2}(N.v_A)(N.v_B) + \bar\gamma_{AB}(\vec v_A - \vec v_B)^2\right]$$

$$-\frac{{G_{AB}}^2\, m_A m_B}{2R^2}\left[m_A(1+2\bar\beta_B) + m_B(1+2\bar\beta_A)\right]$$

$L^{\rm EMD}_{2PN}$ is given by $L^{\rm ST}_{2PN}$ with some replacements and modulo 3 coefficients yet to be found.

$L^{\rm EMD}_{2PN}$ depends on the positions, velocities *and* accelerations of A and B
It is *allowed* to replace them by $\vec A_A \to -\vec N G_{AB} m_B^0 / R^2$

[This amounts to change the coordinate system :
T. Ohta, H. Okamura, T. Kimura, K. Hiida, 1974 *vs* T. Damour ND, 1981
Problem solved by Schäfer 1983, Damour-Schäfer 1991.]

In the centre-of-mass frame $(M = m_A + m_B,\ \mu = m_A m_B / M)$:

$$H = M + \left(\frac{P^2}{2\mu} - G_{AB}\frac{\mu M}{R}\right) + H^{1\mathrm{PN}} + H^{2\mathrm{PN}} + \cdots$$

$$\frac{H^{1\mathrm{PN}}}{\mu} = (h_1^{1\mathrm{PK}}\hat{P}^4 + h_2^{1\mathrm{PK}}\hat{P}^2\hat{P}_R^2 + h_3^{1\mathrm{PK}}\hat{P}_R^4) + \frac{\left(h_4^{1\mathrm{PK}}\hat{P}^2 + h_5^{1\mathrm{PK}}\hat{P}_R^2\right)}{\hat{R}} + \frac{h_6^{1\mathrm{PK}}}{\hat{R}^2}$$

$$\frac{H^{2\mathrm{PN}}}{\mu} = (h_1^{2\mathrm{PK}}\hat{P}^6 + h_2^{2\mathrm{PK}}\hat{P}^4\hat{P}_R^2 + h_3^{2\mathrm{PK}}\hat{P}^2\hat{P}_R^4 + h_4^{2\mathrm{PK}}\hat{P}_R^6)$$

$$\frac{(h_5^{2\mathrm{PK}}\hat{P}^4 + h_6^{2\mathrm{PK}}\hat{P}_R^2\hat{P}^2 + h_7^{2\mathrm{PK}}\hat{P}_R^4)}{\hat{R}} + \frac{\left(h_8^{2\mathrm{PK}}\hat{P}^2 + h_9^{2\mathrm{PK}}\hat{P}_R^2\right)}{\hat{R}^2} + \frac{h_{10}^{2\mathrm{PK}}}{\hat{R}^3}$$

where $G_{AB} = 1 + \alpha_A\alpha_B - e_A e_B$ with $e_A = (q_A/m_A)\, e^{\varphi_\infty}$

$$m_A = m_A|_{\varphi_\infty},\ \alpha_A = (m'_A/m_A)|_\infty,\ \beta_A = \alpha'_A|_{\varphi_\infty}$$ and β'_A

$H^{1\mathrm{PN}}$ known for EMD black holes ; $H^{2\mathrm{PN}}$ known for scalar theories

The 17 $h_a^{i\mathrm{PK}}$ depend on the 8 (+2) parameters characterizing the theory.

State-of-the-art in general relativity

slides from T Damour, Berlin conference 2015

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\rm N}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2}\frac{G m_1 m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$\begin{aligned} c^2 H_{\rm 1PN}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{G m_1 m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right) \\ & + \frac{1}{4}\frac{G m_1 m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \leftrightarrow 2), \end{aligned}$$

$$\begin{aligned} c^4 H_{\rm 2PN}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{G m_1 m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right. \\ & \left. - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right) \\ & + \frac{1}{4}\frac{G^2 m_1 m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right) \\ & - \frac{1}{8}\frac{G m_1 m_2}{r_{12}}\frac{G^2(m_1^2 + 5 m_1 m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2), \end{aligned}$$

2-body Taylor-expanded 3PN Hamiltonian [JS 98, DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = &-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\Bigg(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2}+6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\\
&-10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}\\
&+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}+\frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}\\
&+\frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\
&-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\Bigg)+\frac{G^2m_1m_2}{r_{12}^2}\Bigg(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\\
&-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
&-\frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}\\
&+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\
&-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Bigg)+\frac{G^3m_1m_2}{r_{12}^3}\Bigg(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\\
&+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
&+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Bigg)\\
&+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right)+(1\leftrightarrow 2).
\end{aligned}
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2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$
\begin{aligned}
c^8 H^{\mathrm{local}}_{4\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
& + \frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
& + \frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
& + \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2), \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2} \\
& - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3} \\
& + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} \\
& - \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
& - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4} \\
& - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} \\
& - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4} \\
& - \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4}, \qquad \text{(A4a)}
\end{aligned}
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\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64 m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
& + \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
& + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2} \\
& - \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} \\
& - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3} \\
& + \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} \\
& + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3} \\
& + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4} \\
& - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4} \\
& - \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8 m_2^6}, \qquad \text{(A4b)}
\end{aligned}
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\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2} \\
& + \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2} \\
& - \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2} \\
& + \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32 m_2^4}, \qquad \text{(A4c)}
\end{aligned}
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\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) = {} & \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
& + \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} \\
& + \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2} \\
& - \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
& + \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2} \\
& + \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}, \qquad \text{(A4d)}
\end{aligned}
$$

$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800 m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2} + \frac{105\mathbf{p}_2^2}{32 m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2}, \qquad \text{(A4e)}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) = {} & \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\
& + \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2} \\
& + \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}, \qquad \text{(A4f)}
\end{aligned}
$$

$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3 m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2 m_2^2. \qquad \text{(A4g)}
$$

$$
\begin{aligned}
H^{\mathrm{nonloc}}_{4\mathrm{PN}}(t) = {} & -\frac{1}{5}\frac{G^2 M}{c^8} I^{(3)}_{ij}(t) \\
& \times \mathrm{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{\mathrm{d}v}{|v|} I^{(3)}_{ij}(t+v), \qquad 12
\end{aligned}
$$

The (conservative) dynamics
of an EMD black hole binary

Mapping to an effective-one-body (EOB) hamiltonian

(following Buonanno-Damour 1998)

The effective-one-body (EOB) “strategy”

- Start from the best available PN Hamiltonian. At 2 PN, 17 coefficients

$$H(Q,P) = M + \left(\frac{P^2}{2\mu} - G_{AB}\frac{\mu M}{R}\right) + H^{1\mathrm{PN}} + H^{2\mathrm{PN}} + \cdots$$

$$\frac{H^{1\mathrm{PN}}}{\mu} = (h_1^{1\mathrm{PK}}\hat{P}^4 + h_2^{1\mathrm{PK}}\hat{P}^2\hat{P}_R^2 + h_3^{1\mathrm{PK}}\hat{P}_R^4) + \frac{\left(h_4^{1\mathrm{PK}}\hat{P}^2 + h_5^{1\mathrm{PK}}\hat{P}_R^2\right)}{\hat{R}} + \frac{h_6^{1\mathrm{PK}}}{\hat{R}^2}$$

$$\frac{H^{2\mathrm{PN}}}{\mu} = (h_1^{2\mathrm{PK}}\hat{P}^6 + h_2^{2\mathrm{PK}}\hat{P}^4\hat{P}_R^2 + h_3^{2\mathrm{PK}}\hat{P}^2\hat{P}_R^4 + h_4^{2\mathrm{PK}}\hat{P}_R^6)$$

$$\frac{(h_5^{2\mathrm{PK}}\hat{P}^4 + h_6^{2\mathrm{PK}}\hat{P}_R^2\hat{P}^2 + h_7^{2\mathrm{PK}}\hat{P}_R^4)}{\hat{R}} + \frac{\left(h_8^{2\mathrm{PK}}\hat{P}^2 + h_9^{2\mathrm{PK}}\hat{P}_R^2\right)}{\hat{R}^2} + \frac{h_{10}^{2\mathrm{PK}}}{\hat{R}^3}$$

- Canonically transform it $H(Q,P) \to H(q,p)$

  At 2PN order the generic generating function depends on 9 parameters

$$\frac{G(Q,p)}{R\,p_r} = \left(\alpha_1\mathcal{P}^2 + \beta_1\hat{p}_r^2 + \frac{\gamma_1}{\hat{R}}\right) + \left(\alpha_2\mathcal{P}^4 + \beta_2\mathcal{P}^2\hat{p}_r^2 + \gamma_2\hat{p}_r^4 + \delta_2\frac{\mathcal{P}^2}{\hat{R}} + \epsilon_2\frac{\hat{p}_r^2}{\hat{R}} + \frac{\eta_2}{\hat{R}^2}\right)$$

- Define $H_e(q,p)$ through the quadratic relation $(\nu = \mu/M)$

$$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(q,p)-M}{\mu}\right)\right] \qquad \text{(Damour 2016)}$$

- Impose $H_e(q,p)$ to be the Hamiltonian for geodesic motion in a static, spherically symmetric spacetime

$$ds_e^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2 \quad , \quad H_e(q,p) = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{\hat{r}^2}\right)}$$

At 2PN order $A(r)$ and $B(r)$ depend on 5 coefficients :

$$A(r) = 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \cdots \quad , \quad B(r) = 1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots$$

Hence : 17-(9+5)= 3 constraints (at 2PN) :

It works for ST tensor theories (Julié ND 2017)

$$A(r) = 1 - 2\left(\frac{G_{AB}M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \left[2\nu + \delta a_3^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^3 + \cdots$$

$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right) + \left[2(2-3\nu) + \delta b_2^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \cdots$$

- Resummation

We started from $$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(q,p)-M}{\mu}\right)\right]$$

we showed $$H_e(q,p) = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{\hat{r}^2}\right)}$$

By inversion one finally obtains the resummed EOB Hamiltonian

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)} \quad \text{where} \quad H_e = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{r^2}\right)}$$

The dynamics deduced from $H_{\rm EOB}$ and the 2-body Hamiltonian H are, by construction, equivalent up to 2PN order

Moreover $H_{\rm EOB}$ defines a very simple resummed dynamics which can be extended to the strong field regime at coalescence.

The (conservative) dynamics of an EMD black hole binary

A first flavour of possible tests

Location of the ISCO

Location of the ISCO

The 2 BH dynamics reduces to geodesic motion in

$$ds_e^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2 \quad , \quad H_e(q,p) = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{\hat{r}^2}\right)}$$

Location and orbital frequency of the last stable circular orbit (ISCO)

$$\frac{A''}{A'} = \frac{(Au^2)''}{(Au^2)'} \quad , \quad \Omega = \frac{j u^2 A}{G_{AB} M E \sqrt{1+2\nu(E-1)}}$$

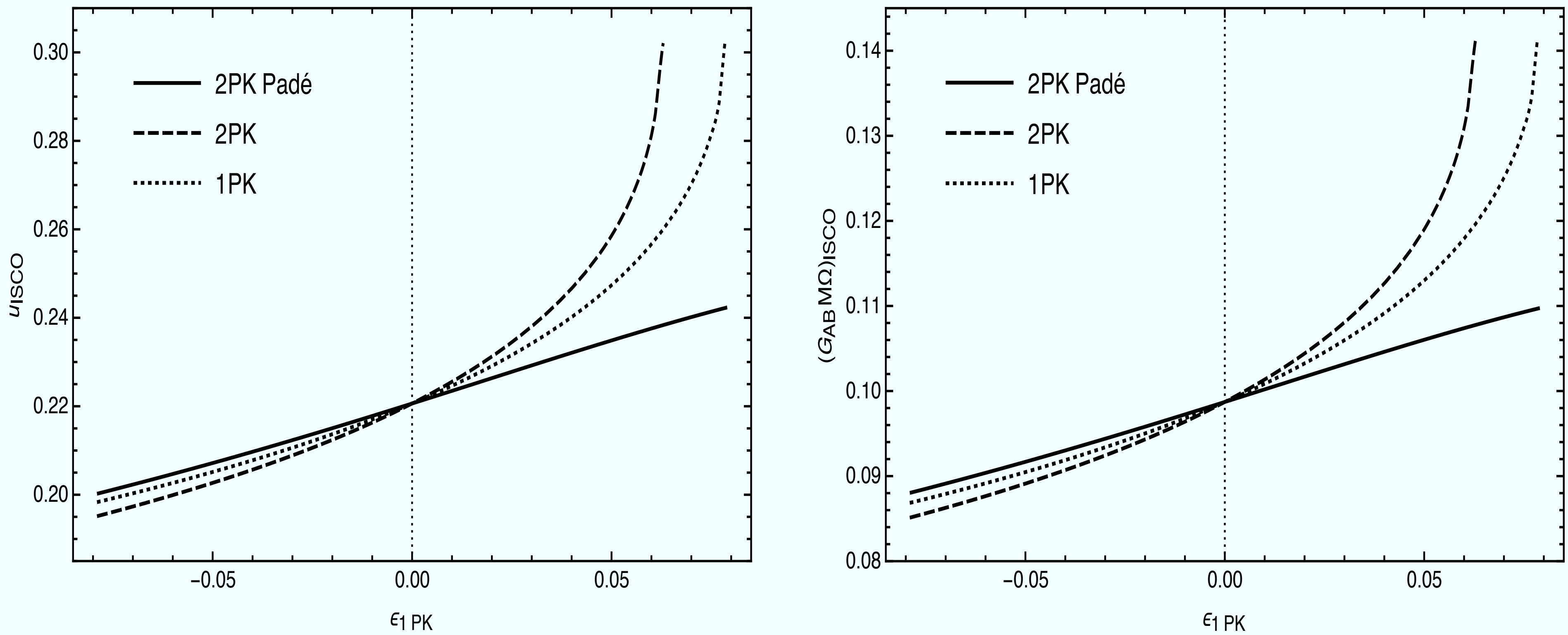
with $u = \frac{G_{AB}M}{r}$, $\quad j^2(u) = -\frac{A'}{(Au^2)'}$, $\quad E(u) = A\sqrt{\frac{2u}{(Au^2)'}}$

$$A(u\,;\nu) = A^{\mathrm{GR}}_{\mathrm{EOBNR}}(u\,;\nu) + 2\epsilon_{\mathrm{1PK}} u^2 + (\epsilon^0_{\mathrm{2PK}} + \nu\epsilon^\nu_{\mathrm{2PK}})u^3$$

For EMD black holes $\epsilon_{\mathrm{1PK}} \equiv \langle\bar{\beta}\rangle - \bar{\gamma}_{AB}$ is a simple function of

$$m_A(\varphi) = \sqrt{\frac{S_A}{4\pi} + Q_A^2 \frac{e^{2\varphi}}{2}}$$

A typical strong-field feature : orbital frequency at the ISCO



[equal-mass case ($\nu = 1/4$), setting $\epsilon_{1\rm PK} = \epsilon^0_{2\rm PK} = \epsilon^\nu_{2\rm PK}$]

$$A = \mathcal{P}^1_5[A^{\rm GR}_{\rm EOBNR}(u\,;\nu) + 2\epsilon_{1\rm PK}u^2 + (\epsilon^0_{2\rm PK} + \nu\epsilon^\nu_{2\rm PK})u^3]$$

Recapitulation

The (conservative) dynamics of an EMD black hole binary
vs “state-of-the-art” in scalar-tensor theories and GR

- Lagrangian and Hamiltonian for the relative motion
- Mapping to an effective-one-body (EOB) hamiltonian
- A first flavour of possible tests

What next ?

- Radiation reaction forces and full dynamics
- Waveforms
- Other models...

Conclusion

Coalescing binary black holes are ideal celestial systems to test theories of gravity.

Predicting the gravitational wave signatures of coalescing "hairy" black holes will give new constraints on modified gravity theories and help to better understand General Relativity

Thank you

for your attention