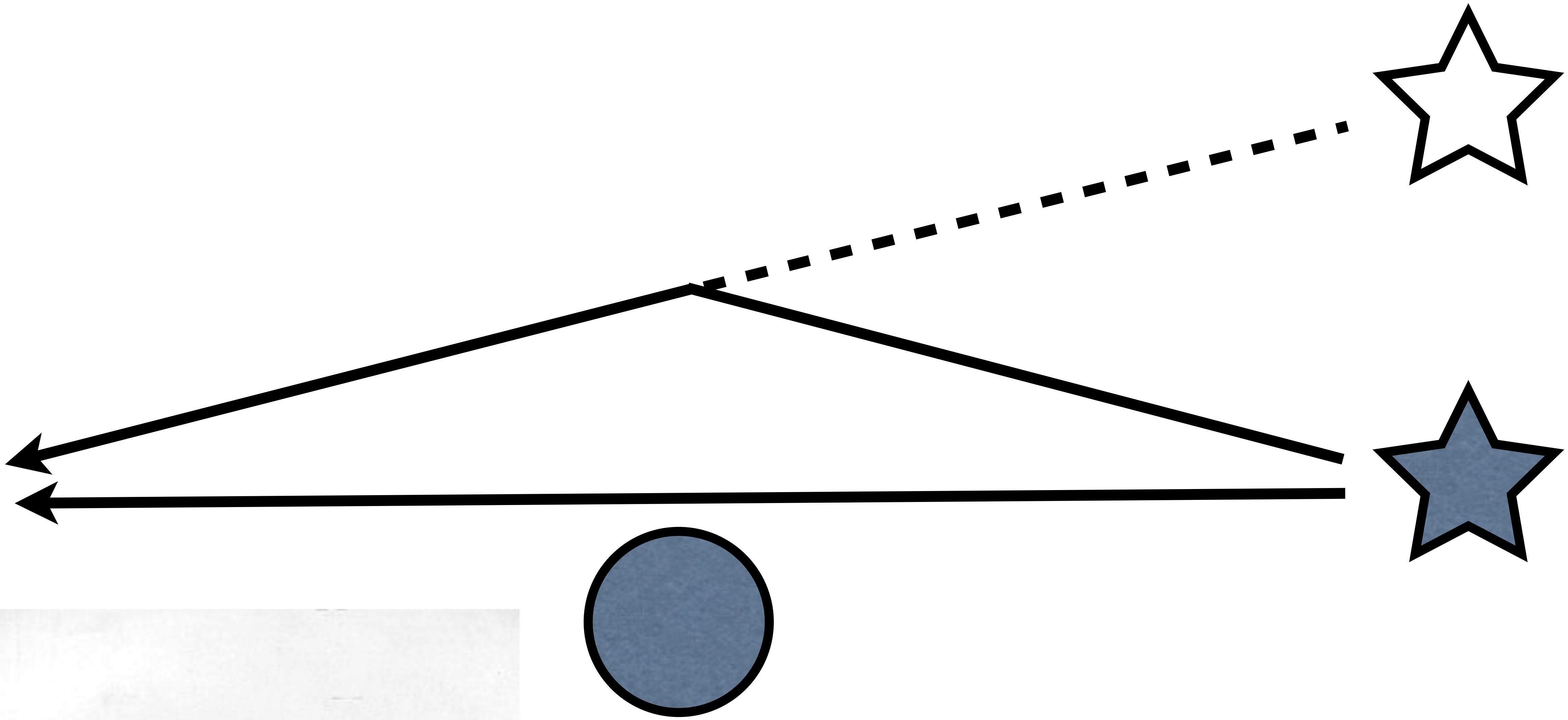
“Gravity and Cosmology 2018” YITP 27 Feb 2018

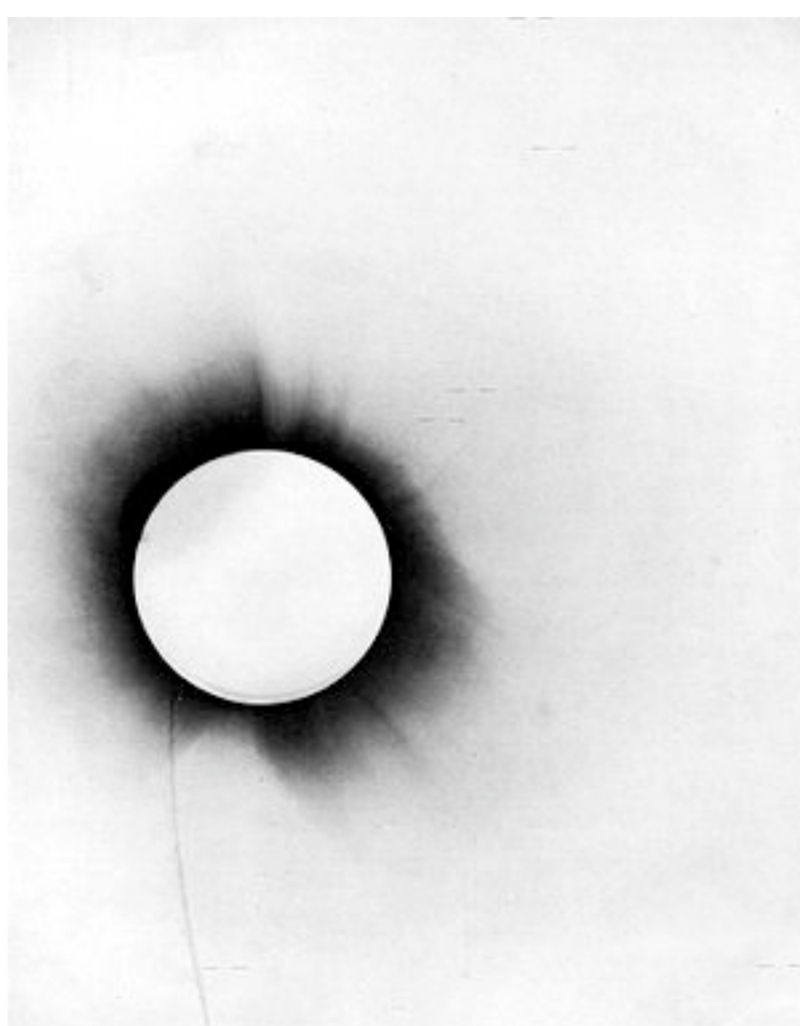
Revisiting the gravitational lensing with Gauss Bonnet theorem

Hideki Asada (Hirosaki)

Ishihara, Ono, HA, PRD 94, 084015 (2016)
PRD 95, 044017 (2017)

Ono, Ishihara, HA, PRD 96, 104037 (2017)





Eddington 1919

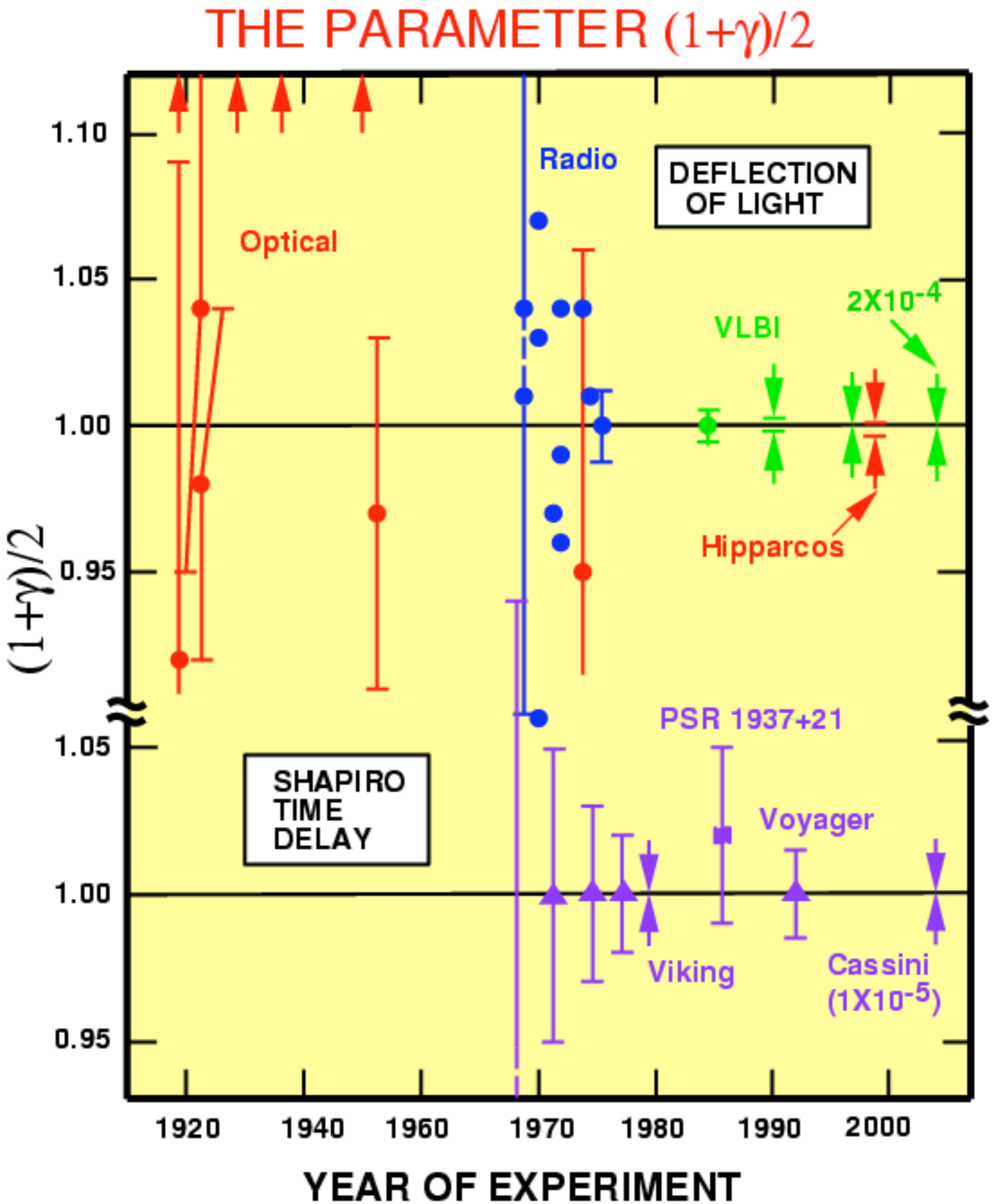
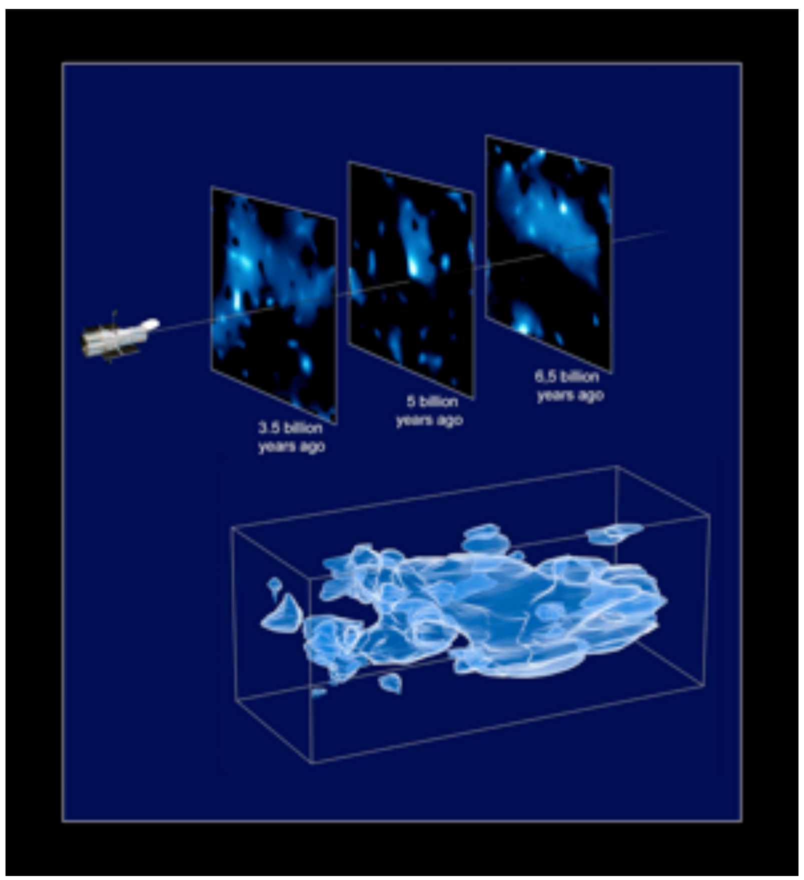
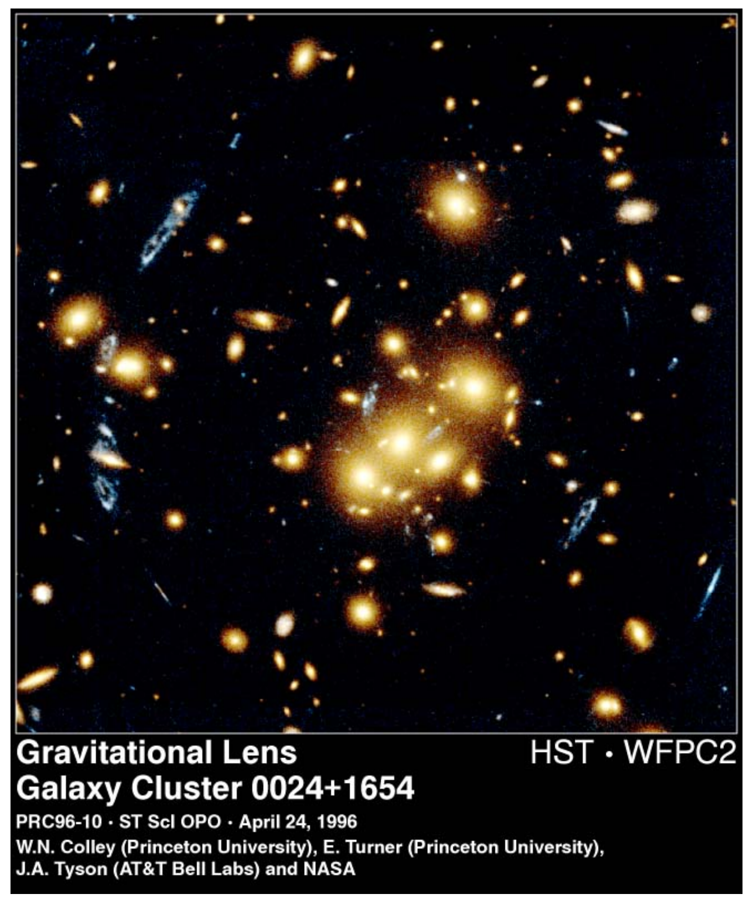


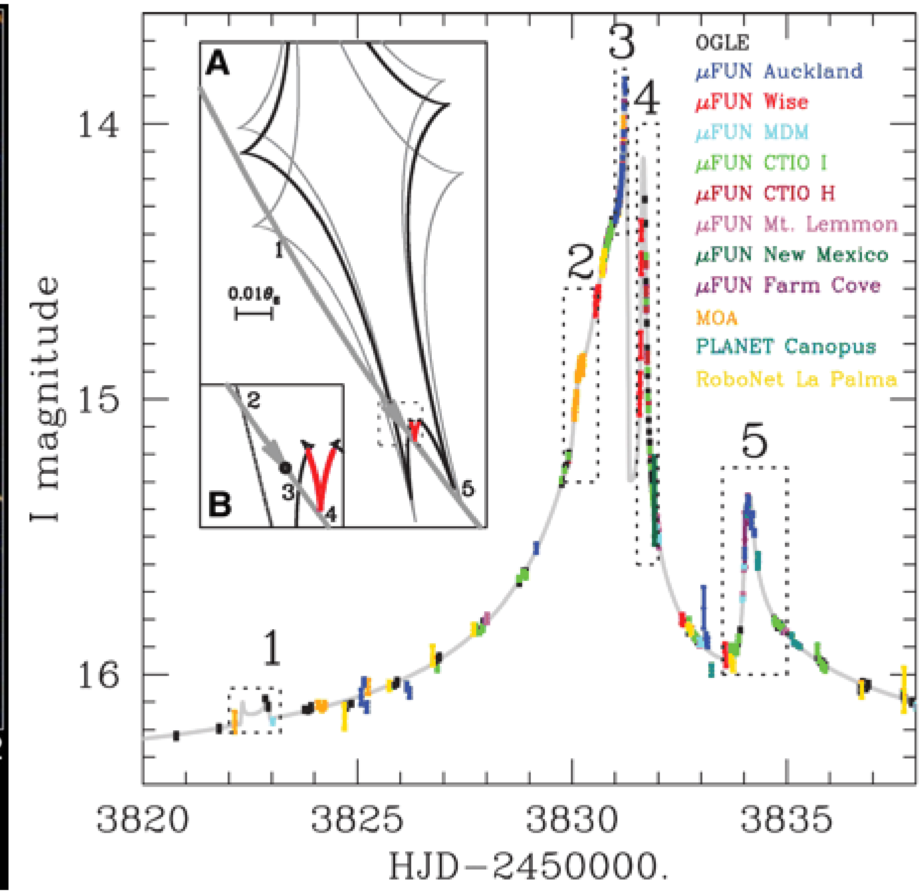
Figure 5: Measurements of the coefficient (1 + γ)/2 from light deflection and time delay measurements. Its GR value is unity. The arrows at the top denote anomalously large values from early eclipse expeditions. The Shapiro time-delay measurements using the Cassini spacecraft yielded an agreement with GR to 10^{-3} percent, and VLBI light deflection measurements have reached 0.02 percent. Hipparcos denotes the optical astrometry satellite, which reached 0.1 percent.

Will, LRR (06)

Gravitational deflection angle of light provides a powerful tool "Gravitational Lens"





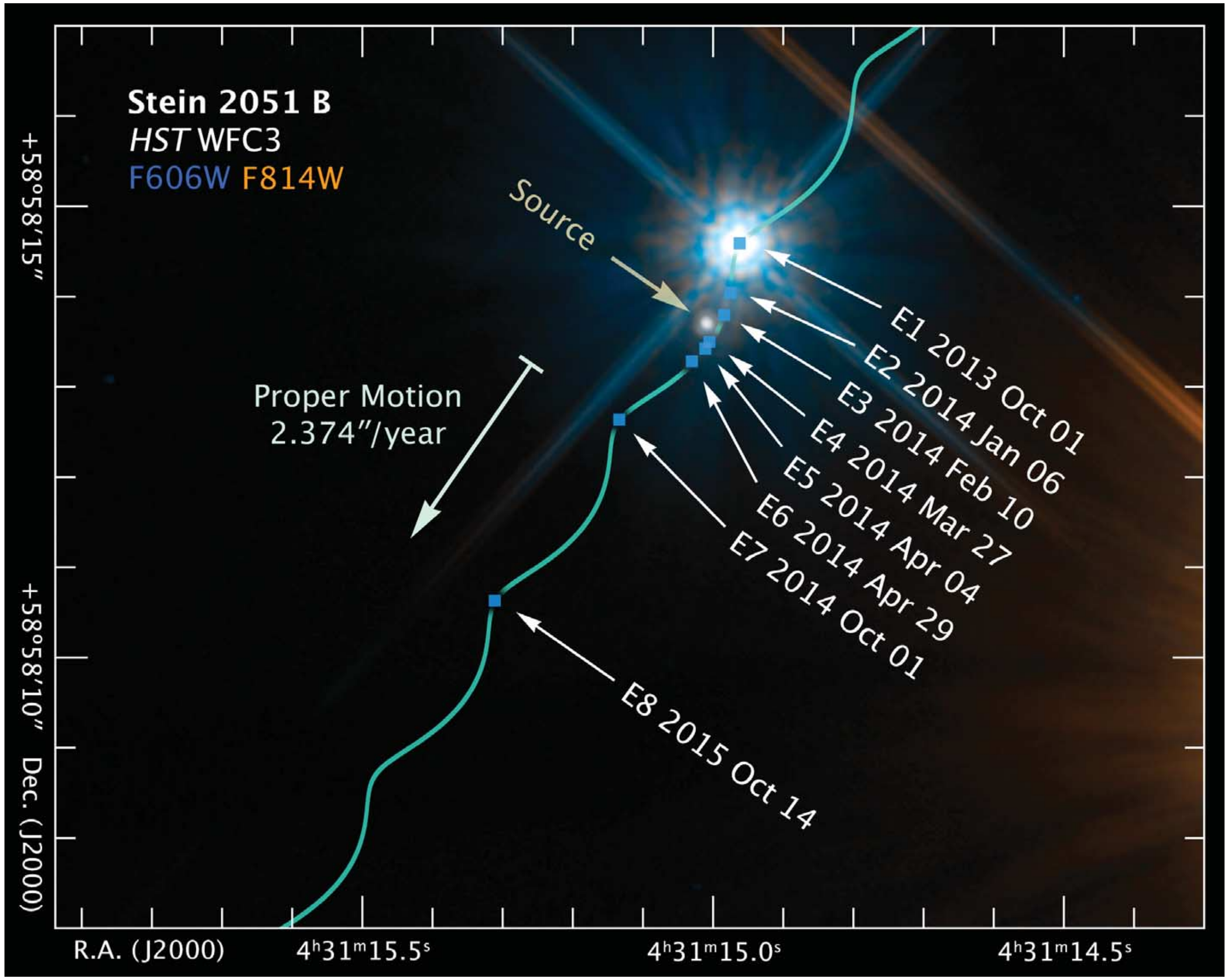


NASA/HST

Gaudi et al. Science (08)

Sahu *et al.*, *Science* **356**, 1046–1050 (2017)   9 June 2017

Fig. 1. Hubble Space Telescope image showing the close passage of the nearby white dwarf Stein 2051 B in front of a distant source star. This color image was made by combining the F814W (orange) and F606W (blue) frames, obtained at epoch E1. The path of Stein 2051 B across the field due to its proper motion toward southeast, combined with its parallax due to the motion of Earth around the Sun, is shown by the wavy cyan line. The small blue squares mark the position of Stein 2051 B at each of our eight observing epochs, E1 through E8. Its proper motion in 1 year is shown by an arrow. Labels give the observation date at each epoch. The source is also labeled; the motion of the source is too small to be visible on this scale. Linear features are diffraction spikes from Stein 2051 B and the red dwarf star Stein 2051 A, which falls outside the lower right of the image. Stein 2051 B passed 0.103 arcsec from the source star on 5 March 2014. Individual images taken at all the eight epochs, and an animated video showing the images at all epochs are shown in fig. S1 and movie S1 (*24*).



First measure of gravitational deflection angle of the nearby white dwarf (Stein 2051B)

Gravitational bending of light
(Gravitational Lens)

1) Testing gravity theories

2) Astronomical tool (natural telescope)

Derivation of Standard formula (at textbook level)

$$\alpha = \frac{4GM}{bc^2}$$

assumes asymptotic source and observer(receiver).

$$r_R, r_S \longrightarrow \infty$$

However, in practice,

$$r_{RS} \neq \infty$$

Ishihara+(2016)

static and spherically symmetric (SSS) spacetime

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2.$$

Optical metric

$$ds^2 = 0 \quad \Longleftrightarrow \quad \begin{aligned} dt^2 &= \gamma_{ij} dx^i dx^j \\ &= \frac{B(r)}{A(r)} dr^2 + \frac{r^2}{A(r)} d\Omega^2 . \end{aligned}$$

Note $\gamma_{ij} \neq g_{ij}$

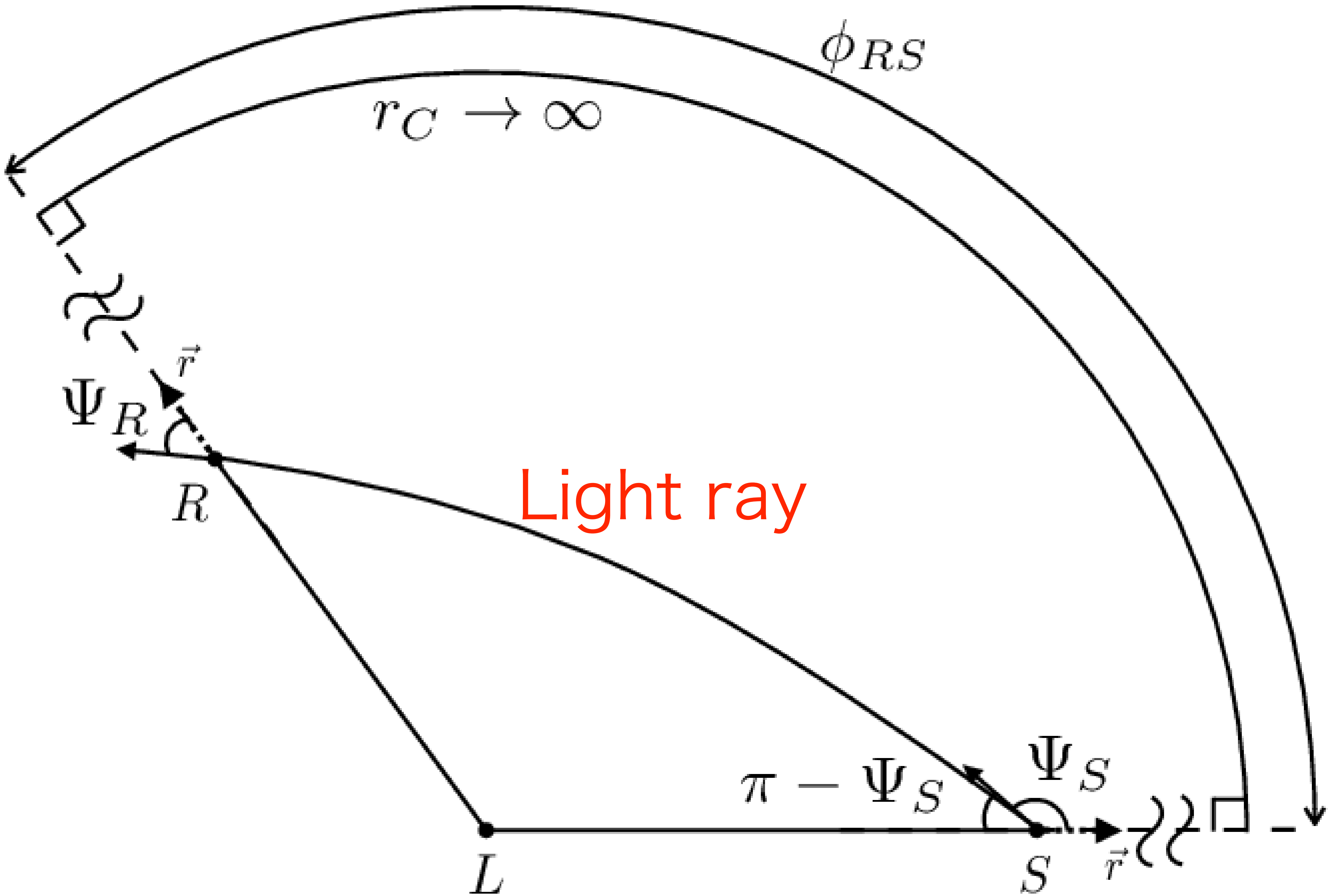
We consider a space defined by optical metric.

Light ray $\delta \int dt = 0$ Fermat's principle

$$\delta \int \sqrt{\gamma_{ij} \left(\frac{dx^i}{dt} \right) \left(\frac{dx^j}{dt} \right)} dt = 0$$

In this space with γ_{ij}, light rays are spatial geodesic.

geometrical configuration



We define

$$\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}$$

This definition seems to make no sense, because

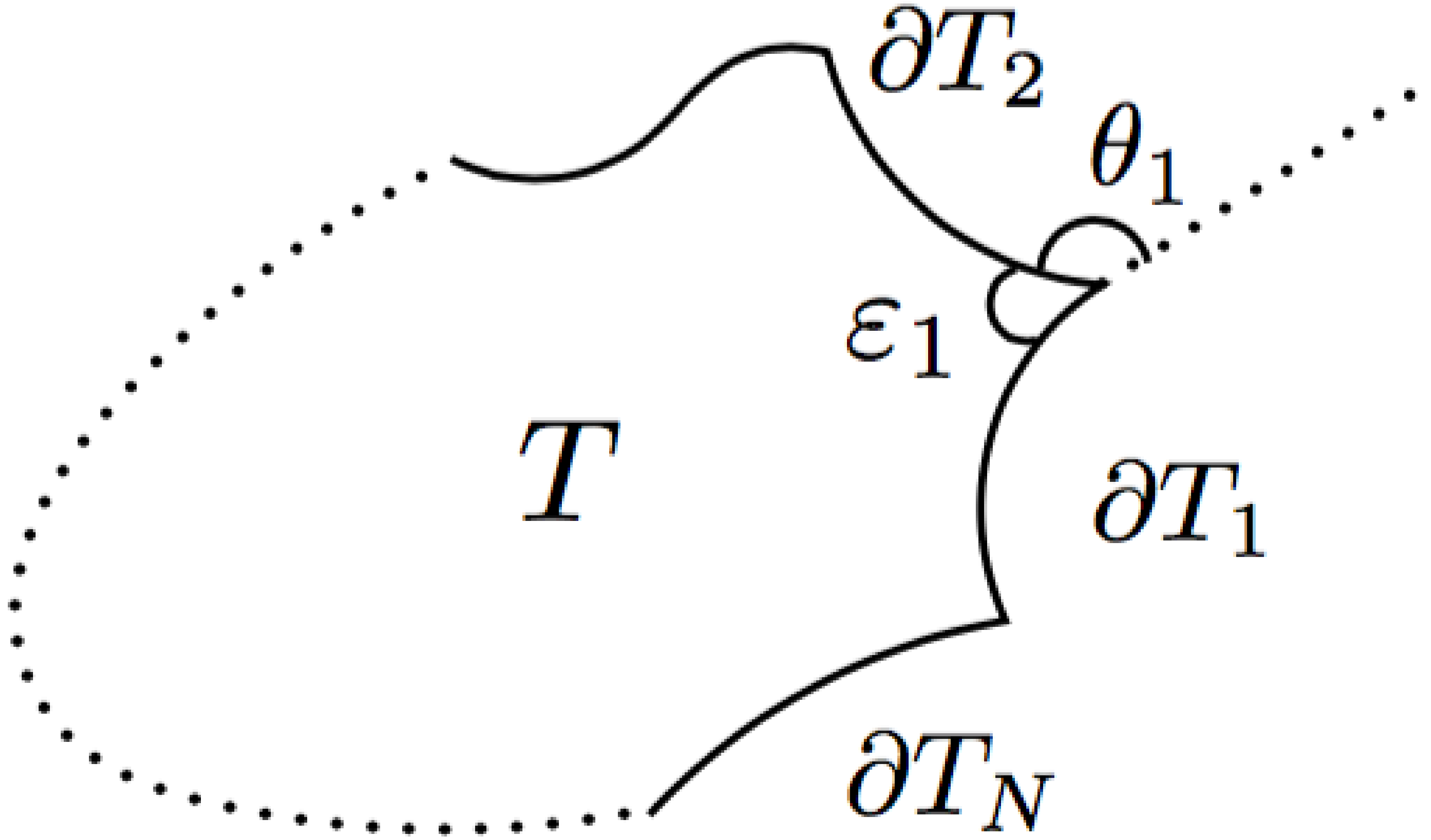
1) Two "Ψ"s are angles at different positions.

2) "Φ" is merely an angular coordinate.

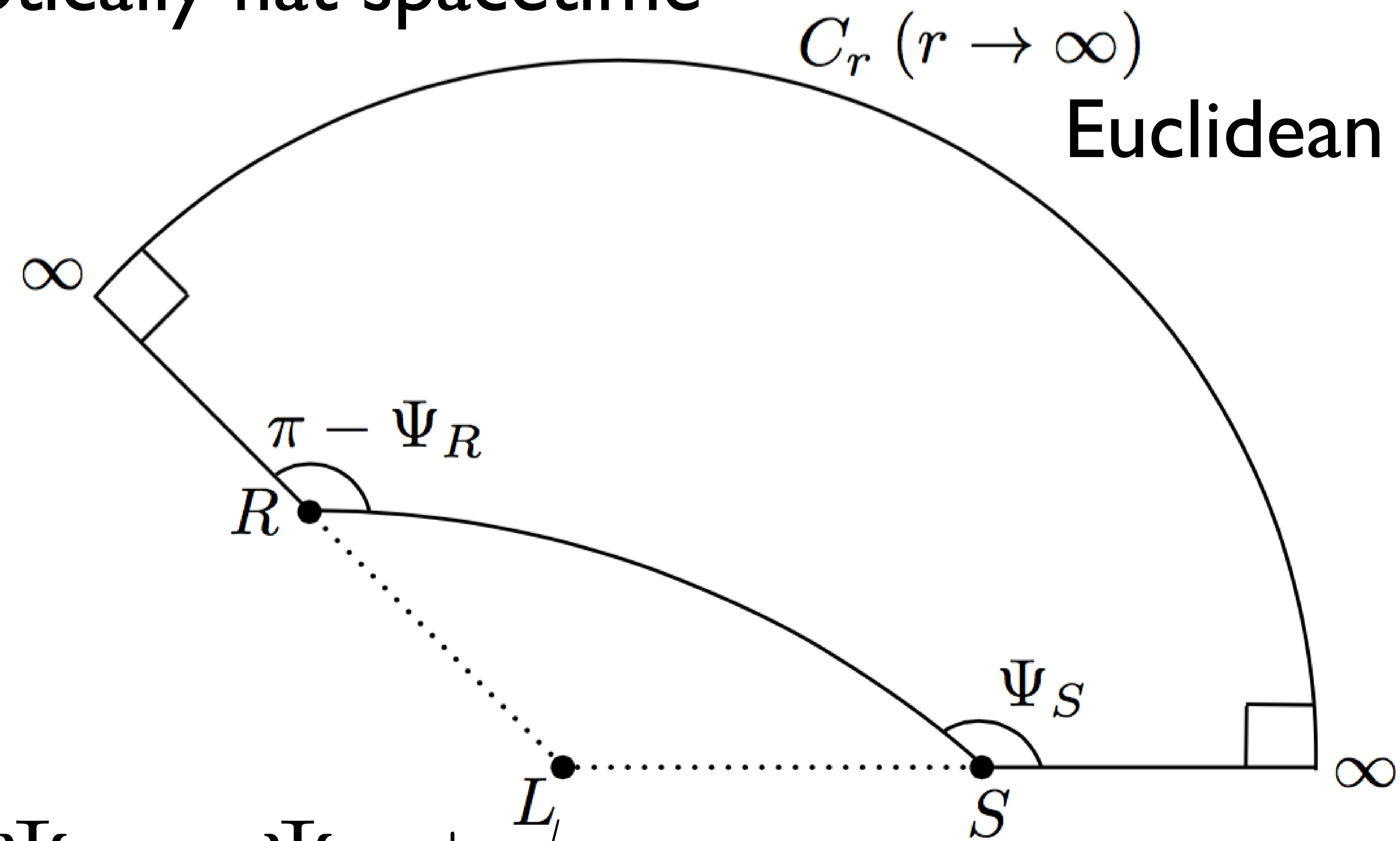
We examine this definition in more detail.

Gauss-Bonnet theorem

$$\iint_T K dS + \int_{\partial T} \kappa_g d\ell + \sum_{a=1}^{N} \theta_a = 2\pi\kappa$$



Asymptotically flat spacetime



$$\alpha = \Psi_R - \Psi_S + \phi_{RS}$$
$$= -\iint_{{}^{\infty}_{R}\square^{\infty}_{S}} K dS.$$

coordinate-invariant

Ishihara et al. (2016)

See also Gibbons&Werner (2008) for r=∞ case

(R and S are in Euclid space)

III. EXAMPLES

we assume $r_R \to \infty$ and $r_S \to \infty$. Then, $\Psi_R = 0$ and $\Psi_S = \pi$

$$\alpha = \phi_{RS} - \pi.$$

agrees with the textbook calculations

B. Approximations

Schwarzschild metric

Correction by finite distance

$$\delta\alpha = \alpha - \alpha_{\infty}$$

For both weak and strong deflection limits,

$$\delta\alpha \sim O\left(\frac{Mb}{{r_S}^2} + \frac{Mb}{{r_R}^2}\right)$$

Examples

Sun

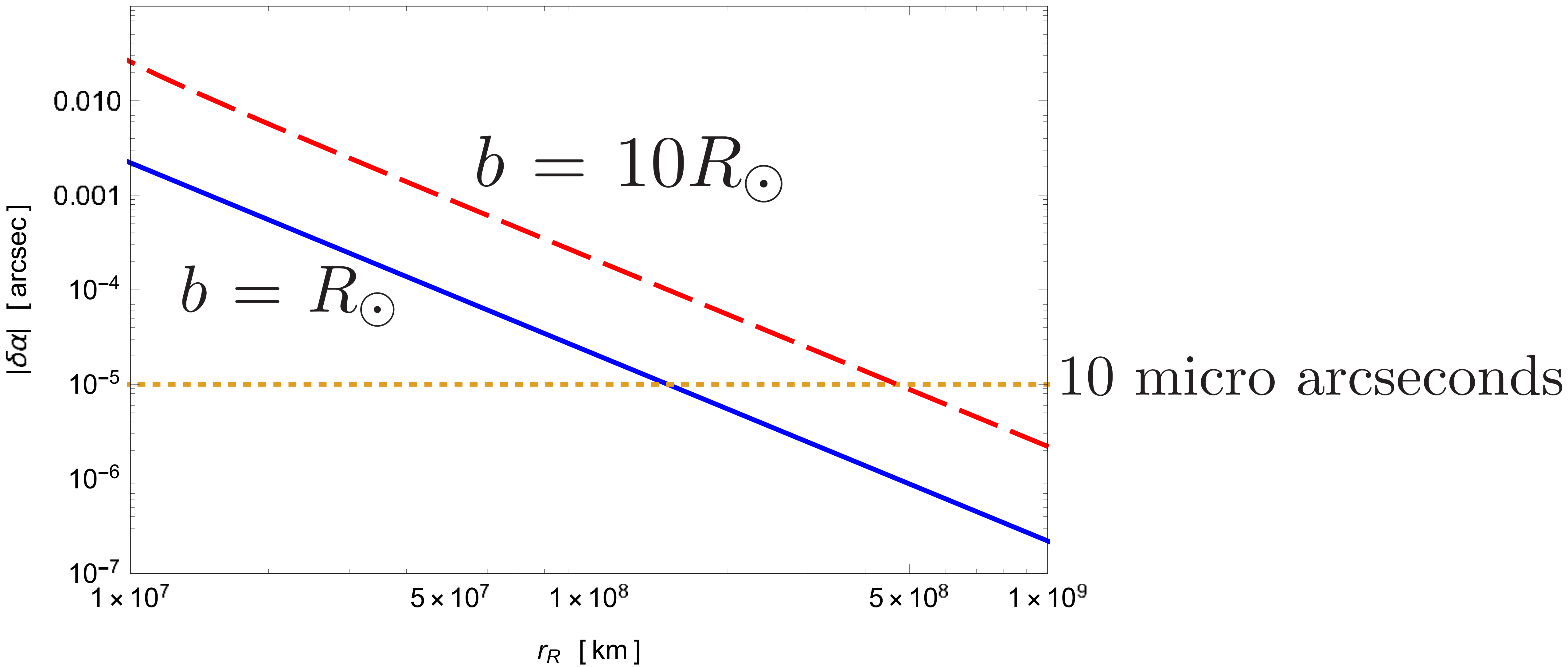
$$\delta\alpha \sim \frac{Mb}{{r_R}^2}$$

$$\sim 10^{-5}\text{arcsec.} \times \left(\frac{M}{M_\odot}\right)\left(\frac{b}{R_\odot}\right)\left(\frac{1\text{AU}}{r_R}\right)^2$$

Sgr A*

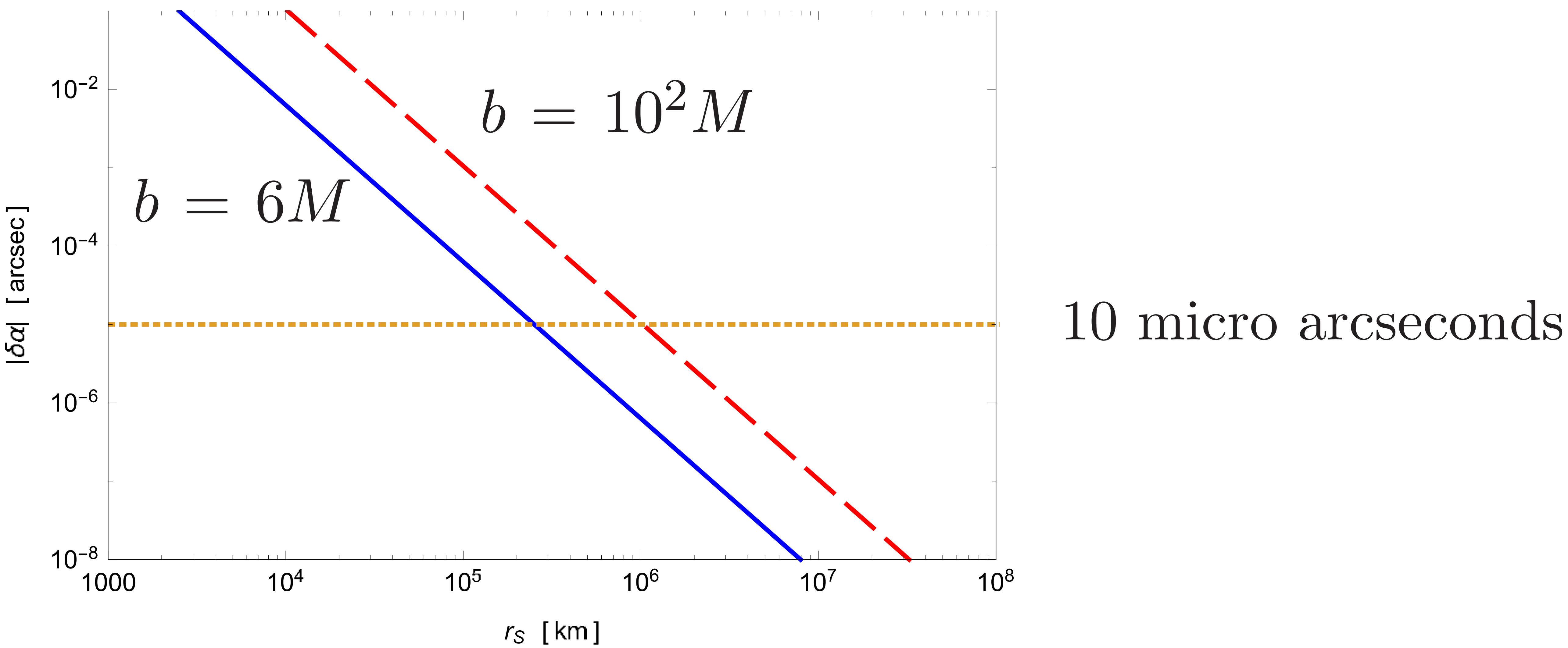
$$\delta\alpha \sim \frac{Mb}{{r_S}^2}$$

$$\sim 10^{-5}\text{arcsec.} \times \left(\frac{M}{4\times 10^6 M_\odot}\right)\left(\frac{b}{3M}\right)\left(\frac{0.1\text{pc}}{r_S}\right)^2$$



0.010
0.001
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$|\delta\alpha|$ [arcsec]
$b = 10R_\odot$
$b = R_\odot$
10 micro arcseconds
1×10^7
5×10^7
1×10^8
5×10^8
1×10^9
r_R [km]

Sun


$b = 10^2 M$
$b = 6M$
10 micro arcseconds
$|\delta\alpha|$ [arcsec]
10^{-2}
10^{-4}
10^{-6}
10^{-8}
1000
10^4
10^5
10^6
10^7
10^8
r_S [km]

Sgr A*

Other BH models

Kottler (Schwarzschild de-Sitter) in GR

$$\begin{aligned}\alpha =&\frac{r_g}{b}\left[\sqrt{1-b^2u_R^2}+\sqrt{1-b^2u_S^2}\right]\\&-\frac{\Lambda b}{6}\left[\frac{\sqrt{1-b^2u_R^2}}{u_R}+\frac{\sqrt{1-b^2u_S^2}}{u_S}\right]\\&+\frac{r_g\Lambda b}{12}\left[\frac{1}{\sqrt{1-b^2u_R^2}}+\frac{1}{\sqrt{1-b^2u_S^2}}\right]+O(r_g^2,\Lambda^2)\end{aligned}$$

Weyl conformal gravity

$$\begin{aligned}\alpha =&\frac{2m}{b}\left(\sqrt{1-b^2u_R^2}+\sqrt{1-b^2u_S^2}\right)\\&-m\gamma\left(\frac{bu_R}{\sqrt{1-b^2u_R^2}}+\frac{bu_S}{\sqrt{1-b^2u_S^2}}\right)+O(m^2,\gamma^2)\end{aligned}$$

Ishihara et al. (2017)

Strong deflection limit > 2π

Darwin(1959), Bozza(2002) and so on

1 loop case

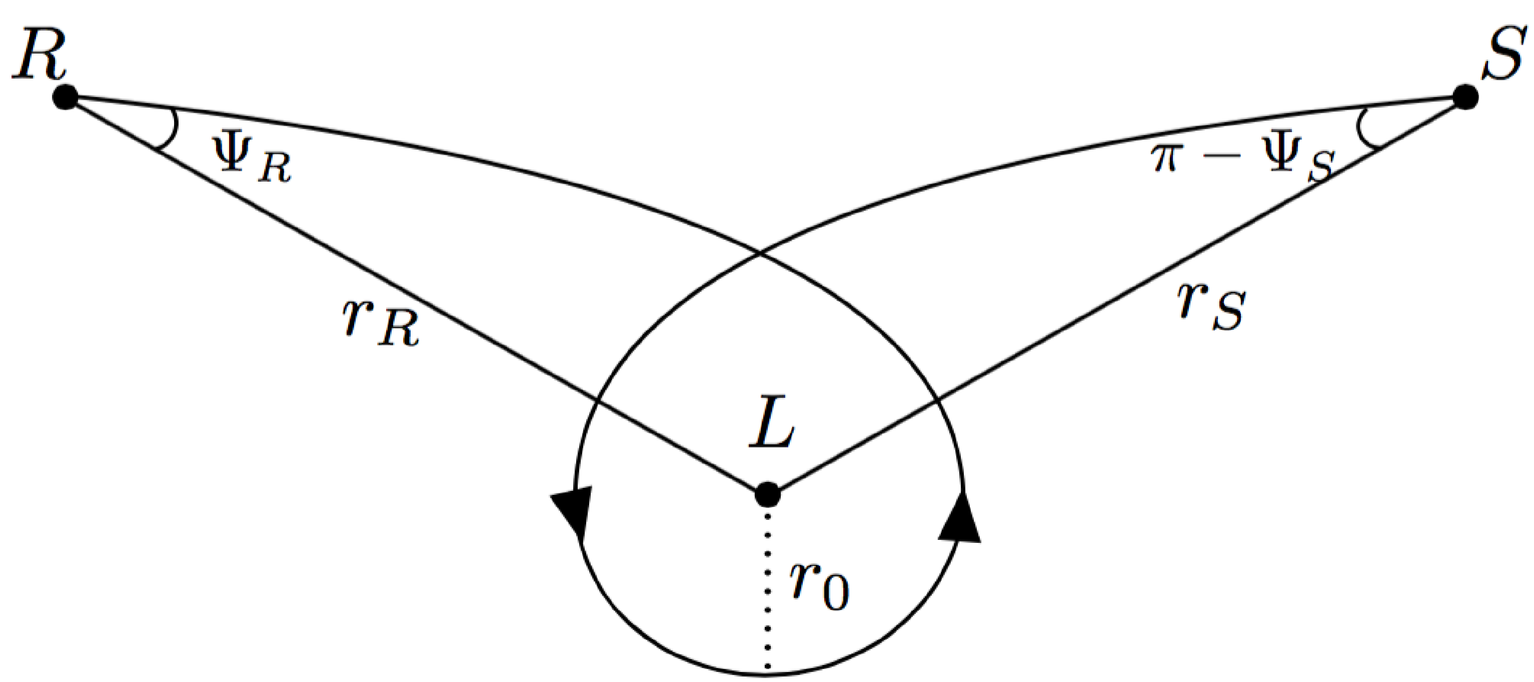
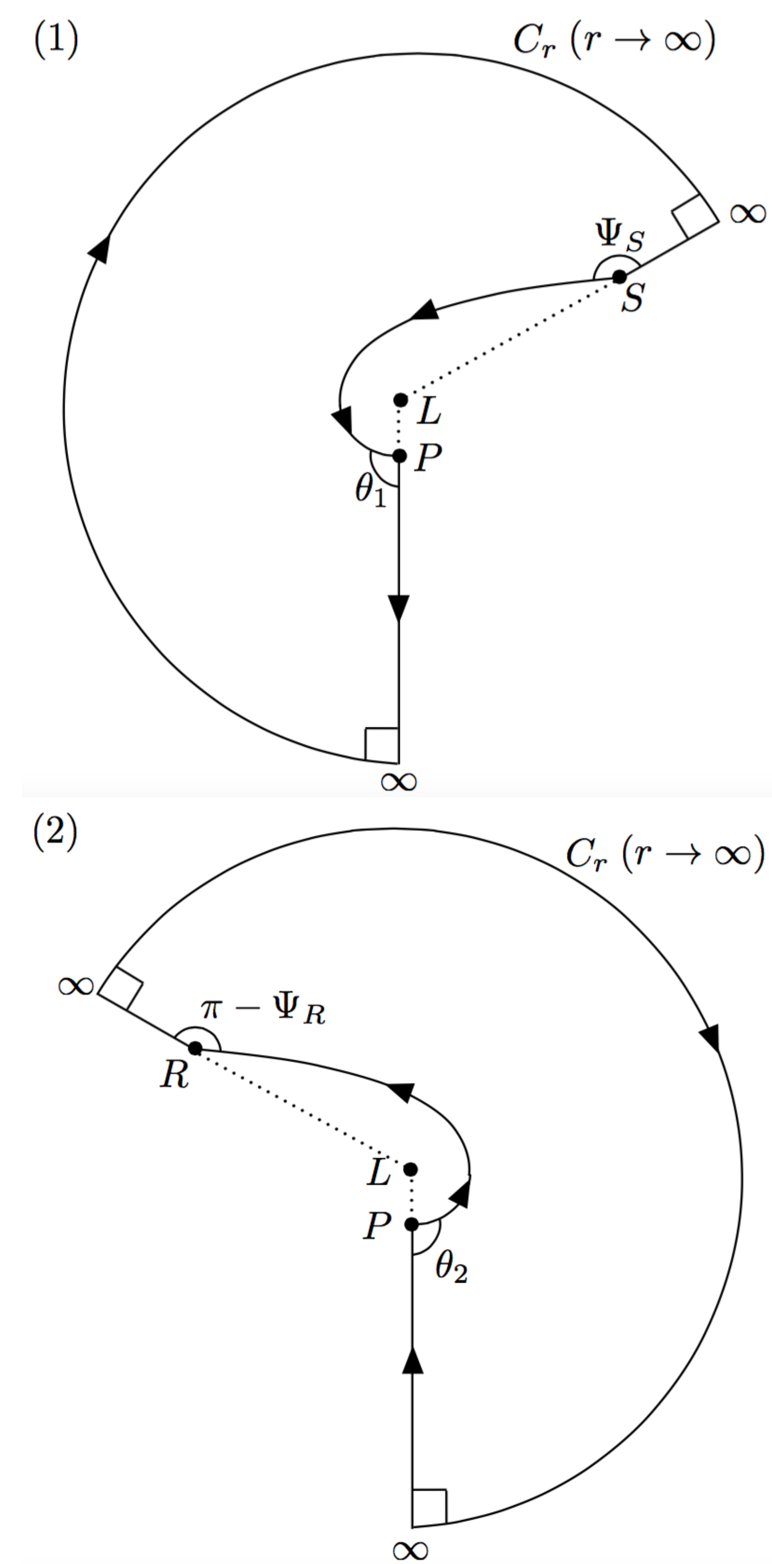


FIG. 3: One-loop diagram for the photon trajectory in $M^{\rm opt}$.



By induction, one can prove
for any winding number

the coordinate invariance of

$$\alpha = \Psi_R - \Psi_S + \phi_{RS}$$

Ono et al. (2017)

Stationary, axisymmetric spacetime

Lewis(1932), Levy and Robinson (1963), Papapetrou (1966)

$$
\begin{aligned}
ds^2 =& g_{\mu\nu} dx^\mu dx^\nu \\
=& -A(y^p, y^q) dt^2 - 2H(y^p, y^q) dt d\phi \\
&+ F(y^p, y^q)(\gamma_{pq} dy^p dy^q) + D(y^p, y^q) d\phi^2
\end{aligned}
$$

p, q = 1, 2

We choose spherical coordinates

(Cylindrical coordinates => Weyl-Lewis-Papapetrou form)

$$ds^2 = -A(r,\theta)dt^2 - 2H(r,\theta)dtd\phi$$

$$+ B(r,\theta)dr^2 + C(r,\theta)d\theta^2 + D(r,\theta)d\phi^2$$

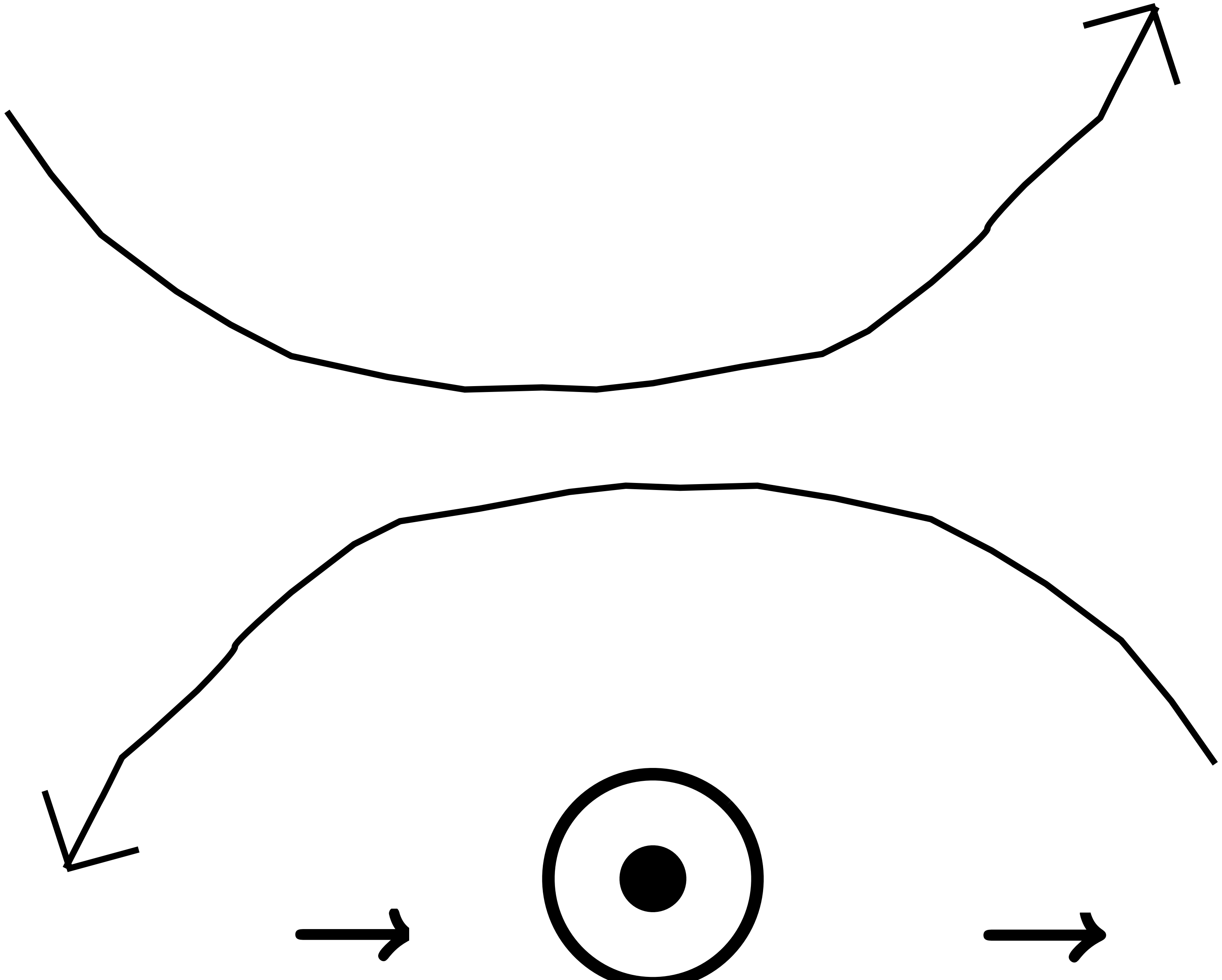
$$dt = \sqrt{\gamma_{ij}dx^i dx^j} + \underline{\beta_i dx^i}$$

Induced by rotation

cf. charged particle in magnetic field

$$L = \frac{1}{2}mv^2 - q\vec{v}\cdot\vec{A}$$

Lorentz (Lorentz-like) force is direction-dependent



$\vec{B}$ or $\vec{B}_g$

Let us consider the photon orbits on the equatorial plane.

Again, we define

$$\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}$$

We use the Gauss-Bonnet theorem...

$$\alpha = -\iint_{{}_{R}^{\infty}\square_{S}^{\infty}} K dS - \int_{R}^{S} \kappa_g d\ell$$

New correction

caused by rotation
(gravitomagnetic effect)

coordinate-invariant

Prograde

$$\alpha_{prog} = \frac{2M}{b}\left(\sqrt{1-b^2{u_S}^2}+\sqrt{1-b^2{u_R}^2}\right)$$
$$-\frac{2aM}{b^2}\left(\sqrt{1-b^2{u_R}^2}+\sqrt{1-b^2{u_S}^2}\right)+O\left(\frac{M^2}{b^2}\right)$$

infinity limit

$$\alpha_{\infty\, prog} \to \frac{4M}{b}-\frac{4aM}{b^2}+O\left(\frac{M^2}{b^2}\right)$$

agrees with the known result

Retrograde

$$\alpha_{retro} = \frac{2M}{b}\left(\sqrt{1-b^2{u_S}^2}+\sqrt{1-b^2{u_R}^2}\right)$$
$$+\frac{2aM}{b^2}\left(\sqrt{1-b^2{u_R}^2}+\sqrt{1-b^2{u_S}^2}\right)+O\left(\frac{M^2}{b^2}\right)$$

infinity limit

$$\alpha_{\infty\, retro} \to \frac{4M}{b}+\frac{4aM}{b^2}+O\left(\frac{M^2}{b^2}\right)$$

agrees with the known result

Summary

The gravitational deflection angle of light
by using the GB theorem

stationary and axisymmetric

Extensions are future work

Thank you!

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