General relativistic weak-field limit and Newtonian N-body simulations

Kazuya Koyama

University of Portsmouth

ICG Portsmouth

with **Christian Fidler, Cornelius Rampf, Thomas Tram**, Rob Crittenden, David Wands

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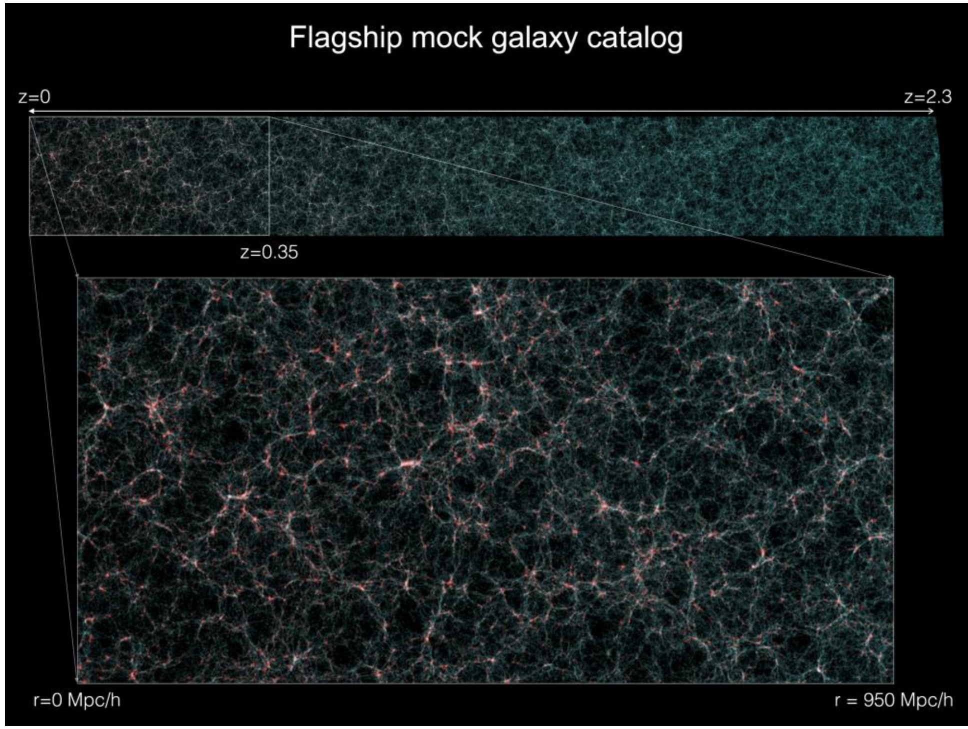
Motivation

- Future surveys (DESI, LSST, Euclid, SKA …)

  These surveys will go wider and deeper,
  probing near horizon perturbations



- N-body simulations

  These surveys require large volume simulations
  cf. Euclid flagship simulation L=3.8 Gpc, N=12600^3
  mock galaxies up to z=2.3

- Limitations of Newtonian simulations

  Newtonian dynamics is based on "action-at-a-distance" in absolute space and time

Questions

- Are Newtonian N-body simulations consistent with weak-field limit of GR?
- If so, how do we interpret Newtonian simulations in a relativistic framework?
- How do we include relativistic effects missing in simulations (e.g. radiation perturbations)

Newtonian simulations

- Initial conditions

$$\boldsymbol{x}(\boldsymbol{q},\eta) = \boldsymbol{q} + \boldsymbol{\psi}(\boldsymbol{q},\eta) \qquad\qquad -\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_\alpha = \delta_\alpha$$

- N-body simulations

$$\rho_{\rm count} = \frac{1}{a^3}\sum_{\rm particles} m\,\delta_{\rm D}^{(3)}(\boldsymbol{x}-\boldsymbol{x}_p)$$

$$k^2\Phi^{\rm N} = 4\pi G a^2\bar{\rho}_{\rm cdm}\delta^{\rm N}_{\rm cdm}$$

$$\ddot{\boldsymbol{x}}_i = -\nabla\Phi^{\rm N}$$

linearisation

$$k^2\Phi^{\rm N} = 4\pi G a^2\bar{\rho}^{\rm N}_{\rm cdm}\delta^{\rm N}_{\rm cdm}\,,$$

$$\dot{\delta}^{\rm N}_{\rm cdm} + k v^{\rm N}_{\rm cdm} = 0\,,$$

$$\left[\partial_\tau + \mathcal{H}\right] v^{\rm N}_{\rm cdm} = -k\Phi^{\rm N}\,,$$

N-body gauge (linear perturbations) Fidler et.al. arXiv:1505.04756

- N-body gauge

$$g_{00} = -a^2(1+2A)\,,$$
$$g_{0i} = -a^2 B_i\,,$$
$$g_{ij} = a^2\left[\delta_{ij}\left(1+2H_{\rm L}\right) - 2H_{{\rm T}\,ij}\right]$$

$$B^{\rm Nb} = v^{\rm Nb} \quad H_{\rm L}^{\rm Nb} = 0$$

$$\dot\delta_{\rm cdm}^{\rm Nb} + k v_{\rm cdm}^{\rm Nb} = 0\,,$$
$$\left[\partial_\tau + \mathcal{H}\right] v_{\rm cdm}^{\rm Nb} = -k\left(\Phi + \gamma^{\rm Nb}\right)$$
$$\Phi \equiv H_{\rm L} + \frac{1}{3}H_{\rm T} + \mathcal{H}k^{-1}\left(B - k^{-1}\dot H_{\rm T}\right)$$
$$-k^2\gamma \equiv \left(\partial_\tau + \mathcal{H}\right)\dot H_{\rm T} - 8\pi G a^2 \bar p \Pi$$ (anisotropic stress)

Cold Dark Matter (CDM) + C.C. $\quad H_{\rm T}^{\rm Nb} = 3\zeta = \text{constant} \quad \Pi = 0 \Rightarrow \gamma$ vanishes

$$\zeta = H_{\rm L} + \frac{1}{3}H_{\rm T} + \mathcal{H}k^{-1}(B - v)$$

- Relativistic density

$$\rho = (1 - 3H_{\rm L})\rho_{\rm count}$$

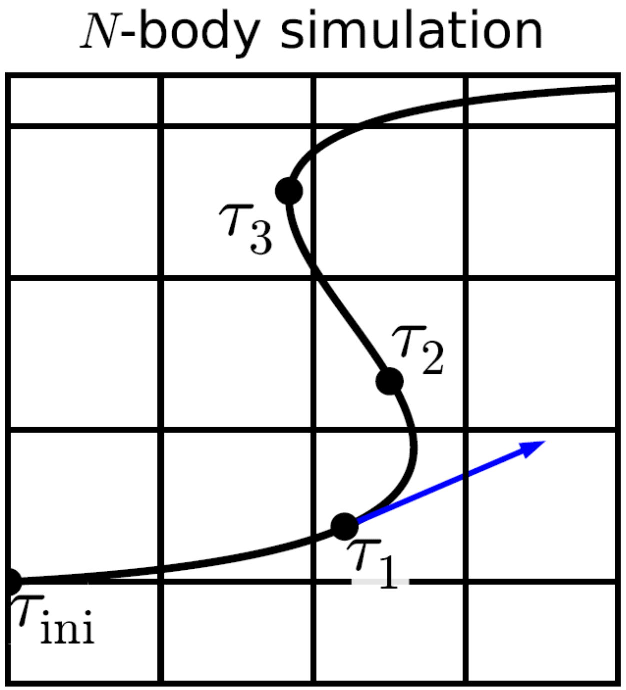
Traceless part of 3-metric does not distort volume

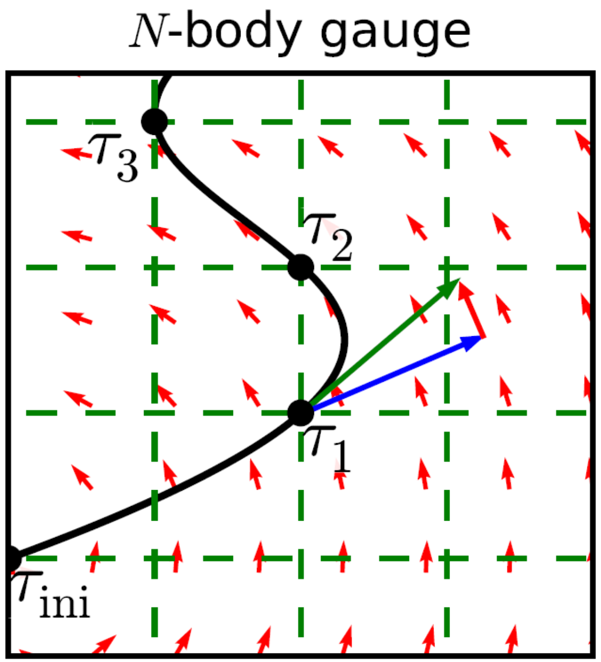
Radiation perturbations

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Radiation perturbations

$$[\partial_\tau + \mathcal{H}]\, \boldsymbol{v}_{\rm cdm}^{\rm Nb} = \nabla\Phi + \nabla\gamma^{\rm Nb}$$

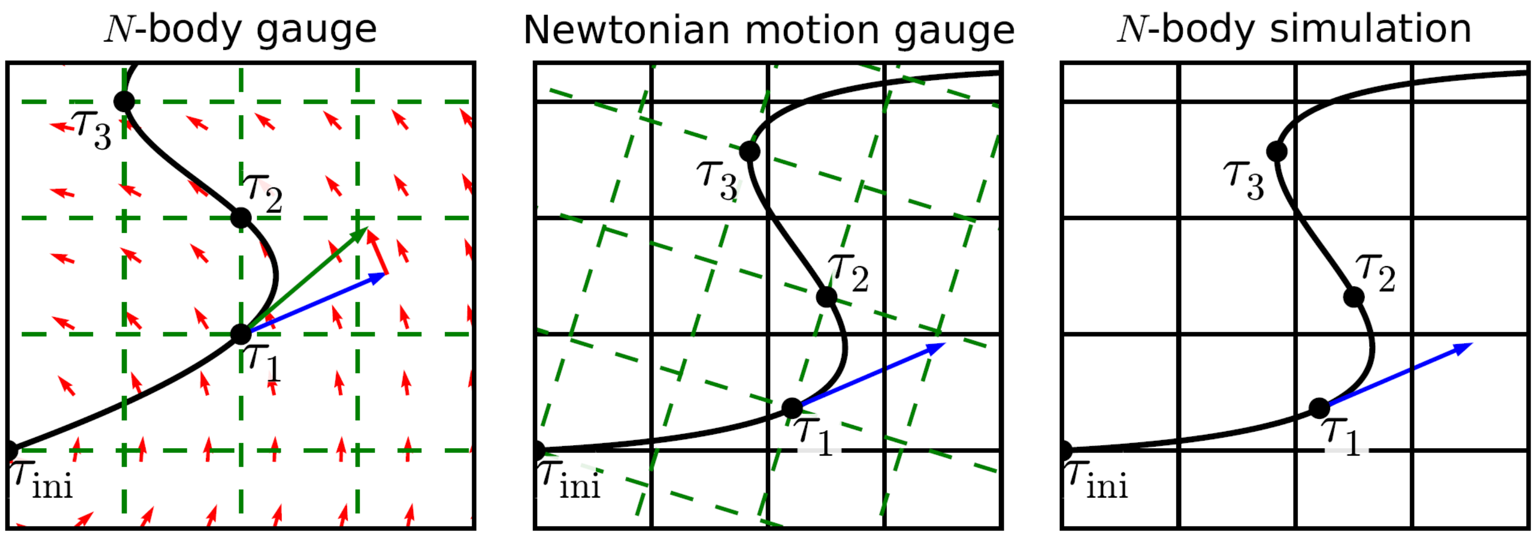




Newtonian motion gauge

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Newtonian motion gauge



$$k^2\Phi^{\rm N} = 4\pi G a^2 \bar\rho_{\rm cdm}\delta^{\rm N}_{\rm cdm}$$

$$\delta^{\rm N}_{\rm cdm} \equiv \delta^{\rm Nm}_{\rm cdm} + 3H^{\rm Nm}_{\rm L}$$

$$H_{\rm T} = \tilde H_{\rm T} + kL$$

$$[\partial_\tau + \mathcal{H}]\,\boldsymbol{v}^{\rm Nb}_{\rm cdm} = \nabla\Phi + \nabla\gamma^{\rm Nb} \quad [\partial_\tau + \mathcal{H}]\,v^{\rm Nm}_{\rm cdm} = -k\Phi^{\rm N}$$

space threading

$$\gamma^{\rm Nm} = \Phi^{\rm N} - \Phi \;\Rightarrow\; (\partial_\tau + \mathcal{H})\,\dot H_{\rm T} - 4\pi G a^2\bar\rho_{\rm cdm}\,(H_{\rm T} - 3\zeta) = S$$

$$S = 4\pi G a^2\left(\bar\rho_{\rm other}\delta_{\rm other} + 3\mathcal{H}k^{-1}(\bar\rho + \bar p)_{\rm other}(v - B) + 2\bar p\Pi\right)$$

Newtonian motion gauge spacetime

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Time slicing $B^{\mathrm{Nm}} = v^{\mathrm{Nm}}$ (not a unique choice)

$$
\begin{aligned}
g_{00} &= -a^2 \left(1 + 2A^{\mathrm{Nm}}\right) \,, \\
g_{0i} &= a^2 \, \mathrm{i}\hat{k}_i B^{\mathrm{Nm}} \,, \\
g_{ij} &= a^2 \left[\delta_{ij} \left(1 + 2H_{\mathrm{L}}^{\mathrm{Nm}}\right) + 2\left(\delta_{ij}/3 - \hat{k}_i \hat{k}_j\right) H_{\mathrm{T}}^{\mathrm{Nm}}\right]
\end{aligned}
$$

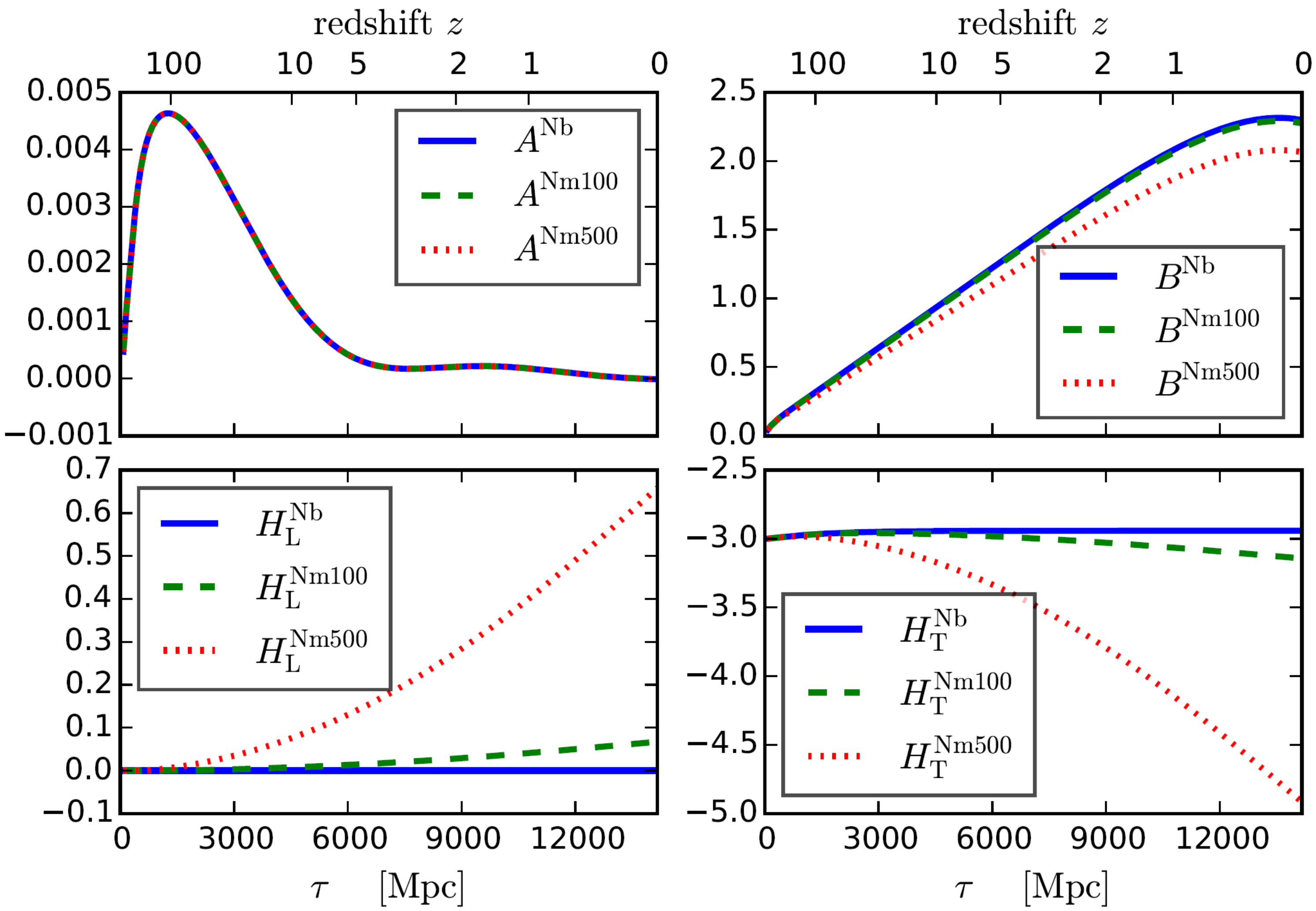
$$
(\bar{\rho} + \bar{p}) \, A^{\mathrm{Nm}} = \frac{2}{3}\bar{p}\Pi - \delta p^{\mathrm{Nm}}
$$

$$
-4\pi G a^2 \bar{\rho}_{\mathrm{cdm}} (3H_{\mathrm{L}}^{\mathrm{Nm}} + \delta_{\mathrm{cdm}}^{\mathrm{Nm}}) = k^2 A^{\mathrm{Nm}} + (\partial_\tau + \mathcal{H}) \, k B^{\mathrm{Nm}}
$$

$$
\frac{1}{3}\dot{H}_{\mathrm{T}}^{\mathrm{Nm}} = \mathcal{H} A^{\mathrm{Nm}} - \dot{H}_{\mathrm{L}}^{\mathrm{Nm}}
$$

Newtonian motion gauge metric

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221



$$\zeta = H_{\mathrm{L}} + \frac{1}{3} H_{\mathrm{T}} + \mathcal{H} k^{-1}(B - v)$$

$$k = 10^{-3}\mathrm{Mpc}^{-1}$$

$$\zeta = -1 \text{ on super-horizon}$$

$$H_{\mathrm{T}}^{\mathrm{Nb}} = 3\zeta = \text{ constant}$$

$$H_{\mathrm{L}}^{\mathrm{Nb}} = 0$$

with no radiation

Application to N-body simulations

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Gauge transformation to N-body gauge

At late times, radiation becomes negligible and N-body simulations are easier to interpret in N-body gauge $H_{\mathrm{T}} = \tilde{H}_{\mathrm{T}} + kL$

$$\boldsymbol{x}^{\mathrm{Nb}} = \boldsymbol{x}^{\mathrm{Nm}} + \boldsymbol{L}^{\mathrm{Nm}\to\mathrm{Nb}}, \qquad \boldsymbol{L}^{\mathrm{Nm}\to\mathrm{Nb}} = -k^{-1}\nabla L^{\mathrm{Nm}\to\mathrm{Nb}}$$

$$\ddot{L}^{\mathrm{Nm}\to\mathrm{Nb}} + \mathcal{H}\dot{L}^{\mathrm{Nm}\to\mathrm{Nb}} - 4\pi G a^2 \bar{\rho}_{\mathrm{cdm}} L^{\mathrm{Nm}\to\mathrm{Nb}} = -k\gamma^{\mathrm{Nb}} - 4\pi G a^2 k^{-1} \bar{\rho}_{\mathrm{other}} \delta^{\mathrm{Nb}}_{\mathrm{other}}$$

$$\delta^{\mathrm{Nb}}_{\mathrm{cdm}} - \delta^{\mathrm{N}} = -kL^{\mathrm{Nm}\to\mathrm{Nb}}$$

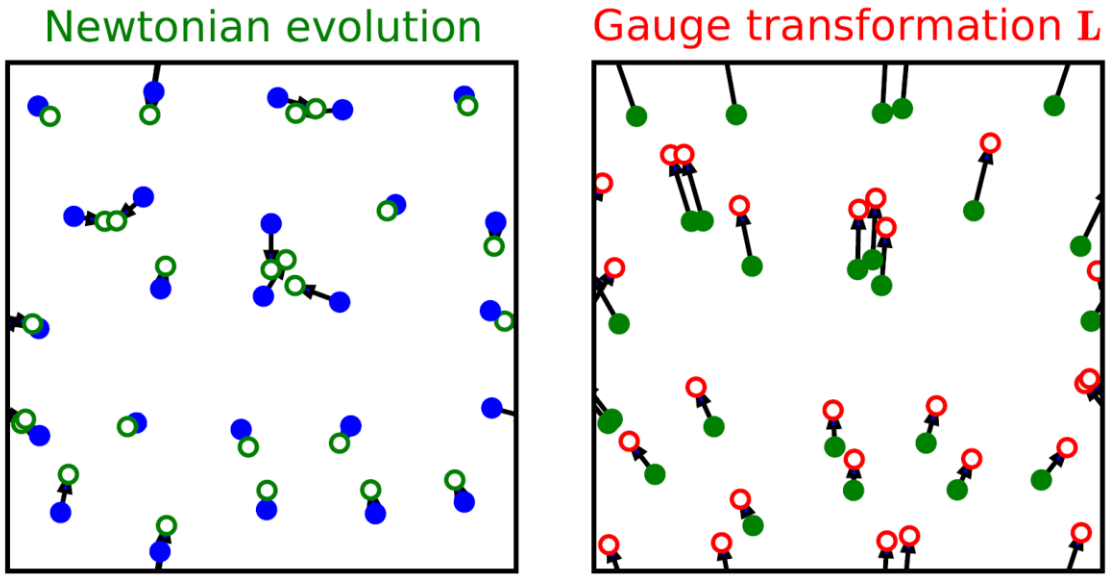
this gauge transformation can be computed by linear Boltzmann code (CLASS)

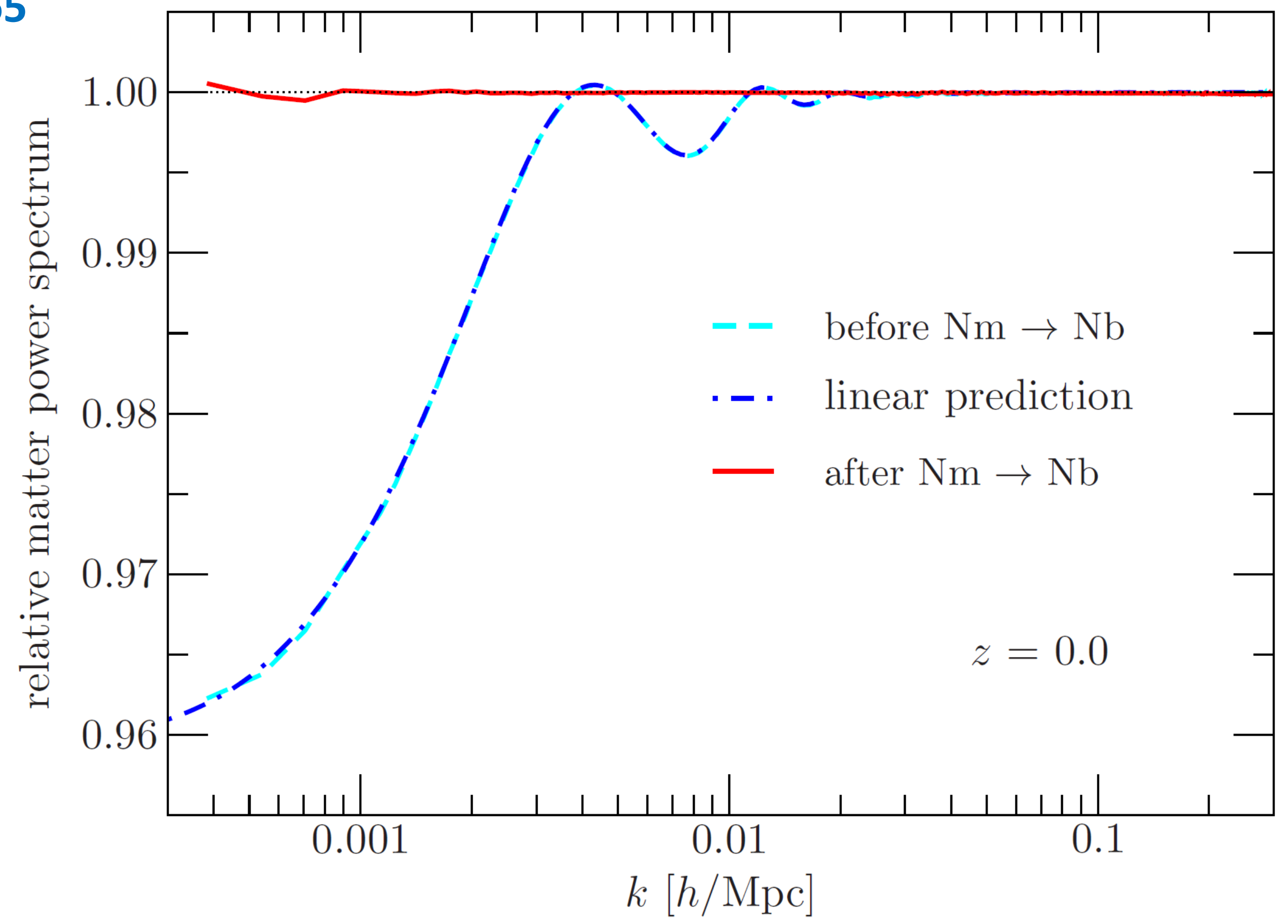
Comparison with relativistic simulations

Adamek et.al. arXiv:1703.08585

- gevolution Adamek et.al. arXiv:1604.06065

  Relativistic simulation code with weak field approximation





Weak field expansion

Fidler et.al. arXiv:1708.07769

- Non-linear density and velocity

  In Newtonian simulations, density and velocity become non-linear but the Newtonian potential remains small

$$\nabla^2\Phi(\boldsymbol{x},\tau) = -4\pi G\bar{\rho}a^2\delta(\boldsymbol{x},\tau)$$

- Weak field expansion

$$\nabla = k_{\rm ref}\mathcal{O}(\kappa) \qquad k_{\rm ref} = aH \equiv \mathcal{H} \qquad \Phi \text{ is of } \mathcal{O}(\epsilon)$$

  - Super/near horizon scales $\kappa^2 = \mathcal{O}(\epsilon)$ $\delta = \mathcal{O}(\epsilon^2)$
  - Under horizon and linear scales $\kappa^2 = \mathcal{O}(1)$ (standard cosmological pert.)
  - Under horizon and "non-linear" scales $\kappa^2 = \mathcal{O}(\epsilon^{-1})$

$$\delta = \mathcal{O}(\kappa^2\epsilon) = \mathcal{O}(1) \qquad \kappa = \mathcal{O}(\epsilon^{-1/2})$$

Weak field expansion

Fidler et.al. arXiv:1708.07769

- Metric perturbations $(A,\, H_{\rm L},\, B,\, B_i,\, H_{\rm T},\, H_{{\rm T}i}$ and $H_{{\rm T}ij})$ are of order ϵ.

$$g_{00} = -a^2\left(1+2A\right)\,,$$

$$g_{0i} = -a^2\left(B_i + \hat{\nabla}_i B\right)\,,$$

$$\hat{\nabla}_i \equiv -(-\Delta)^{-1/2}\nabla_i \qquad \Delta = \nabla^2$$

$$g_{ij} = a^2\left[\delta_{ij}\left(1+2H_{\rm L}\right) + 2\left(\hat{\nabla}_i\hat{\nabla}_j + \frac{\delta_{ij}}{3}\right)H_{\rm T} - \hat{\nabla}_i H_{{\rm T}j} - \hat{\nabla}_j H_{{\rm T}i} - 2H_{{\rm T}ij}\right]$$

- Matter perturbations $\delta = \mathcal{O}(\kappa^2\epsilon) \qquad v = \mathcal{O}(\kappa\epsilon) \qquad \Sigma \sim \Sigma^i \sim \Sigma^{ij} = \mathcal{O}(\epsilon)$

$$T^0{}_0 = -\rho\,,$$

$$T^0{}_i = (\rho+p)(v_i + \hat{\nabla}_i v - B_i - \hat{\nabla}_i B)$$

$$T^i{}_j = p\delta^i_j + \left(\hat{\nabla}^i\hat{\nabla}_j + \frac{\delta^i_j}{3}\right)\Sigma - \frac{1}{2}\left(\hat{\nabla}^i\Sigma_j + \hat{\nabla}_j\Sigma^i\right) - \Sigma^i{}_j + (\rho+p)(v^i + \hat{\nabla}^i v)(v_j + \hat{\nabla}_j v)$$

Weak field Newtonian motion gauge

Fidler et.al. arXiv:1708.07769

- Temporal gauge

  $v = B$ violates our assumption

  we adopt $\quad \mathfrak{K}B = \dot{H}_{\mathrm{T}} \qquad \mathfrak{K} = (-\Delta)^{1/2}$

- Newtonian variables

$$4\pi G a^2 \delta\rho^{\mathrm{cdm}}_{\mathrm{count}} = \mathfrak{K}^2 \Phi^{\mathrm{N}} \qquad \delta^{\mathrm{cdm}}_{\mathrm{count}} = \delta + 3H_{\mathrm{L}} = \delta^{\mathrm{N}} \qquad v^{\mathrm{cdm}} = v^{\mathrm{N}}$$

- Newtonian motion gauge

$$A + (\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}} = -\Phi^{\mathrm{N}}$$

This spatial gauge condition realises Newtonian (non-linear) Euler equation

$$(\partial_\tau + \mathcal{H})\,(v_i^{\mathrm{cdm}} + \hat{\nabla}_i v^{\mathrm{cdm}}) - (v^j_{\mathrm{cdm}} + \hat{\nabla}^j v^{\mathrm{cdm}})\hat{\nabla}_j \mathfrak{K}(v_i^{\mathrm{cdm}} + \hat{\nabla}_i v^{\mathrm{cdm}}) + \hat{\nabla}_i \mathfrak{K}\Phi^{\mathrm{N}}$$
$$= -\frac{1}{\rho}\left(\frac{2}{3}\hat{\nabla}_i \mathfrak{K}\Sigma^{\mathrm{cdm}} - \frac{1}{2}\mathfrak{K}\Sigma_i^{\mathrm{cdm}}\right).$$

Linear Boltzmann + Newtonian N-body

Fidler et.al. arXiv:1708.07769

- Gauge conditions

$$(\partial_\tau + \mathcal{H})\dot{H}_{\mathrm{T}} = 4\pi G a^2 (\delta\rho_\gamma + 3\mathcal{H}(\rho_\gamma + p_\gamma)\mathfrak{K}^{-1}(v - \mathfrak{K}^{-1}\dot{H}_{\mathrm{T}}) - \rho_{\mathrm{cdm}}(3\zeta - H_{\mathrm{T}})) + 8\pi G a^2 \Sigma$$

$$\dot{H}_{\mathrm{T}} = \mathfrak{K} B$$

These are $\mathcal{O}(\epsilon)$ equations so can be computed using a linear Boltzmann code

Other variables can be computed using simulation quantities δ^{N} v^{N} Φ^{N}

$$A = -\Phi^{\mathrm{N}} - (\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}}\,,$$

$$H_{\mathrm{L}} = \Phi^{\mathrm{N}} - \frac{1}{3}H_{\mathrm{T}} - \gamma\,,$$

$$v = v^{\mathrm{N}}\,,$$

$$\delta = \delta^{\mathrm{N}} - 3H_{\mathrm{L}} = \delta^{\mathrm{N}} - 3\Phi^{\mathrm{N}} + H_{\mathrm{T}} + 3\gamma \qquad \gamma \equiv -(\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}} + 8\pi G a^2 \mathfrak{K}^{-2}\Sigma$$

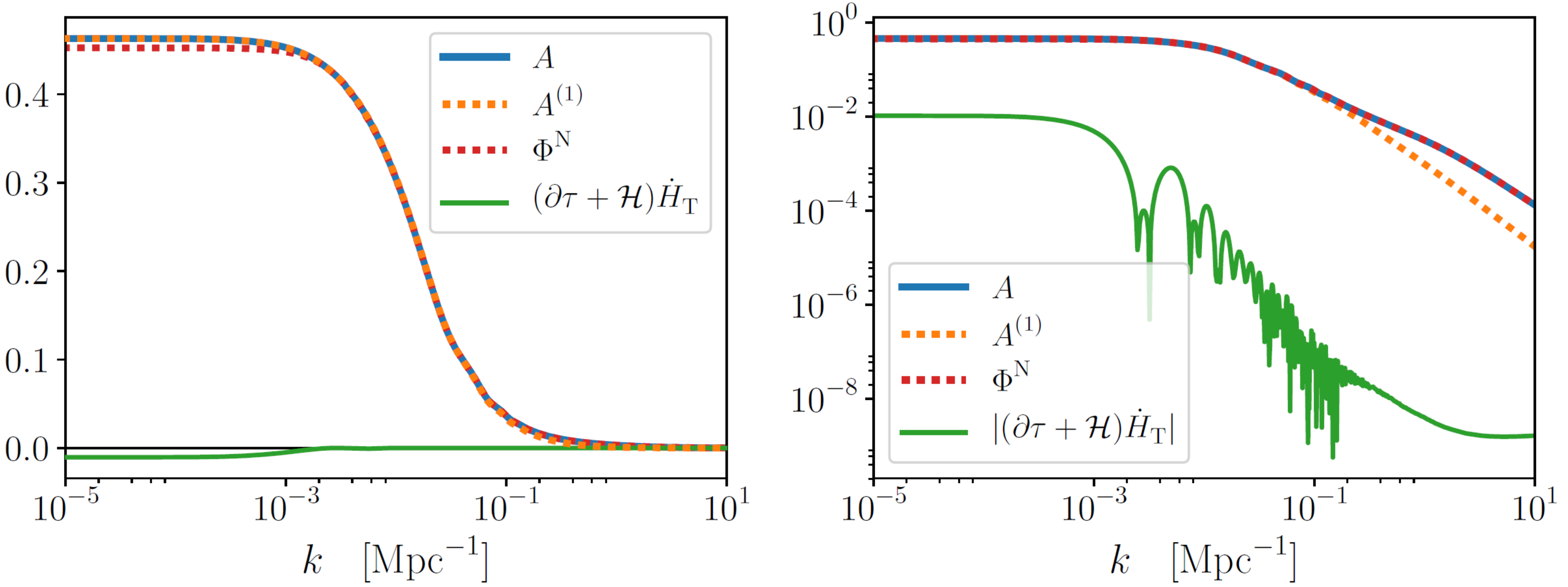
Linear Boltzmann + Newtonian N-body

Fidler et.al. arXiv:1708.07769

- Relativistic corrections (radiation) + non-linear corrections

$$A = -\boxed{\Phi^{\rm N}} - \boxed{(\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\rm T}}$$

Relativistic correction



Poisson gauge

Fidler et.al. arXiv:1708.07769

- Relation to Poisson gauge (time slicing is the same)

$$\Psi \equiv A^{\mathrm{P}} = A = -\Phi^{\mathrm{N}} - (\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}}$$

$$\delta^{\mathrm{P}} = \delta = \delta^{\mathrm{N}} - 3\Phi^{\mathrm{N}} + H_{\mathrm{T}} + 3\gamma\,.$$

$$\Phi \equiv H_{\mathrm{L}}^{\mathrm{P}} = \Phi^{\mathrm{N}} - \gamma$$

$$v^{\mathrm{P}} = v^{\mathrm{N}} - \mathfrak{K}^{-1}\dot{H}_{\mathrm{T}}$$ (space threading is different)

- At late time (C.C. + CDM)

$$\delta = \delta^{\mathrm{N}} - 3\left(\Phi^{\mathrm{N}} + \zeta\right)\,, \quad v = v^{\mathrm{N}}\,, \quad A = -\Phi^{\mathrm{N}}\,, \quad B = 0\,, \quad H_{\mathrm{L}} = \Phi^{\mathrm{N}} + \zeta\,, \quad H_{\mathrm{T}} = 3\zeta$$

$$\delta^{\mathrm{P}} = \delta^{\mathrm{N}} - 3\Phi^{\mathrm{N}} + 3\zeta \qquad \Phi = \Phi^{\mathrm{N}}\,,$$

$$v^{\mathrm{P}} = v^{\mathrm{N}} \qquad \Psi = -\Phi = -\Phi^{\mathrm{N}}$$

Ray tracing

Fidler et.al. arXiv:1708.07769

- N-body results should be interpreted in Nm gauge

photon displacement

$$\delta x^0 = \left[-\delta a_0 + \Phi_0^{\mathrm{N}} + v_{||}^0\right]\chi - 2\int_0^{\chi_s} \mathrm{d}\chi\left[\Phi^{\mathrm{N}} + (\chi_s - \chi)\dot{\Phi}^N\right] + \int_0^{\tau_0} \mathrm{d}\tau\Phi^{\mathrm{N}}(0,\tau)$$

$$\delta x^i = \left[\delta a_0 n^i + \Phi_0^{\mathrm{N}} n^i - v_0^i - \hat{\nabla}^i v_0\right]\chi + 2\int_0^{\chi_s} \mathrm{d}\chi\left[-\Phi^{\mathrm{N}} n^i + (\chi_s - \chi)\nabla^i\Phi^{\mathrm{N}}\right]$$

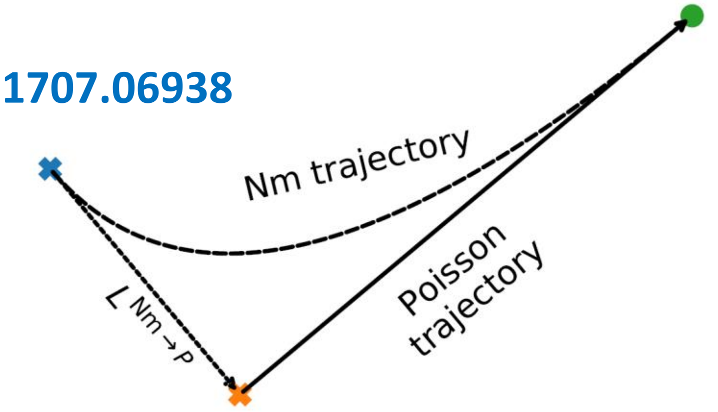
$$+\boxed{\hat{\nabla}^i\mathfrak{K}^{-1}(H_{\mathrm{T}e} - H_{\mathrm{T}0})}.$$

Integrated Coordinate Shift

see also Adamek arXiv:1707.06938

- Chisari & Zaldarriaga description

$$\boldsymbol{x}^{\mathrm{P}} = \boldsymbol{x}^{\mathrm{Nm}} + \delta\boldsymbol{x}_{\mathrm{in}}, \quad \nabla\cdot\delta\boldsymbol{x}_{\mathrm{in}} = -5\Phi_{\mathrm{in}}^{\mathrm{N}} = -\ 3\zeta$$

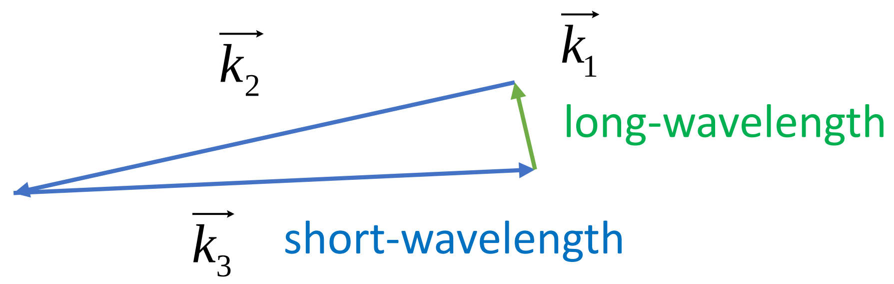


Chisari & Zaldarriaga arXiv:1101.3555

Going beyond linear order

- Bispectrum including relativistic corrections
  - To compute quantities like bispectrum, we need a scheme to combine second order relativistic perturbations + Newtonian N-body simulations

    e.g. squeezed limit $B(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ $\quad \vec{k}_1 \to 0$



- 2nd order Boltzmann code is required

- Initial conditions
  - In GR, the constraint equation becomes non-linear at second order

- Scalar-vector-tensor mixing
  - At second order, scalar, vector and tensor mix.

Second order Poisson density

- Second order density in Poisson gauge

$$\frac{1}{2}\delta_P^{(2)}(\mathbf{k}) = \mathcal{C}_{\mathbf{k}}\left\{\mathcal{K}_P(k_1,k_2,k)\delta_P^{(1)}(\mathbf{k}_1)\,\delta_P^{(1)}(\mathbf{k}_2)\right\},$$

$$\mathcal{K}_{P,\mathrm{VR}} \equiv \frac{(\beta_{P,\mathrm{VR}}-\alpha_{P,\mathrm{VR}}) + \frac{\beta_{P,\mathrm{VR}}}{2}\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2\left(\frac{k_2}{k_1}+\frac{k_1}{k_2}\right) + \alpha_{P,\mathrm{VR}}\left(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2\right)^2 + \gamma_{P,\mathrm{VR}}\left(\frac{k_1}{k_2}-\frac{k_2}{k_1}\right)^2}{\left(1+3f\frac{\mathcal{H}^2}{k_1^2}\right)\left(1+3f\frac{\mathcal{H}^2}{k_2^2}\right)}$$

full solutions with CDM + C.C. is known **Villa & Rampf arXiv:1505.04782**

EdS limit Newtonian solutions

$$\alpha_{\mathrm{P,EdS}} = \boxed{\frac{2}{7}} + \frac{59\mathcal{H}^2}{14k^2} + \frac{45\mathcal{H}^4}{2k^4}$$

$$\beta_{\mathrm{P,EdS}} = \boxed{1} - \frac{\mathcal{H}^2}{2k^2} + \frac{54\mathcal{H}^4}{k^4},$$

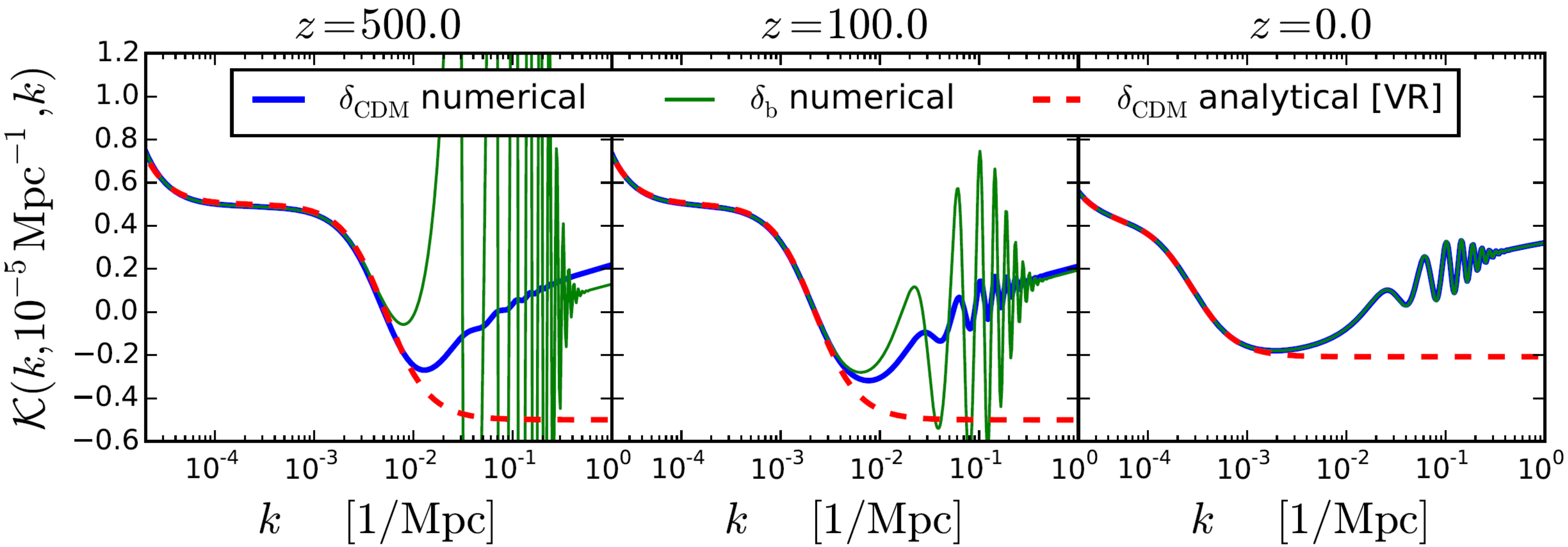
$$\gamma_{\mathrm{P,EdS}} = -\frac{3\mathcal{H}^2}{2k^2} + \frac{9\mathcal{H}^4}{2k^4}.$$

The dominant term in the squeezed limit

Full solutions from 2nd order Boltzmann code

Tram et.al. arXiv:1602.05933

- Comparison with **SONG*** Pettinari et.al. arXiv:1302.0832; arXiv:1406.2981 Fidler et.al. arXiv:1401.3296



* https://github.com/coccoinomane/song

Squeezed limit from separate universe

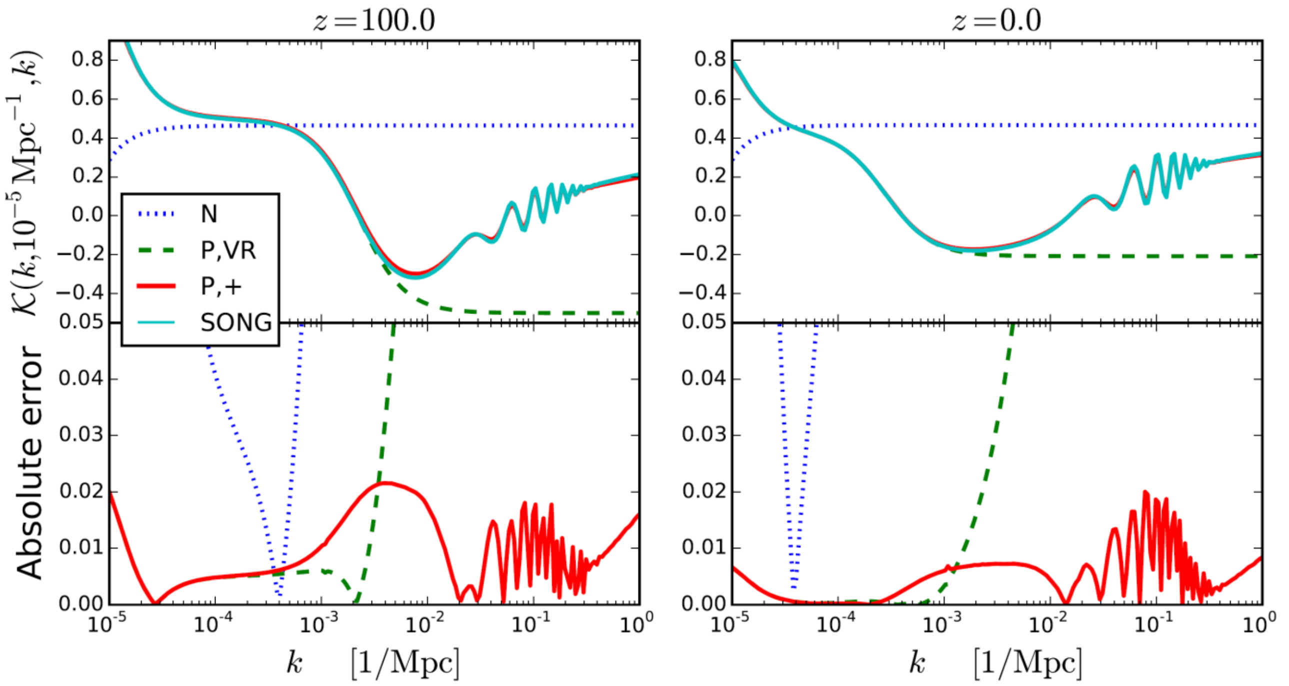
Tram et.al. arXiv:1602.05933

- Squeezed limit
  - The effect of long-modes can be considered as coordinate transformations for short modes

$$\delta\rho_P(\tilde\eta,\tilde x^i) = \delta\rho_P(\eta,x^i) + \frac{\partial\delta\rho_P}{\partial\eta}(\eta,x^i)(\tilde\eta-\eta) + \frac{\partial\delta\rho_P}{\partial x^j}(\eta,x^i)(\tilde x^j - x^j)$$

$$\tilde\eta = \eta + \epsilon(\eta),$$
$$\tilde x^j = x^j(1+\zeta),$$

$$\gamma_{P,\mathrm{sq}} = \frac{1}{2}\left[f - f^2 - 3u\right]\frac{\mathcal{H}^2}{k^2} + \frac{9}{2}\left[f^2 - \frac{1}{2}fu(1-f)\right]\frac{\mathcal{H}^4}{k^4} - \frac{1}{2}\left(f + \frac{3u}{2}\right)\left[\frac{\mathcal{H}^2}{k^2} + 3f\frac{\mathcal{H}^4}{k^4}\right]\frac{\partial\log T}{\partial\log k}.$$



Initial conditions

- Comoving synchronous gauge

$$ds^2 = a(\tau)^2\left[-d\tau^2 + \gamma_{ij}(\tau, q^k)dq^i dq^j\right] = a(\tau)^2\left[-d\tau^2 + \exp\left(2\zeta(\tau, q^k)\right)\hat{\gamma}_{ij}(\tau, q^l)dq^i dq^j\right]$$

Constraint equation **Bruni et.al. arXiv:1307.1478; arXiv: 1405.7006**

$$\nabla^2\zeta \boxed{-2\zeta\nabla^2\zeta} + \frac{1}{2}(\nabla\zeta)^2 = -\frac{5}{2}a^2H^2\delta$$

Primordial non-Gaussianity **Bartolo et.al. astro-ph/0501614**

$$\Phi_N = \varphi + f_{\mathrm{NL}}\left(\varphi^2 - \langle\varphi^2\rangle\right) \qquad \varphi = \varphi_S + \varphi_L$$

$$(1 + 2f_{\mathrm{NL}}\varphi_L)\nabla^2\varphi_S = -\frac{3}{2}a^2H^2\delta_S$$

There is a subtle but important difference between GR non-linearity and primordial NG **Bartolo, … Sasaki,… et.al. arXiv:1506.00915**

Matter evolution equations

- Continuity and Euler equation

  Up to second order, there exists a gauge in which relativistic corrections appear only at third order **Hwang and Noh astro-ph/9812007, gr-qc/0412128, 0412129**

- Solutions for matter perturbations

$$\delta_C(\mathbf{q},\tau) = \boxed{C(\mathbf{q})D_+(\tau)} + \sum_{n=2}^{\infty} P_n(\mathbf{q})D_{n+}(\mathbf{q},\tau)$$

$$C(\mathbf{q}) \propto \exp(-2\zeta)\Big[-4\nabla^2\zeta - 2(\nabla\zeta)^2\Big]$$

  Non-linearity in the Hamiltonian constraint introduces additional GR corrections

  Bertacca et.al. 1501.03163
  Gong & Yoo 1602.06300

GR N-body

Shibata 1999 Prog. Theor. Phys. 101, 251 and 1199

- Geodesic

$$\frac{dx^i}{dt} = -\beta^i + \frac{\gamma^{ij} u_j}{u^0}$$
$$\frac{du_i}{dt} = -\alpha u^0 \alpha_{,i} + u_j \beta^j{}_{,i} - \frac{u_j u_k}{2u^0} \gamma^{jk}{}_{,i}$$

- Constraints

$$R - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = 16\pi E,$$
$$D_i \tilde{A}^i{}_j - \frac{2}{3} D_j K = 8\pi J_j,$$

- Evolution equations (GR)

$$(\partial_t - \beta^k \partial_k)\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik}\beta^k{}_{,j} + \tilde{\gamma}_{jk}\beta^k{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^k{}_{,k} \quad \cdots\cdots$$

$$ds^2 = (-\alpha^2 + \beta_k \beta^k) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

$$\gamma = \det(\gamma_{ij}) \equiv e^{12\phi},$$
$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \ \text{i.e.,} \ \det(\tilde{\gamma}_{ij}) = 1$$
$$\tilde{A}_{ij} \equiv e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right),$$

$$E = m_p \sum_{a=1}^{N} (u^0)_a \alpha e^{-6\phi} \delta^{(3)}(x^k - x^k_a),$$
$$J_i = m_p \sum_{a=1}^{N} (u_i)_a e^{-6\phi} \delta^{(3)}(x^k - x^k_a),$$

Other approaches

- Weak field expansion

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- Weak field N-body simulations

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- Full GR simulations with dust

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Conclusions

- Newtonian motion gauge

  We provided a framework to interpret and use Newtonian N-body simulations in terms of the weak filed limit of general relativity at leading order

  - inclusion of relativistic perturbations using a linear Einstein-Boltzmann code
  - identification of relativistic corrections to particle positions in N-body simulations

- Going beyond linear order
  - 2^{nd} order Einstein-Boltzmann code is available and tested
  - Relativistic corrections to initial conditions need to be included
  - Still missing “2^{nd} order Newtonian motion gauge”