

Gravity and Cosmology 2018

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Yukawa Institute for Theoretical Physics, Kyoto University

Seeing stochastic inflation

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work with

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27th February 2018



UNIVERSITY OF
PORTSMOUTH



Three applications of stochastic inflation

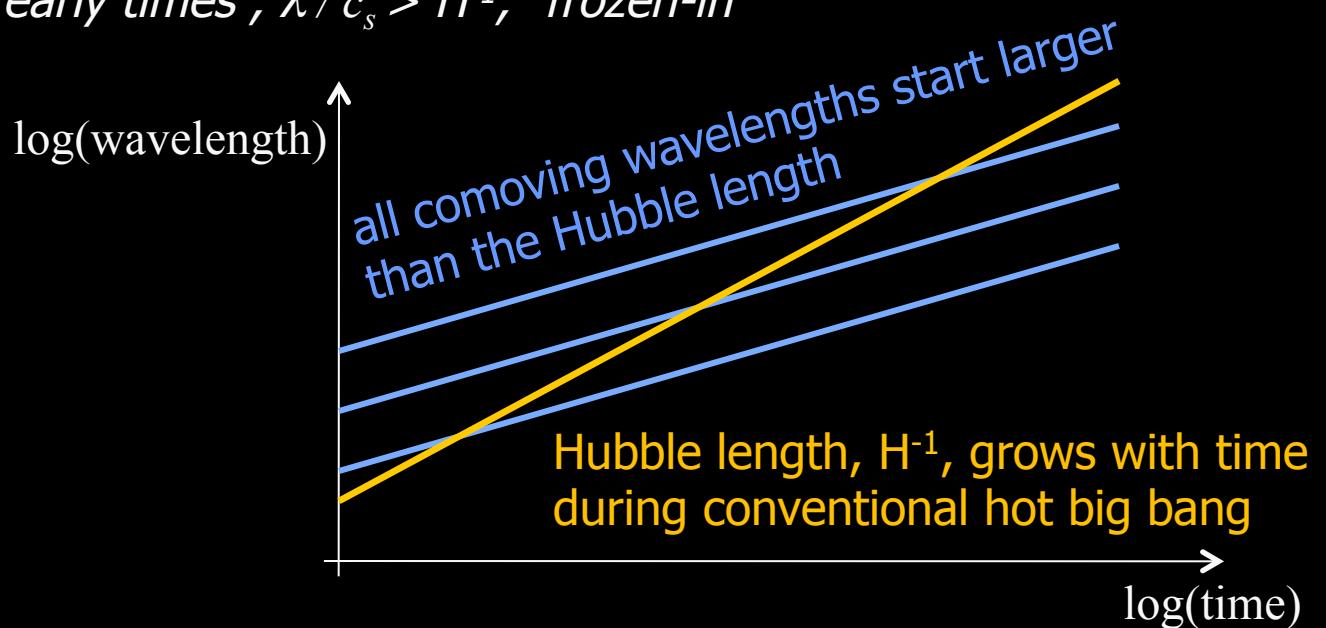
1. Primordial density perturbations can be sensitive to **Planck scale physics**
 - *Assadullahi, Firouzjahi, Noorbala, Vennin & Wands (2016)*
2. Stochastic approach gives **full probability distribution function for density perturbations** needed, e.g., for **primordial black hole abundance**
 - *Pattison, Assadullahi, Vennin & Wands (2017)*
3. Bayesian **model comparison sensitive to stochastic distribution** of multiple fields
 - *Torrado, Byrnes, Hardwick, Vennin & Wands (2017)*

Inflation = origin of super-Hubble structure

$$\delta\ddot{\rho} + 3H\delta\dot{\rho} + (c_s / \lambda)^2 \delta\rho = 0$$

Characteristic timescales for density waves with fixed comoving wavelength

- *small-scales = late times, $\lambda / c_s < H^{-1}$, under-damped oscillator*
- *large-scales = early times, $\lambda / c_s > H^{-1}$, "frozen-in"*

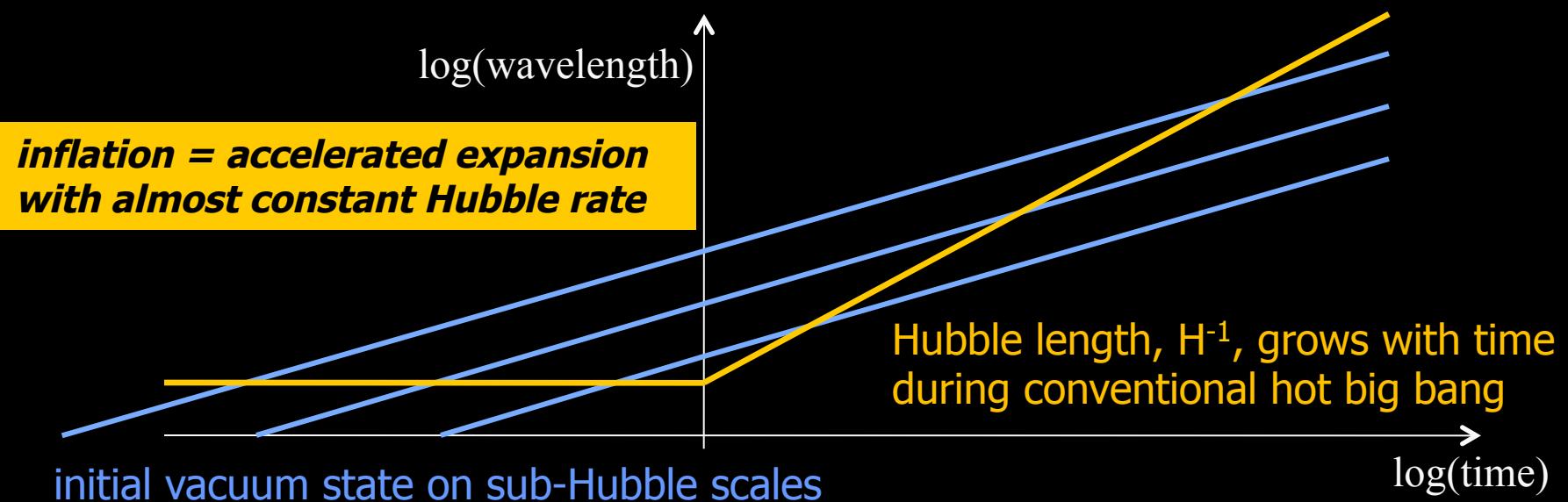


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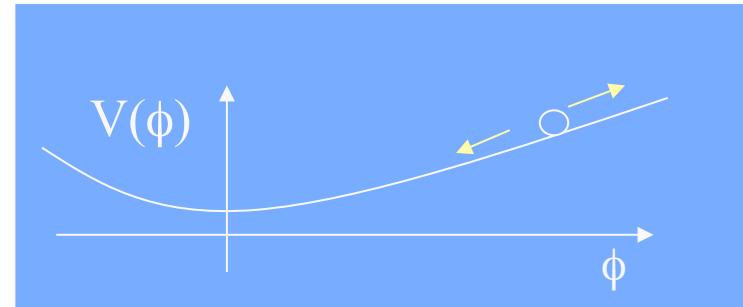
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vacuum fluctuations

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \frac{k^2}{a^2}\delta\varphi = 0$$



- *sub-Hubble/underdamped zero-point oscillations:* $\delta\phi_k \approx \frac{e^{-ik\eta}}{\sqrt{2k}}$
- *super-Hubble/overdamped perturbations in squeezed state:*

$$\langle \delta\phi^2 \rangle_{k=aH} \approx \frac{4\pi k^3 |\delta\phi_k|^2}{(2\pi)^3} = \left(\frac{H}{2\pi}\right)^2$$

- *sub-Hubble vacuum fluctuations are constantly crossing the Hubble scale ($k=aH$) into the coarse-grained super-Hubble field*

δN formalism for primordial density perturbations



Starobinsky '85;

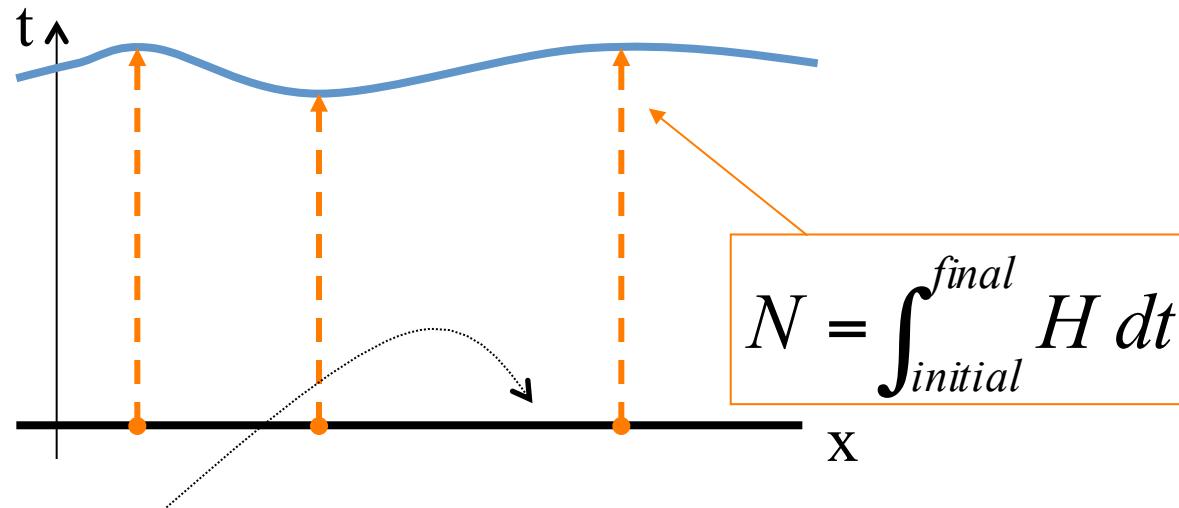


Sasaki & Stewart '96;



Lyth & Rodriguez '05

after inflation: curvature perturbation ζ on uniform-density hypersurface



during inflation: field perturbations $\phi(x, t_i)$ on initial spatially-flat hypersurface

on super-Hubble scales, evolve as “separate universes” (neglect spatial gradients):

$$\zeta = N(\phi_{initial}) - \bar{N} \approx \sum_I \frac{\partial N}{\partial \phi_I} \delta\phi_I$$

Seeing vacuum fluctuations

Planck collaboration 2015

$$\left\langle \frac{\delta T^2}{T^2} \right\rangle_{SW} \approx \frac{1}{25} \left\langle \zeta^2 \right\rangle \approx \frac{1}{25} \left(\frac{H^2}{2\pi\dot{\varphi}} \right)_{k=aH}^2$$

CMB temperature anisotropies probe a narrow range of scales corresponding to field values around Hubble exit during inflation ($N=50-60$)

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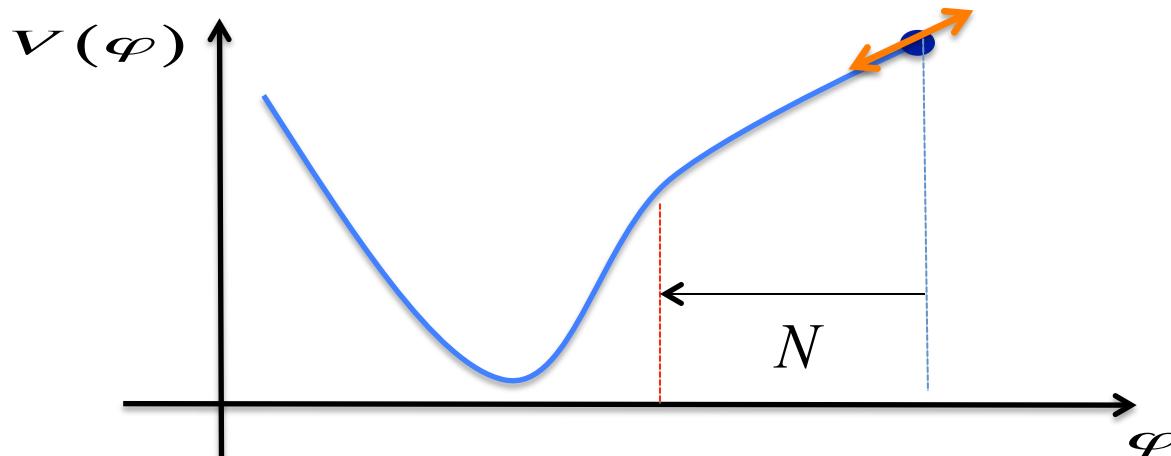
Stochastic inflation

Starobinsky (1986)

- Quantum fluctuations swept up to super-Hubble scales give stochastic kick, $\xi(N)$, to coarse-grained ($k < aH$) field

$$\frac{d\varphi}{dN} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi(N)$$

- Number of e-folds to the end of inflation, \mathcal{N} , from a given field value, φ , becomes a stochastic variable



Stochastic \mathcal{N} Vennin & Starobinsky (2015)

$$\frac{\partial \phi}{\partial N} + \frac{V'}{3H^2} = \frac{H}{2\pi}\xi \longleftrightarrow \frac{\partial}{\partial N}P(\phi, N) = \frac{\partial}{\partial \phi} \left[\frac{V'}{3H^2}P(\phi, N) \right] + \frac{\partial^2}{\partial \phi^2} \left[\frac{H^2}{8\pi^2}P(\phi, N) \right]$$

$= -\mathcal{L}_{\text{FP}} \cdot P(\phi, N)$

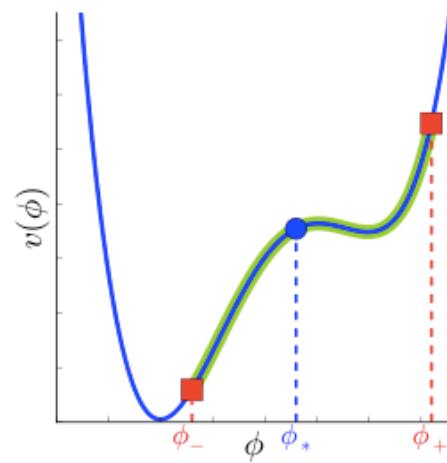
Langevin equation **Fokker-Planck equation**

First Passage Time: Louis Bachelier, 1900

$$\mathcal{L}_{\text{FP}}^\dagger \cdot \langle \mathcal{N} \rangle(\phi_*) = 1$$

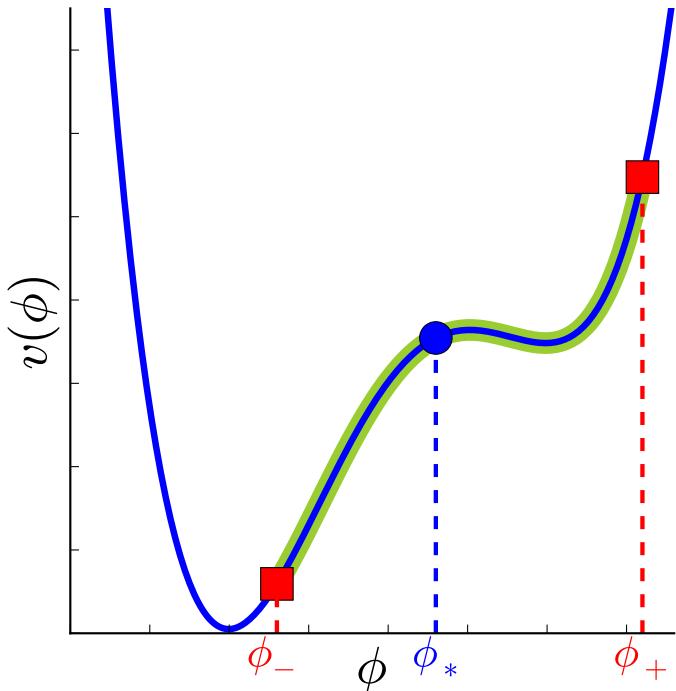
$$\langle \mathcal{N} \rangle'' v - \langle \mathcal{N} \rangle' \frac{v'}{v} = -1 \quad \text{where} \quad v = V/(24\pi^2 M_{\text{Pl}}^4)$$

$$\langle \mathcal{N} \rangle = \int_{\phi_{\text{end}}}^{\phi_*} \frac{dx}{M_{\text{Pl}}} \int_x^{\bar{\phi}} \frac{dy}{M_{\text{Pl}}} \frac{1}{v(y)} \exp \left[\frac{1}{v(y)} - \frac{1}{v(x)} \right]$$



Stochastic δN

Vennin & Starobinsky (2015)



see also Enqvist et al (2008);
Fujita, Kawasaki, Tada & Takesako (2013, 2014)

$$\langle \mathcal{N} \rangle = \int_{\phi_-}^{\phi_*} \frac{dx}{M_{Pl}} \int_x^{\phi_+} \frac{dy}{M_{Pl}} \frac{e^{\frac{1}{v(y)} - \frac{1}{v(x)}}}{v(y)}$$

where dimensionless potential : $v(\phi) \equiv \frac{V(\phi)}{24\pi^2 M_{Pl}^4}$

$$\zeta = \mathcal{N} - \langle \mathcal{N} \rangle$$

- integration domain covers the entire field space

Saddle Point Approximation

$$\langle \mathcal{N} \rangle = \int_{\phi_{\text{end}}}^{\phi_*} \frac{dx}{M_{\text{Pl}}} \int_x^{\bar{\phi}} \frac{dy}{M_{\text{Pl}}} \frac{1}{v(y)} \exp \left[\frac{1}{v(y)} - \frac{1}{v(x)} \right]$$

\downarrow

$$\left| 2v - \frac{v'' v^2}{v'^2} \right| \ll 1$$

$$\langle \mathcal{N} \rangle \simeq \int_{\phi_{\text{end}}}^{\phi_*} \frac{dx}{M_{\text{Pl}}^2} \frac{v(x)}{v'(x)} \left[1 + v(x) - \frac{v''(x) v^2(x)}{v'^2(x)} + \dots \right]$$

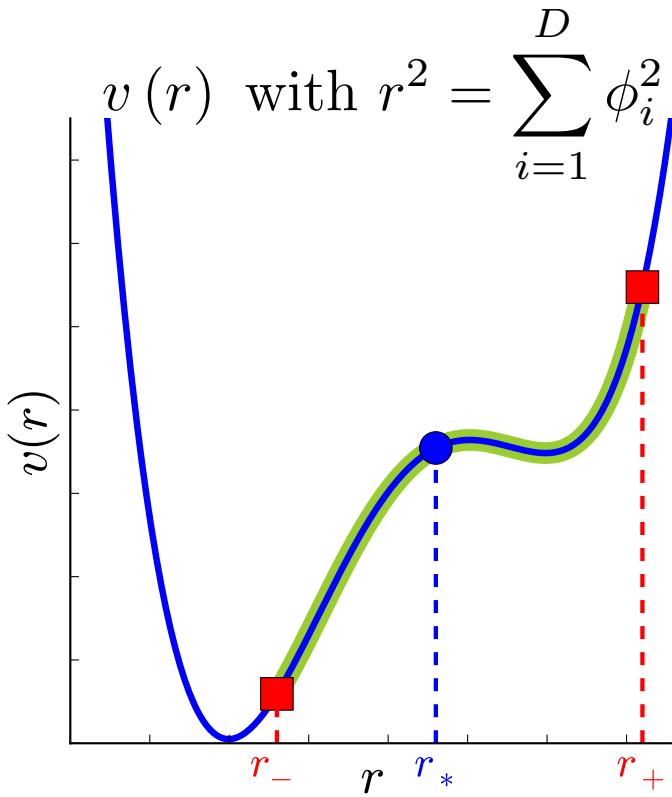


Classical result



First order correction

Infinite inflation and number of fields (D)



Vennin, Assadullahi, Firouzjahi, Noorbala & Wands (2016)

$$\langle \mathcal{N} \rangle = \int_{r_-}^{r_*} \frac{dx}{M_{\text{Pl}}} \int_x^{r_+} \frac{dy}{M_{\text{Pl}}} \frac{e^{\frac{1}{v(y)} - \frac{1}{v(x)}}}{v(y)} \left(\frac{y}{x}\right)^{D-1}$$

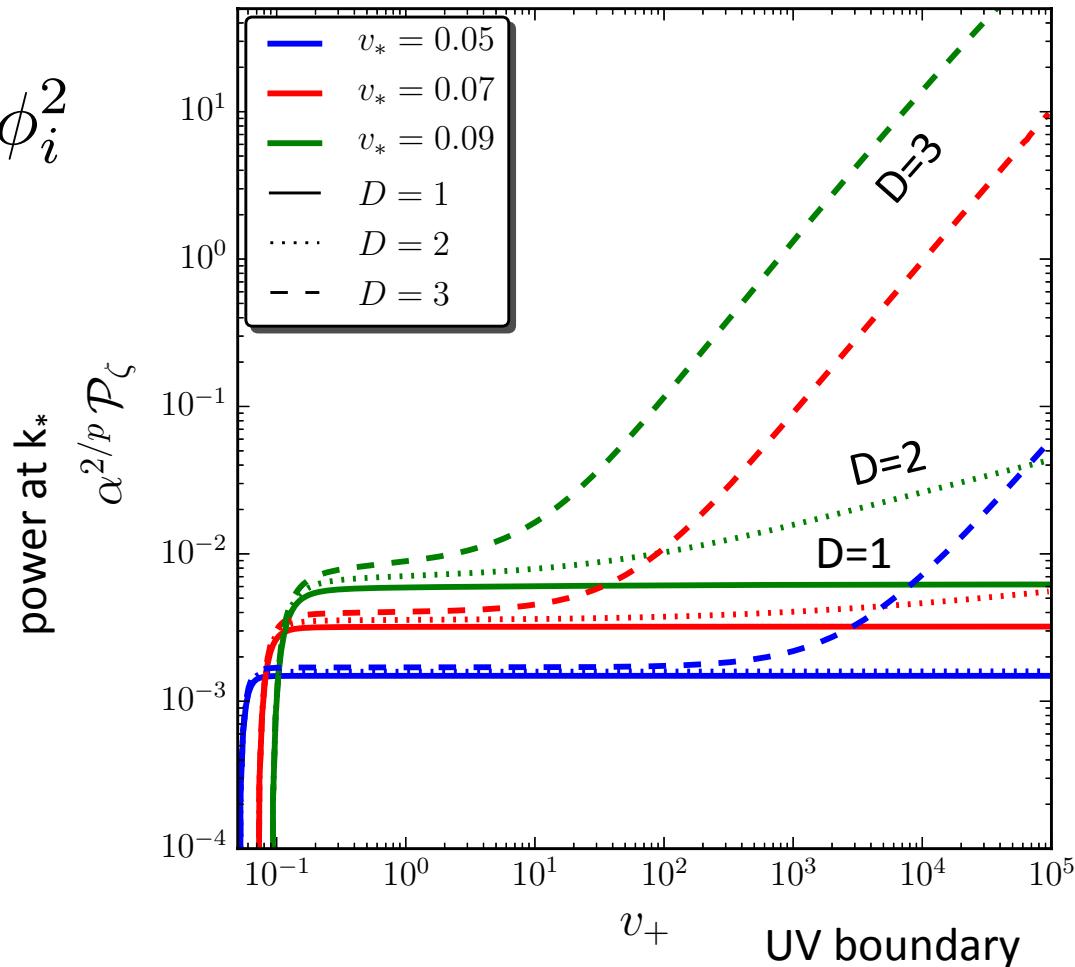
- integration domain covers the entire field space
- what if $r_+ \rightarrow \infty$? $v(r) \propto r^p$ \longrightarrow $\langle \mathcal{N} \rangle = \infty$ if $p \leq D$

Stochastic $\delta\mathcal{N}$

Vennin, Assadullahi, Firouzjahi, Noorbala & Wands (2016)

$D > 2$ requires UV regularisation (boundary) at $v_+ \sim M_{Pl}$

$$v \propto \sum_{i=1}^D \phi_i^2$$

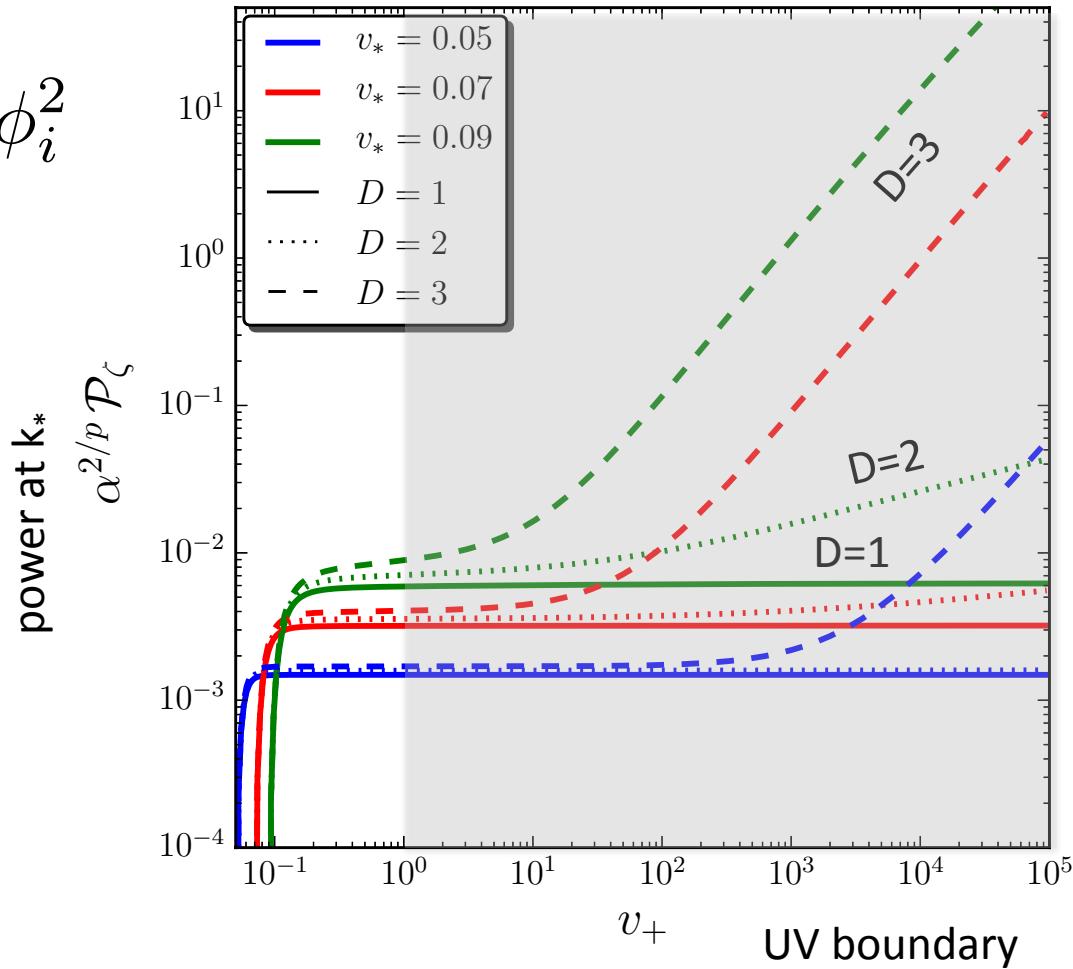


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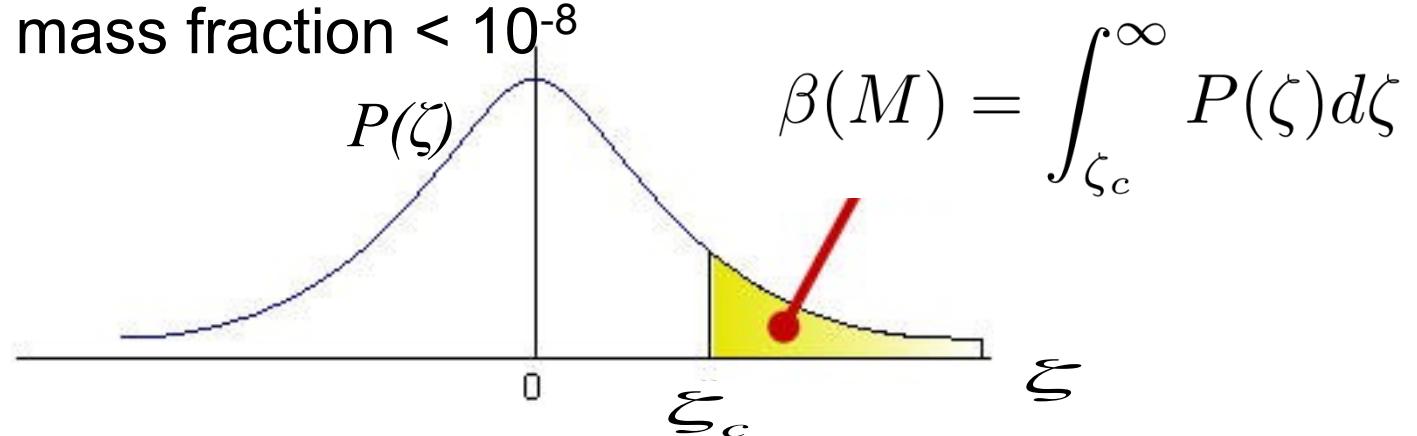
Primordial black holes from inflation

- primordial density perturbations when modes re-enter Hubble-scale after inflation

$$\left. \frac{\delta \rho}{\rho} \right|_{k=aH} \sim \zeta$$

- rare fluctuations exceeding critical value, $\zeta > \zeta_c \sim 1$, collapse to form black holes

- $\beta(M) = \text{mass fraction} < 10^{-8}$

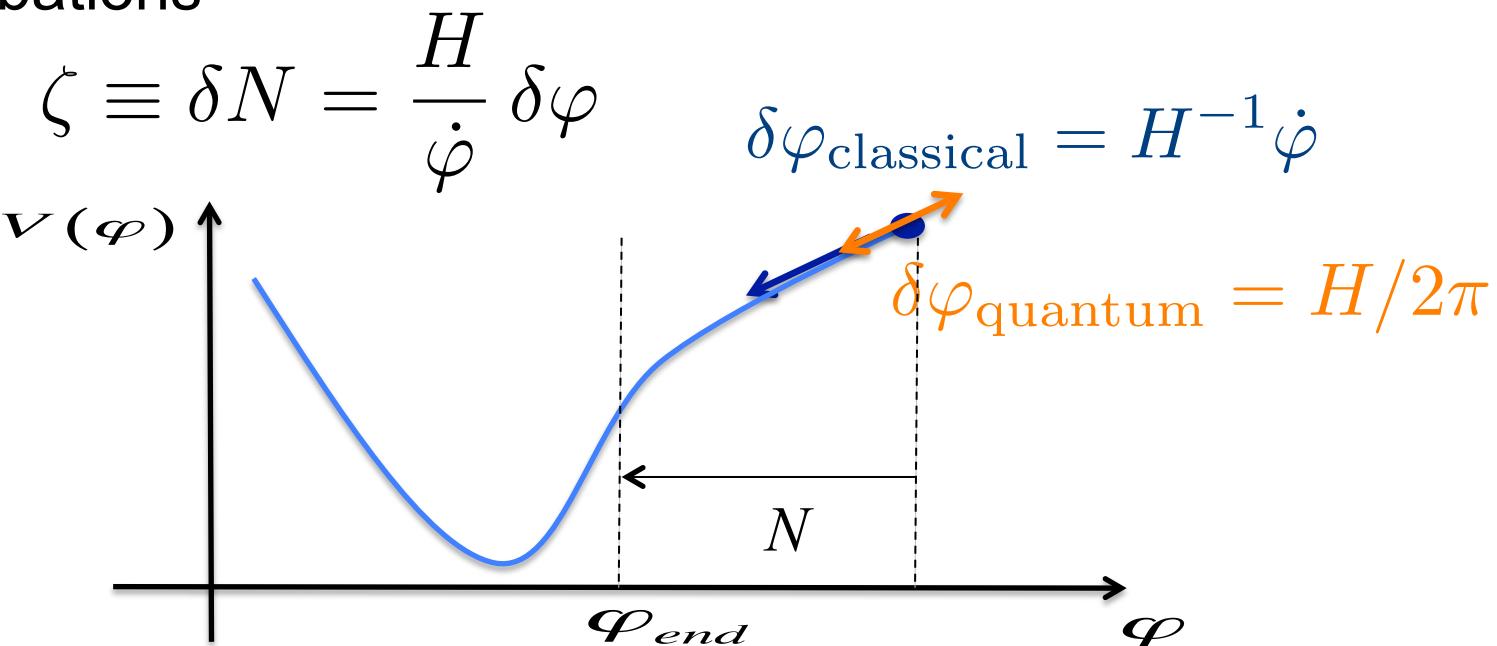


Classical vs quantum inflation

- **Classical** slow-roll inflates universe

$$N = \int H dt = \int \frac{H}{\dot{\varphi}} d\varphi$$

- **Quantum** field fluctuations lead to primordial metric perturbations



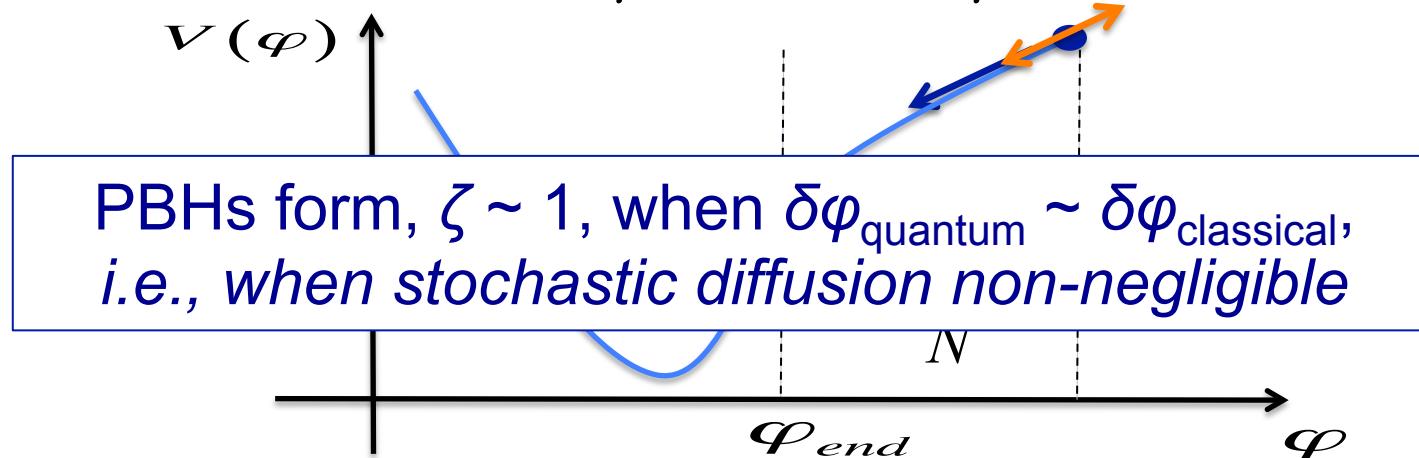
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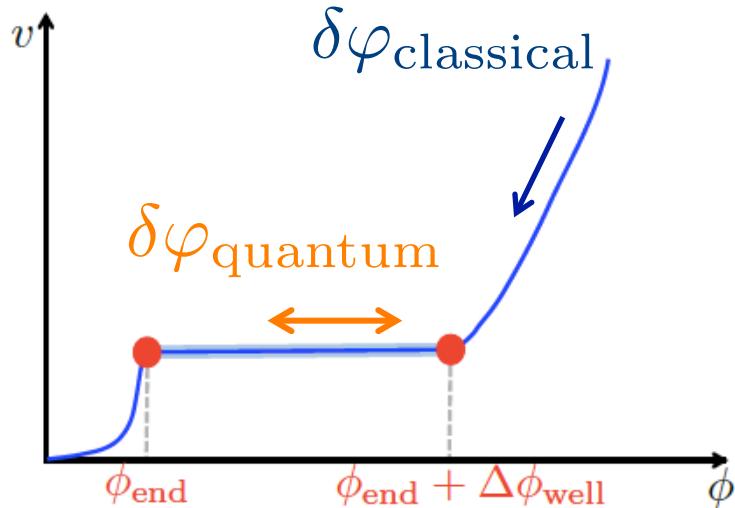
$$\zeta \equiv \delta N = \frac{H}{\dot{\varphi}} \delta\varphi = \frac{\delta\varphi_{\text{quantum}}}{\delta\varphi_{\text{classical}}}$$



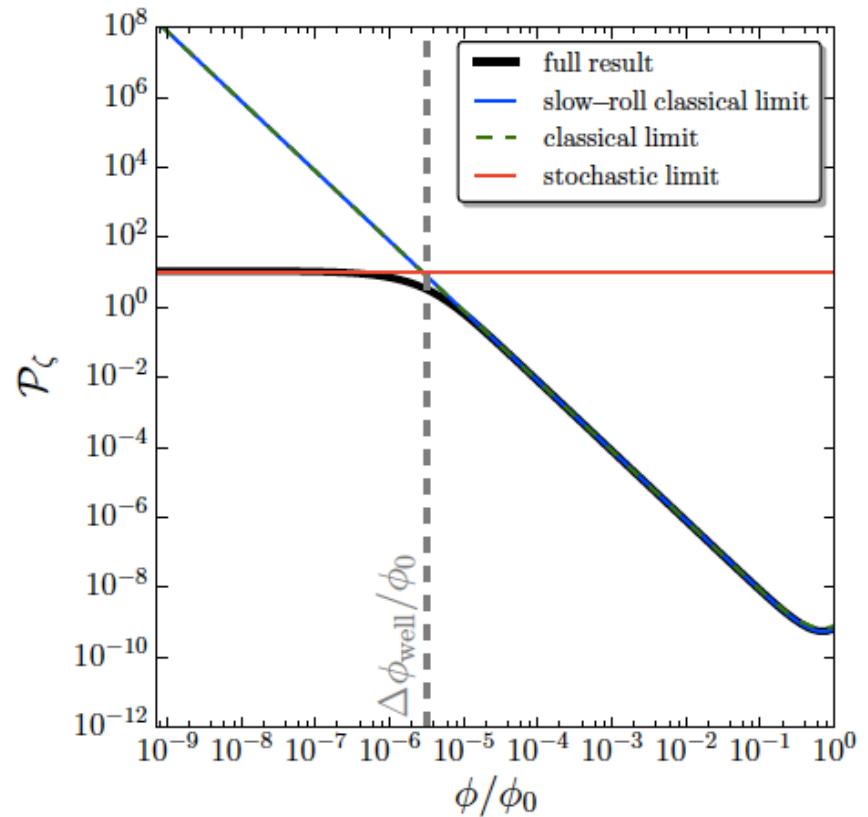
Plateau at end of inflation

Pattison et al (2017)

- large deviations from classical δN when quantum diffusion dominates over classical drift near end of inflation: $\phi < \phi_{\text{end}} + \Delta\phi_{\text{well}}$



$$v = v_0 \left[1 + \left(\frac{\phi}{\phi_0} \right)^p \right]$$



Full PDF for δN Pattison et al (2017)

- number of e-folds is a stochastic variable, $\mathcal{N}(\phi)$

$$\zeta = \mathcal{N} - \langle \mathcal{N} \rangle$$

- define characteristic function (includes all the moments)

$$\chi_{\mathcal{N}}(t, \varphi) = \langle e^{it\mathcal{N}(\varphi)} \rangle = \int e^{it\mathcal{N}(\varphi)} P(\mathcal{N}, \varphi) d\mathcal{N}$$

- obeys differential equation

$$\left(\frac{\partial^2}{\partial \phi^2} - \frac{v'}{v^2} \frac{\partial}{\partial \phi} + \frac{it}{v M_{Pl}^2} \right) \chi_{\mathcal{N}}(t, \phi) = 0$$

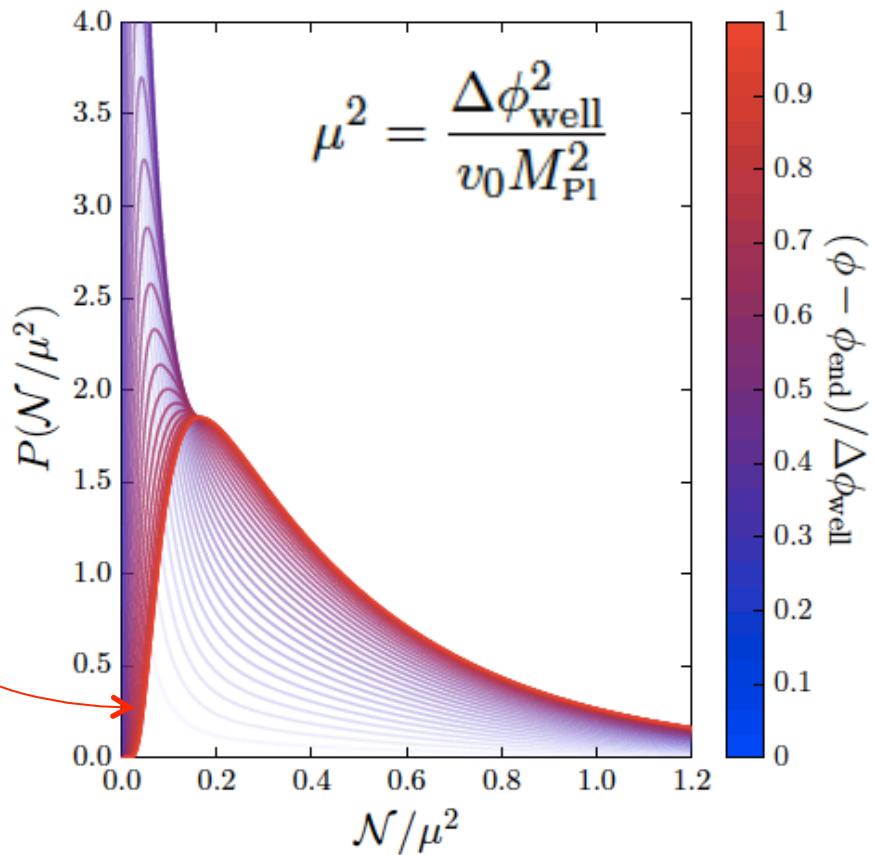
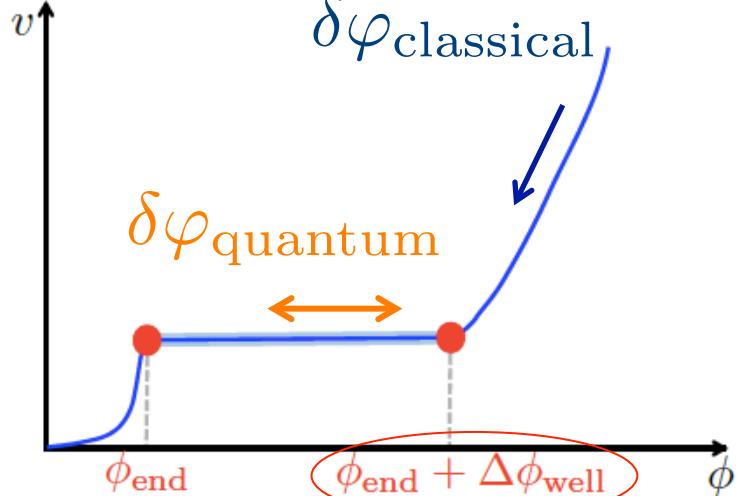
- inverse Fourier transform gives full probability distribution

$$P(\mathcal{N}, \varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\mathcal{N}} \chi_{\mathcal{N}}(t, \varphi) dt$$

Plateau at end of inflation

Pattison et al (2017)

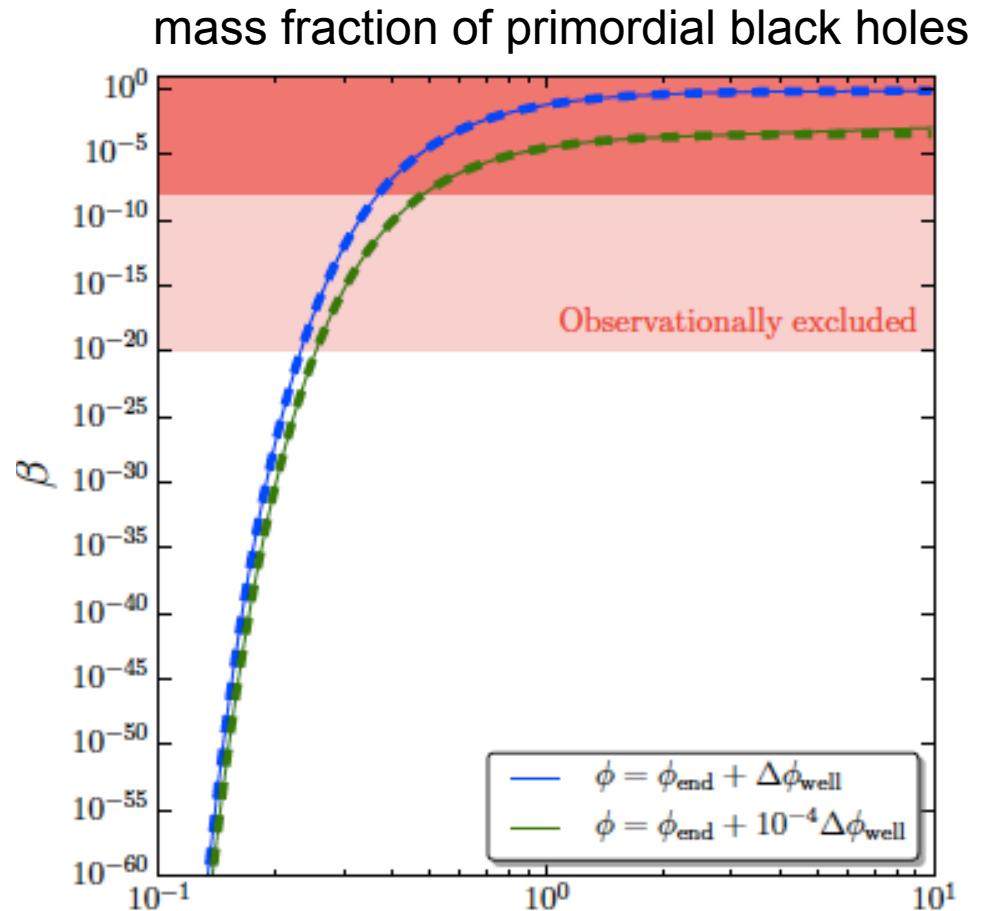
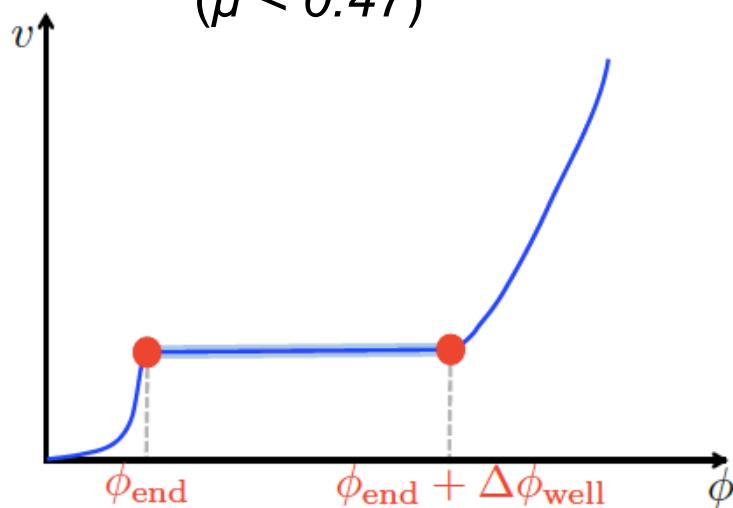
- large deviations from classical δN when quantum diffusion dominates over classical drift \rightarrow non-Gaussian probability distribution



Plateau at end of inflation

Pattison et al (2017)

- require number of e-folds in quantum diffusion dominated regime $N < 1$
 $(\mu < 0.47)$

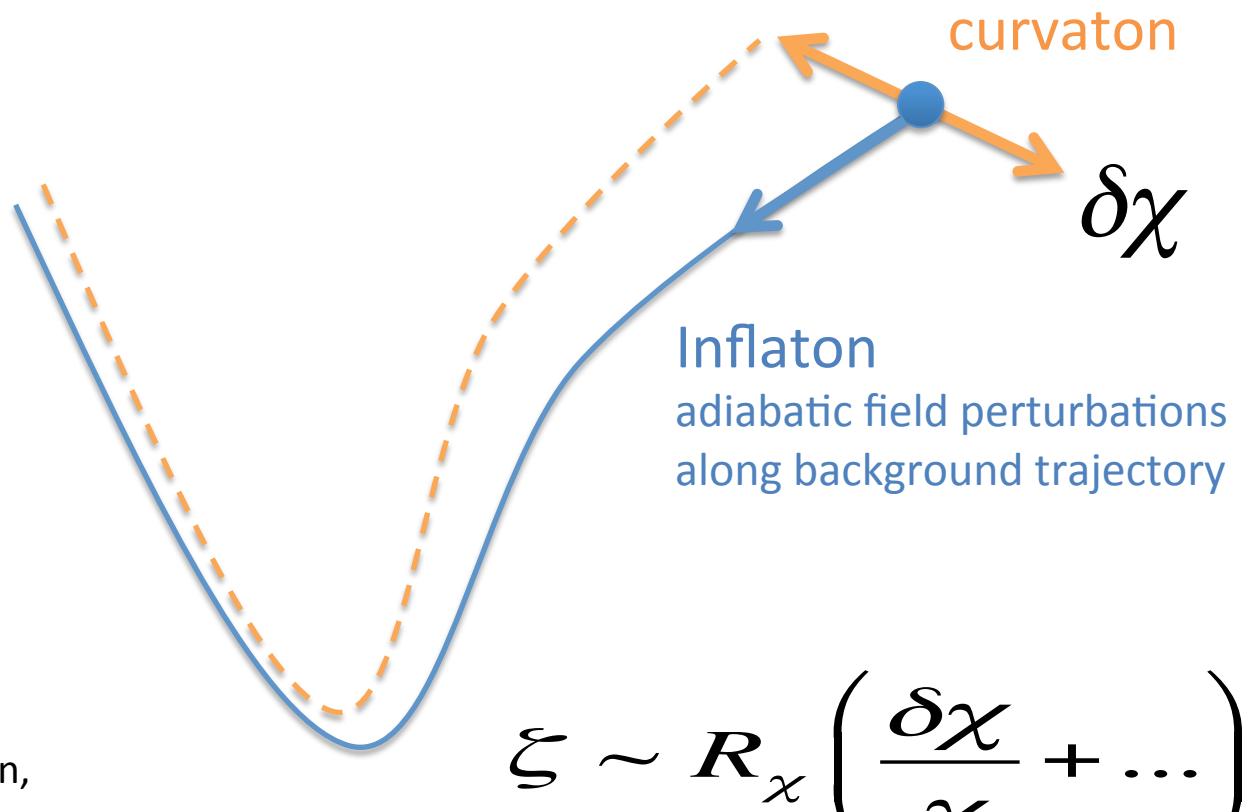


$$\mu^2 = \frac{\Delta\phi_{\text{well}}^2}{v_0 M_{\text{Pl}}^2}$$

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what is the origin of primordial density perturbations?



see also multi-field inflation,
modulated reheating,
inhomogeneous end of
inflation...

Inflaton models + curvaton field, χ

Curvaton scenarios with quadratic potential $V(\phi, \chi) = U(\phi) + m_\chi^2 \chi^2 / 2$

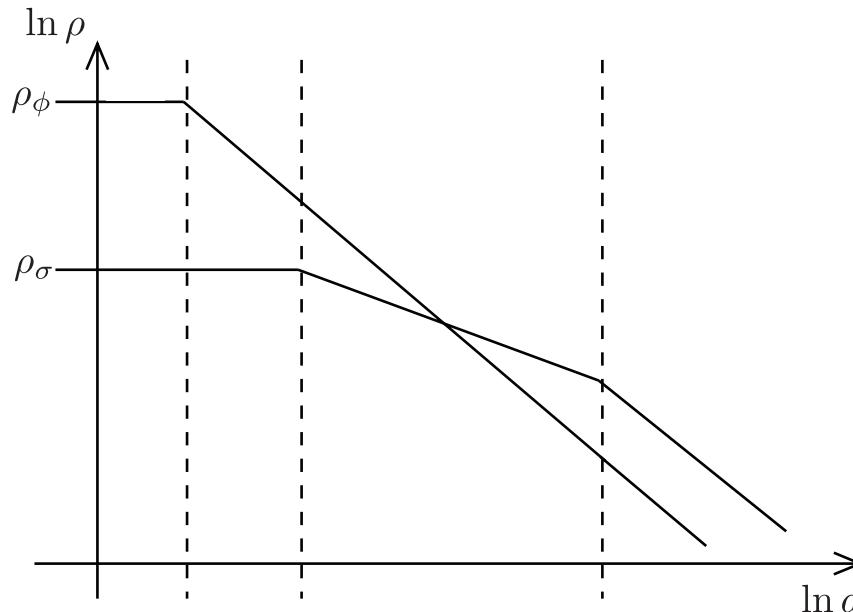
Linde and Mukhanov, 1997

Enqvist and Sloth, 2001

Lyth and Wands, 2001

Moroi and Takahashi, 2001

Langlois and Vernizzi, 2004



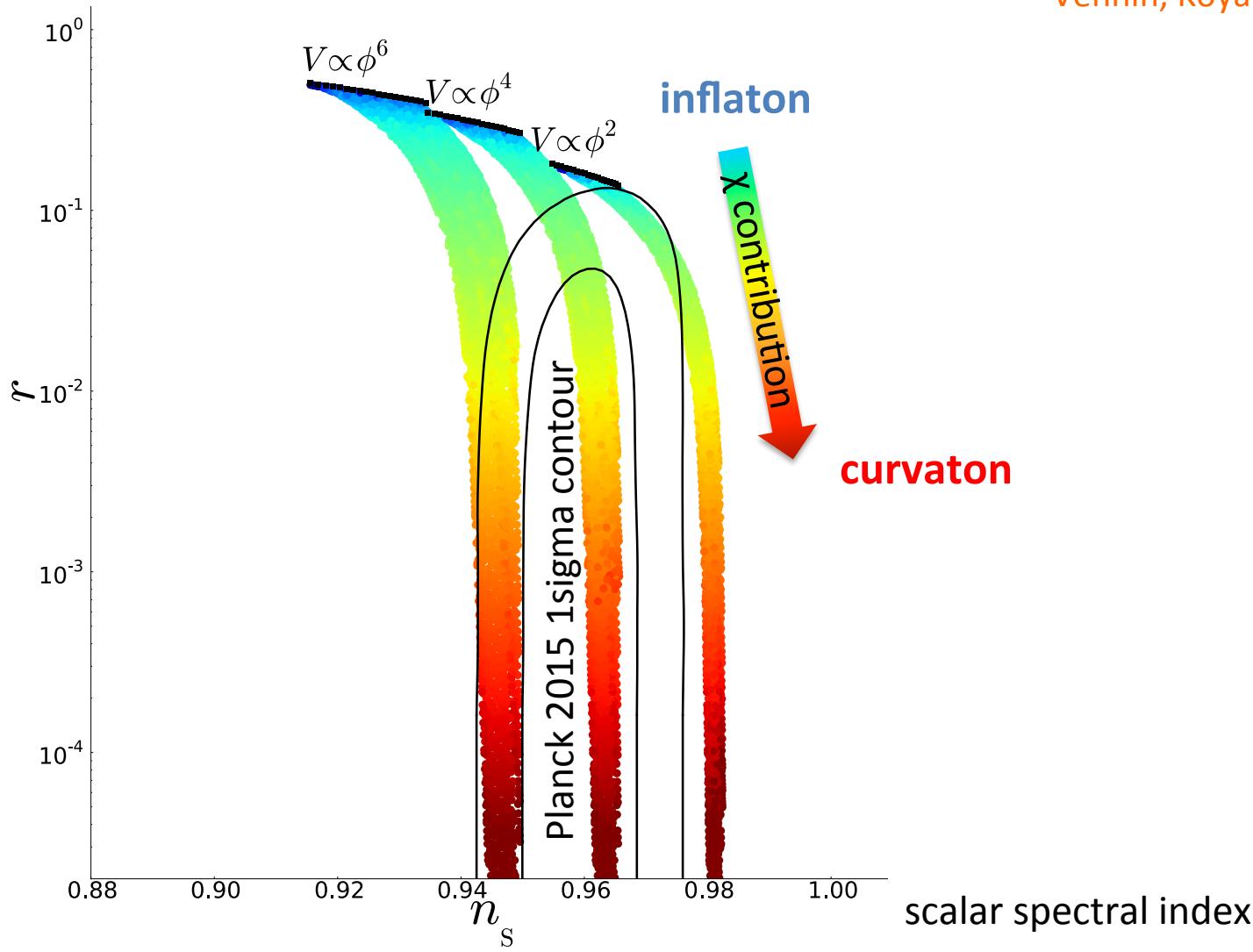
more reheating parameters: $\Gamma_\phi \rightarrow \Gamma_\phi, \Gamma_\chi, m_\chi, \chi_{\text{end}}$

primordial perturbations directly dependent on reheating
(not just through the expansion N_*)

large-field inflaton (LFI) plus quadratic curvaton, χ

tensor-scalar ratio

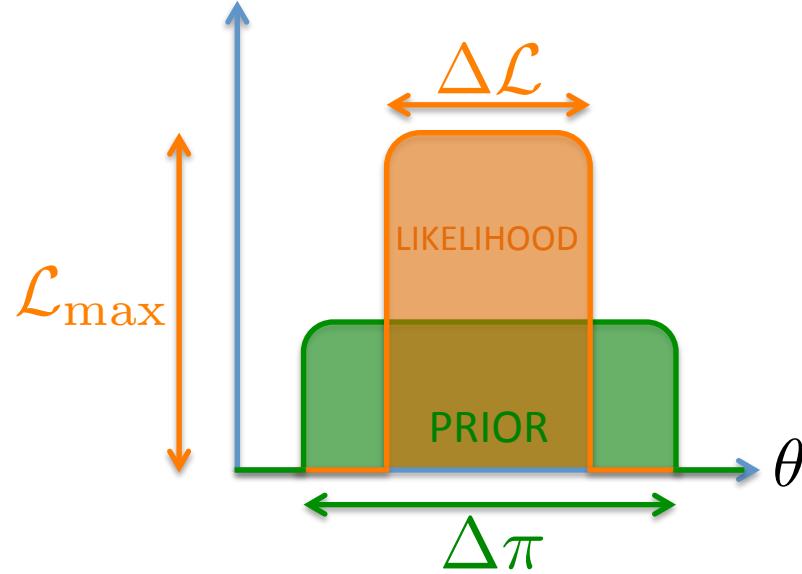
Vennin, Koyama and Wands (2015)



Bayesian Approach

to model comparison

Bayesian evidence: Integral of the likelihood over parameter prior



$$\mathcal{E}(\mathcal{M}) = \mathcal{L}_{\max} \frac{\Delta\mathcal{L}}{\Delta\pi}$$

Compromise between **quality of fit** and **simplicity**

Bayes factor = ratio of evidence

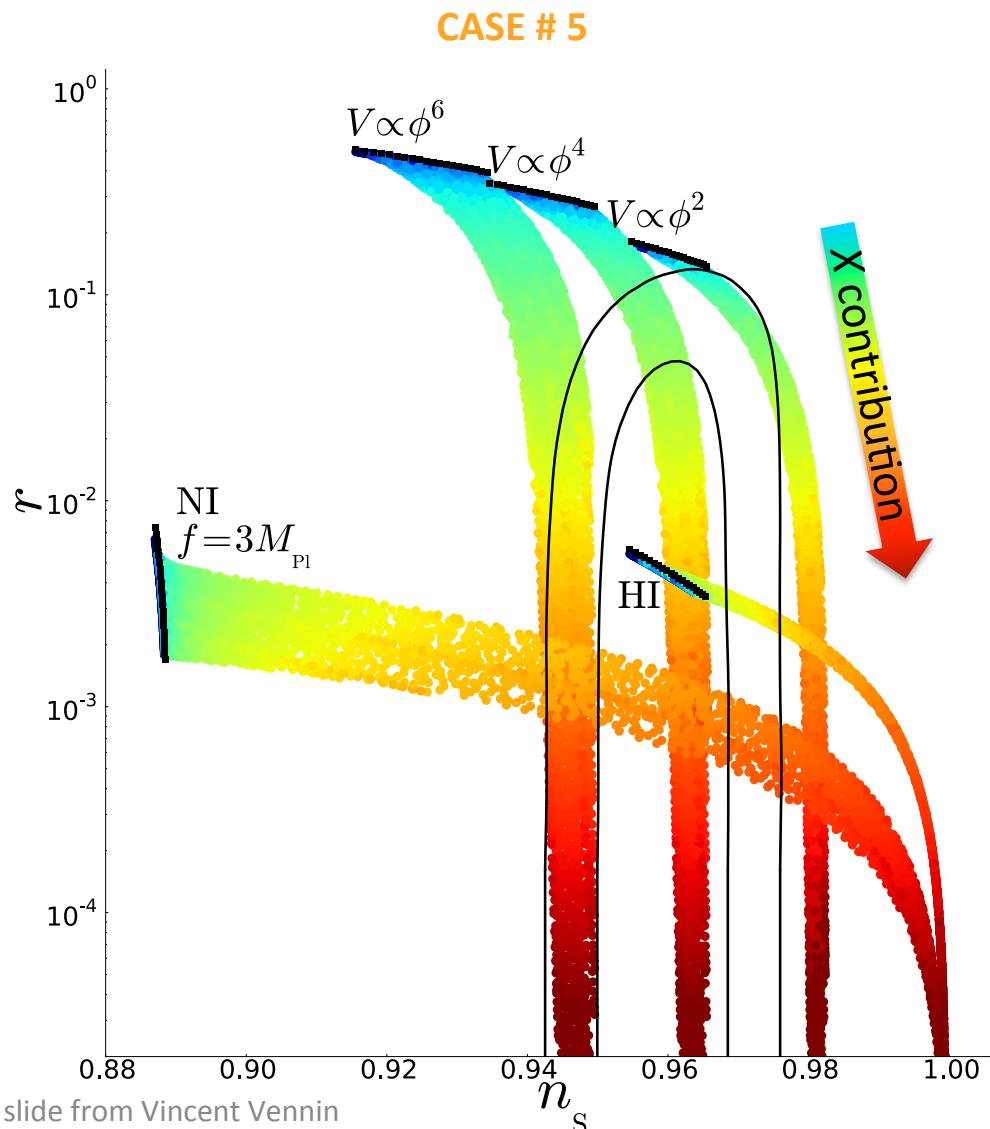
$$B_{ij} = E(M_i) / E(M_j)$$

Jeffreys scale

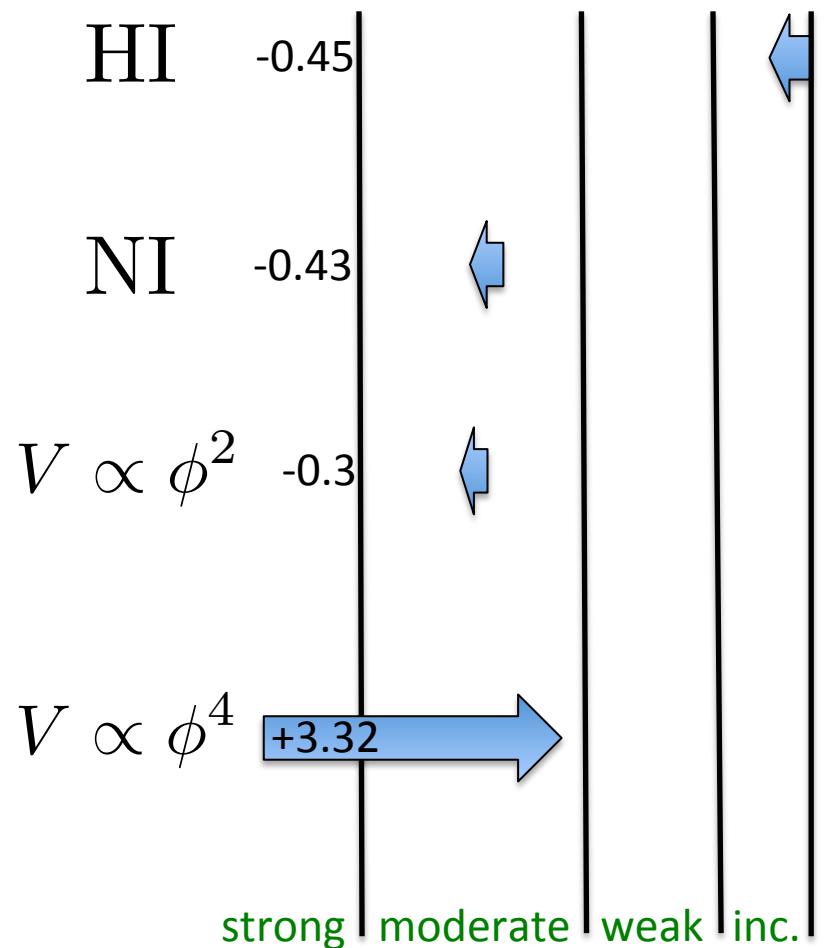
- | | |
|---------------------|---------------------|
| • Strong evidence | $\ln(B_{ij}) > 5$ |
| • Moderate evidence | $\ln(B_{ij}) > 2.5$ |
| • Weak evidence | $\ln(B_{ij}) > 1$ |
| • Inconclusive | $\ln(B_{ij}) < 1$ |

Inflaton models plus weakly-coupled scalar field, χ

Vennin, Koyama and Wands (2015)

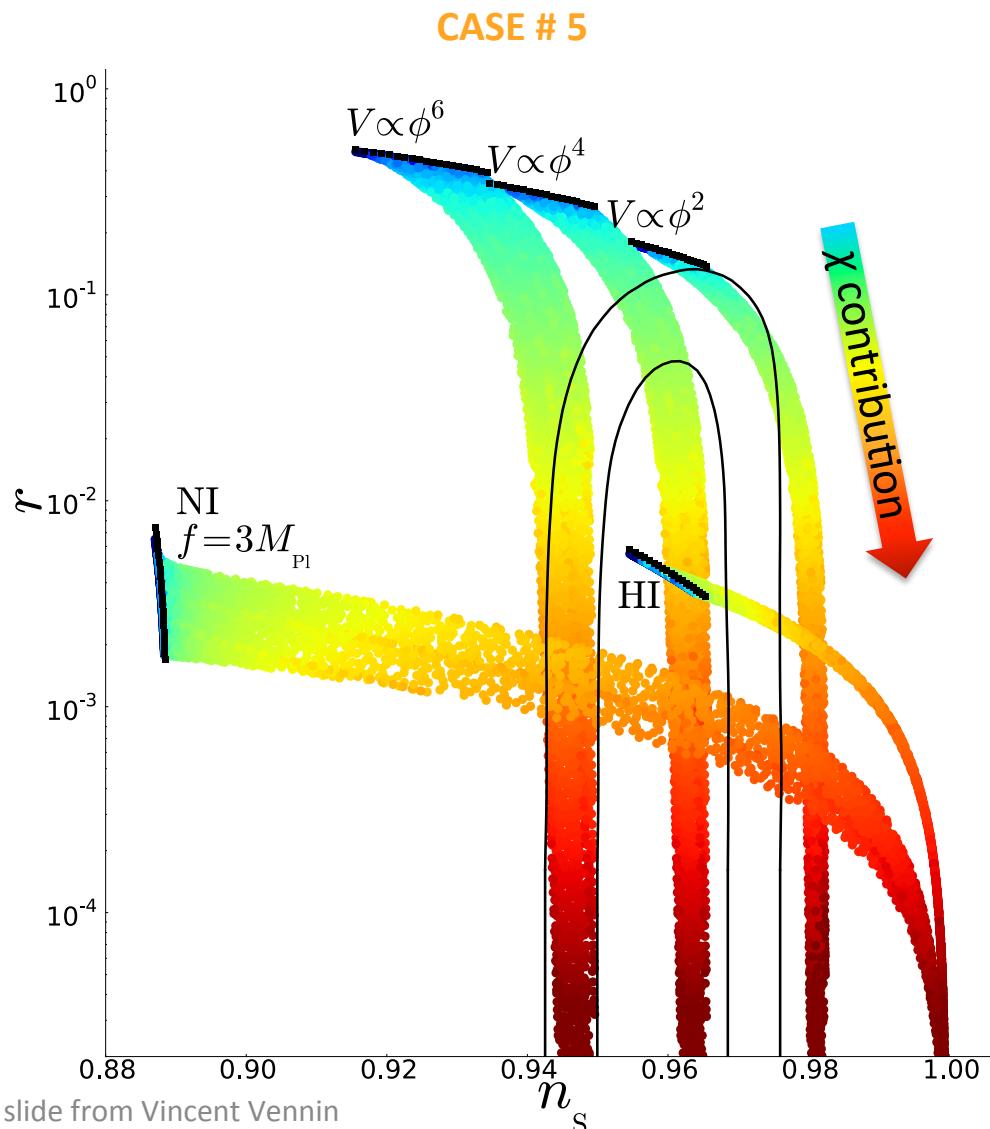


Bayesian evidence, averaging over all cases:

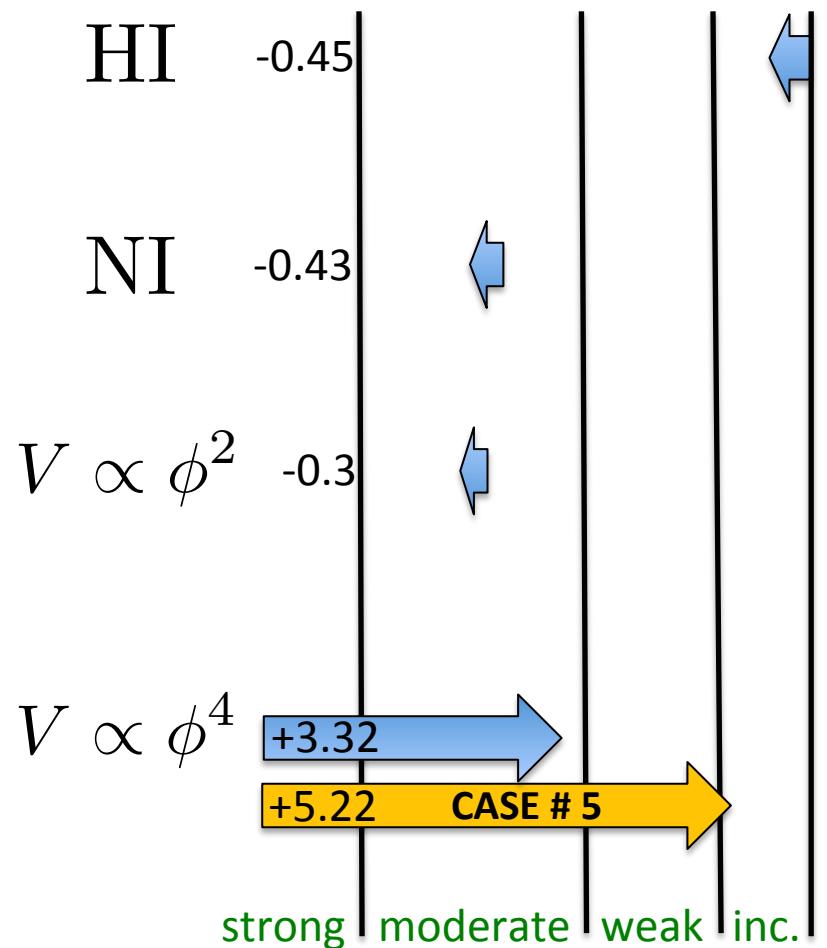


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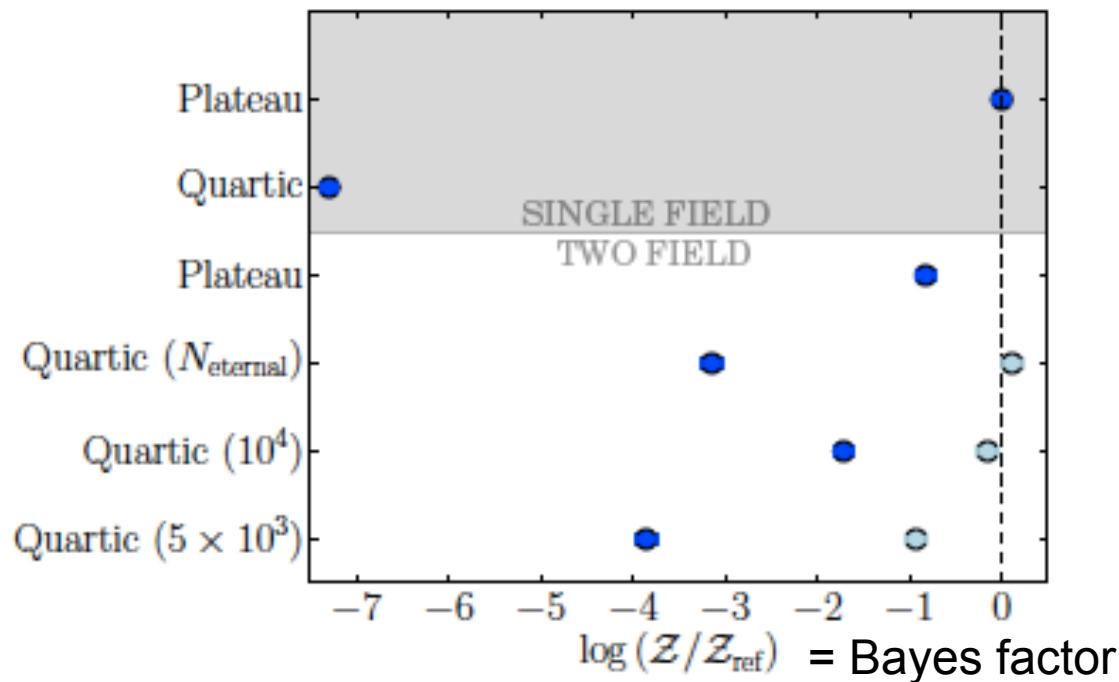


Bayesian evidence, averaging over all cases:



evidence depends on theory priors

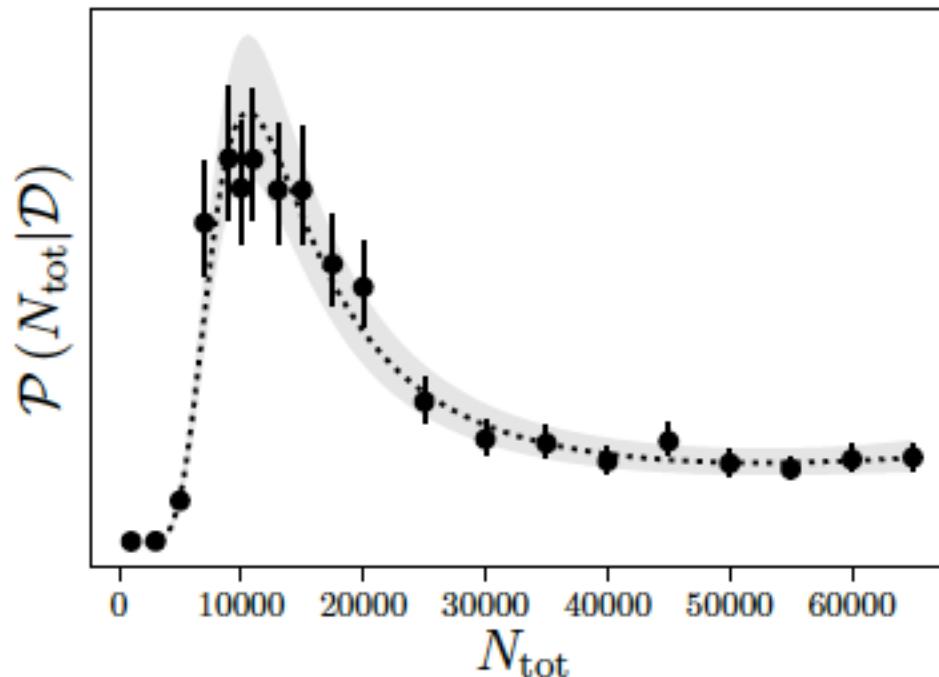
- Stochastic evolution can predict statistical distribution of curvaton field value during inflation
- Weakly-coupled curvaton does not reach the stationary distribution during large-field inflation
- Curvaton variance grows with the duration of inflation



Observing the duration of inflation

Torrado, Byrnes, Hardwick, Vennin & Wands (2017)

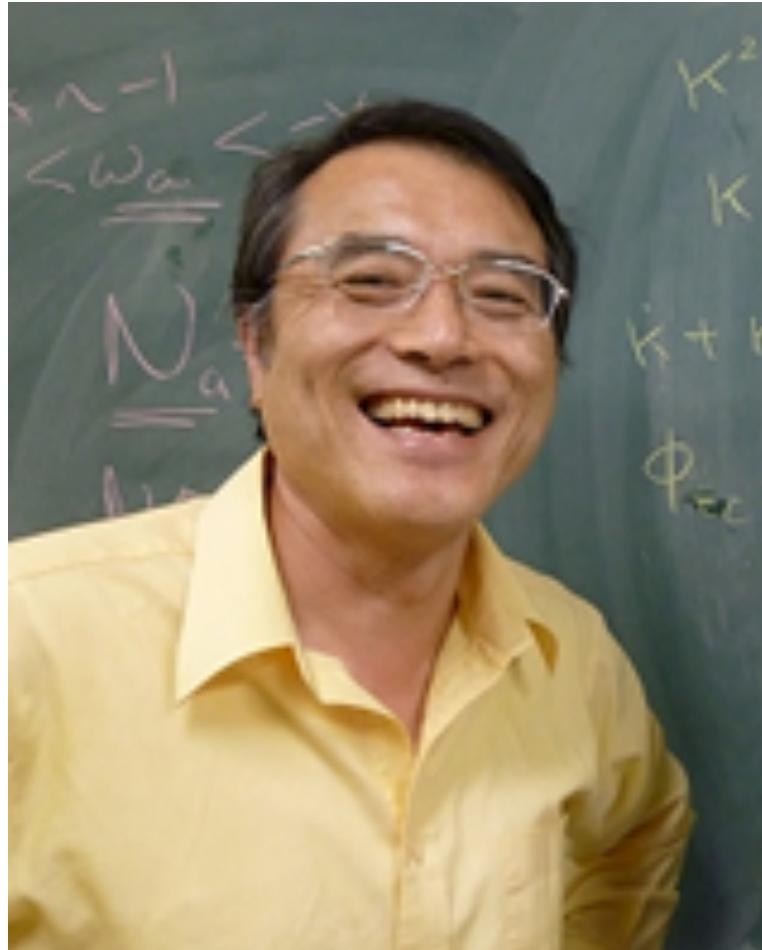
- Curvaton variance grows with duration of inflation
- Observational data can be used to infer likelihood (“observe”) the duration of inflation in the curvaton scenario...



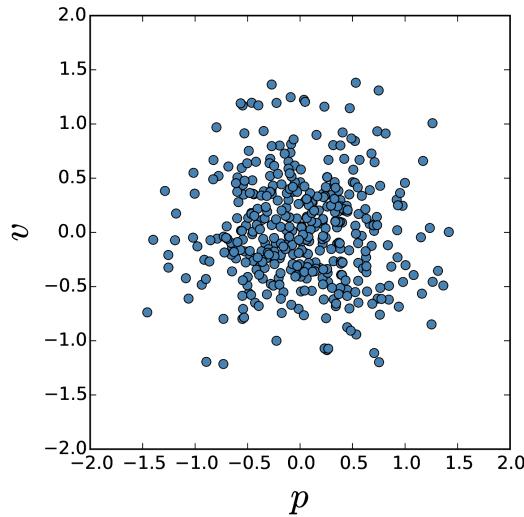
stochastic inflation has observational implications

- (not just our place in the eternally inflating multiverse and all that...)
- Density perturbations beyond the perturbative approach
 - Stochastic δN formalism
 - Enqvist et al (2008); Fujita et al (2013, 2014); Vennin & Starobinsky (2015)
 - Infinite inflation requires UV cut-off
 - *Assadullahi, Firouzjahi, Noorbala, Vennin & Wands, arXiv:1604.04502*
 - Quantum diffusion and PBHs from inflation
 - *Pattison, Vennin, Assadullahi & Wands, arXiv:1707.00537*
- Probability distributions for field values in inflation
 - The stochastic spectator
 - *Hardwick, Vennin, Byrnes, Torrado & Wands, arXiv:1701.06473*
 - Theoretical priors for Bayesian model comparison
 - *Torrado, Byrnes, Hardwick, Vennin & Wands, arXiv:1712.05364*

best wishes to Sasaki-san!



vacuum fluctuations



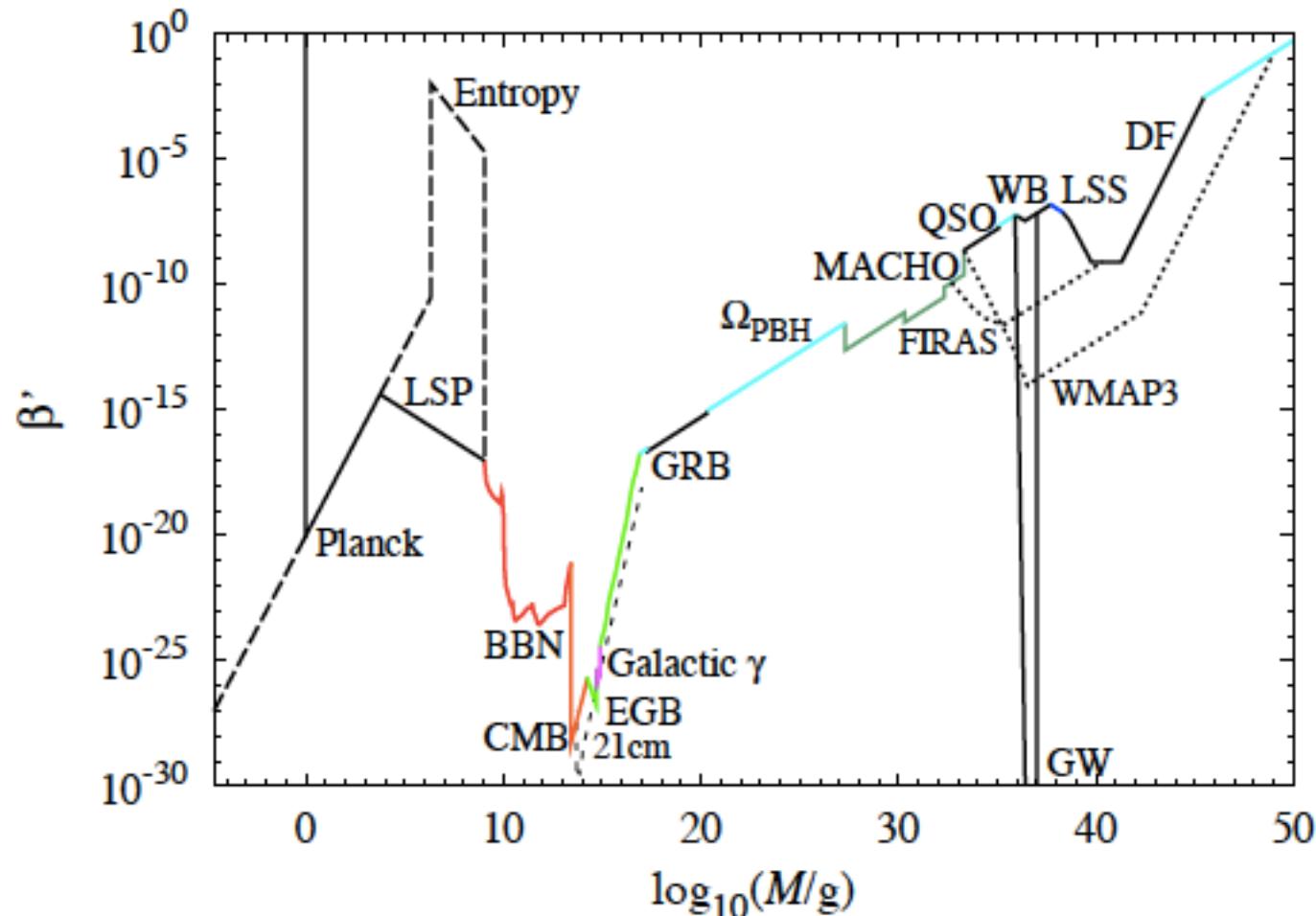
- *phase-space evolution (animation c/o Vincent Vennin)*
- *small-scale/underdamped zero-point oscillations:* $\delta\phi_k \approx \frac{e^{-ik\eta}}{\sqrt{2k}}$
- *large-scale/overdamped perturbations in squeezed state:*

$$\langle \delta\phi^2 \rangle_{k=aH} \approx \frac{4\pi k^3 |\delta\phi_k|^2}{(2\pi)^3} = \left(\frac{H}{2\pi}\right)^2$$

Primordial black holes from inflation

e.g., Carr, Kohri, Sendouda & Yokoyama (2009)

- $\beta(M) = \text{mass fraction}$



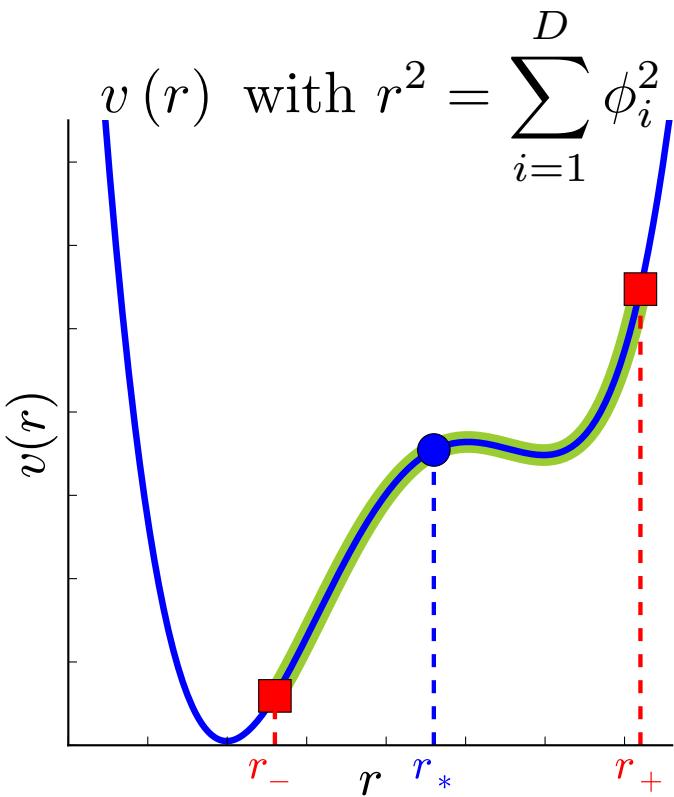
summary

- Stochastic δN needed to calculate primordial density perturbations beyond perturbative approach
- We constructed full probability distribution function
 - solve for characteristic function, then Fourier transform
 - calculated abundance of primordial black holes produced in simple plateau models
 - e.g., running-mass model of inflation
- ***Primordial Black Hole bounds require $N < 1$ in quantum diffusion regime***

further work:

- alternative PBH models
 - transient non-slow-roll backgrounds, e.g., inflection point inflation (e.g., Garcia-Bellido & Ruiz, Germani & Prokopec, Motohashi & Hu 2017)
- explore nature of non-Gaussianity beyond leading order (classical) δN
 - corrections to tail of distribution even close to classical limit?
 - understand consistency of non-Gaussian pdf with absence of correlation between large and small physical scales in single-clock inflation (e.g., Pajer, Schmidt & Zaldarriaga 2013)

Large-Field Exploration & Number of Fields



$$\langle \mathcal{N} \rangle = \int_{r_-}^{r_*} \frac{dx}{M_{\text{Pl}}} \int_x^{r_+} \frac{dy}{M_{\text{Pl}}} \frac{e^{\frac{1}{v(y)} - \frac{1}{v(x)}}}{v(y)} \left(\frac{y}{x}\right)^{D-1}$$

$$p_+ = \frac{\int_{r_-}^{r_*} x^{1-D} e^{-\frac{1}{v(x)}} dx}{\int_{r_-}^{r_+} x^{1-D} e^{-\frac{1}{v(x)}} dx}$$

- integration domain covers the entire field space
- what if $r_+ \rightarrow \infty$? $v(r) \propto r^p \rightarrow \langle \mathcal{N} \rangle = \infty$ if $p \leq D$
 $p_+ > 0$ if $D > 2$

Classical Limit

$$\mathcal{P}_\zeta(\phi_*) = 2 \left\{ \int_{\phi_*}^{\bar{\phi}} \frac{dx}{M_{\text{Pl}}} \frac{1}{v(x)} \exp \left[\frac{1}{v(x)} - \frac{1}{v(\phi_*)} \right] \right\}^{-1} \times \\ \int_{\phi_*}^{\bar{\phi}} \frac{dx}{M_{\text{Pl}}} \left\{ \int_x^{\bar{\phi}} \frac{dy}{M_{\text{Pl}}} \frac{1}{v(y)} \exp \left[\frac{1}{v(y)} - \frac{1}{v(x)} \right] \right\}^2 \exp \left[\frac{1}{v(x)} - \frac{1}{v(\phi_*)} \right]$$

Saddle Point Approximation

$$\left| 2v - \frac{v'' v^2}{v'^2} \right| \ll 1$$

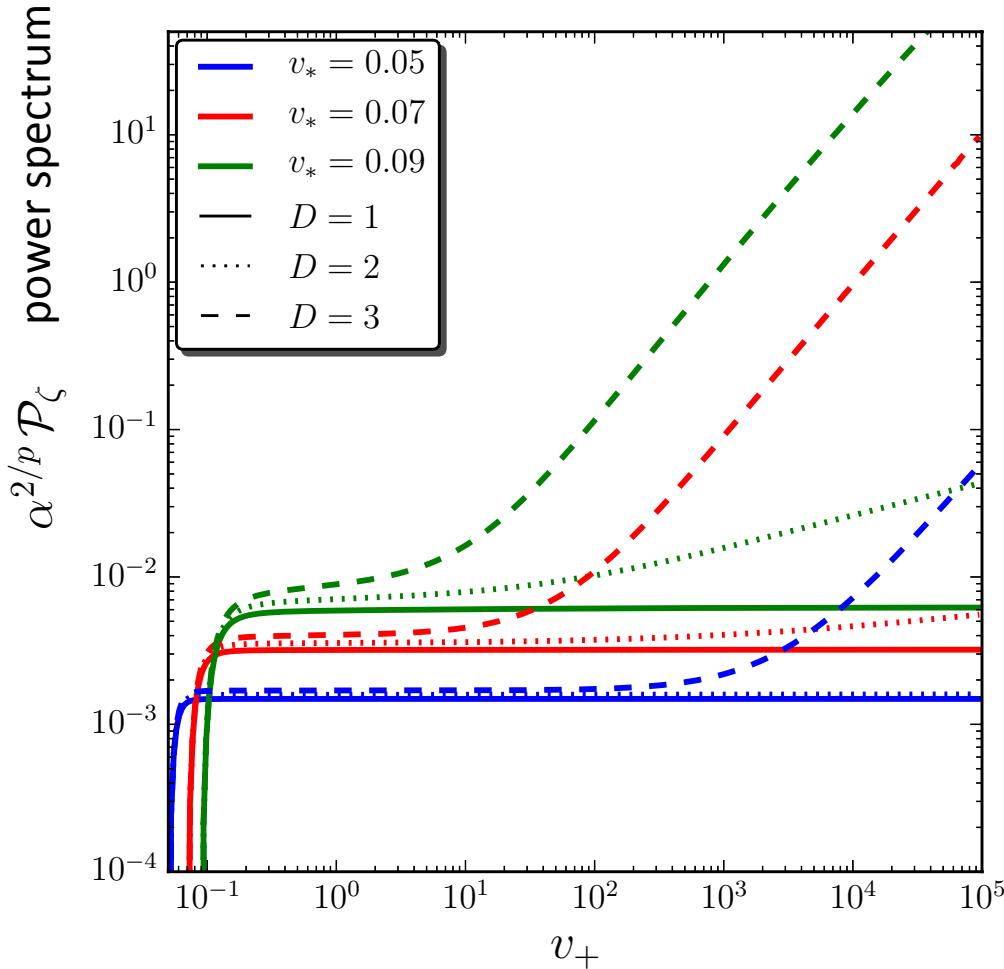
$$\mathcal{P}_\zeta(\phi_*) \simeq \frac{2}{M_{\text{Pl}}^2} \frac{v^3(\phi_*)}{v'^2(\phi_*)} \left[1 + 5v(\phi_*) - 4 \frac{v^2(\phi_*) v''(\phi_*)}{v'^2(\phi_*)} + \dots \right]$$

Classical result

First order correction

Stochastic $\delta\mathcal{N}$

D>2 requires UV regularisation (boundary) at some v_+
 $v \propto r^2$



Monomial Potentials:

$$v_* \ll v_+ \ll e^{\frac{\mathcal{O}(1)}{v_*}}$$

$$D = 2, p = 3 : \\ v_+ \ll 10^{3,474,355,825}$$

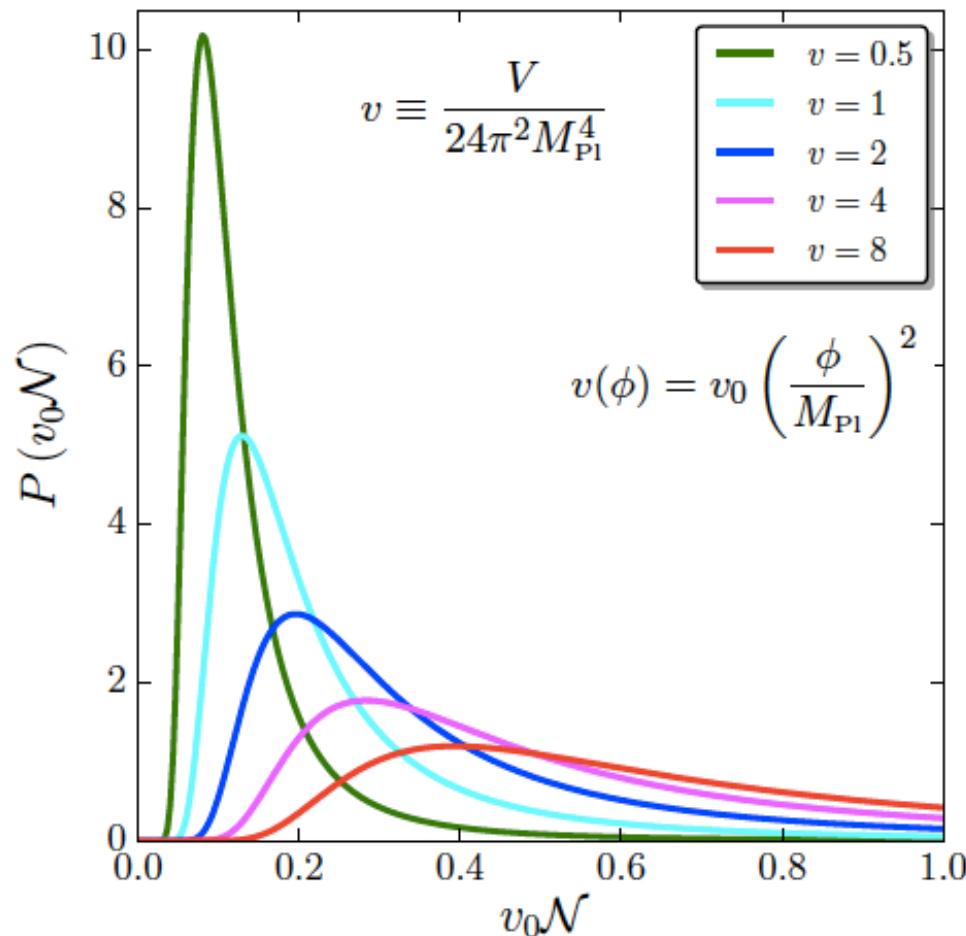
Plateau Potentials

$$r \ll r_+ \ll e^{\mathcal{O}(1)\left(\frac{1}{v_*} - \frac{1}{v_\infty}\right)}$$

$$D = 1, \text{Starobinsky:} \\ r_+ \ll 10^{6,166,453,090} M_{\text{Pl}}$$

Large-field inflation Pattison et al (2017)

- large deviations from classical δN close to Planck energies
 - > non-Gaussian probability distribution



Stochastic δN

- number of e-folds is a stochastic variable, $\mathcal{N}(\phi)$

$$\zeta = \mathcal{N} - \langle \mathcal{N} \rangle$$

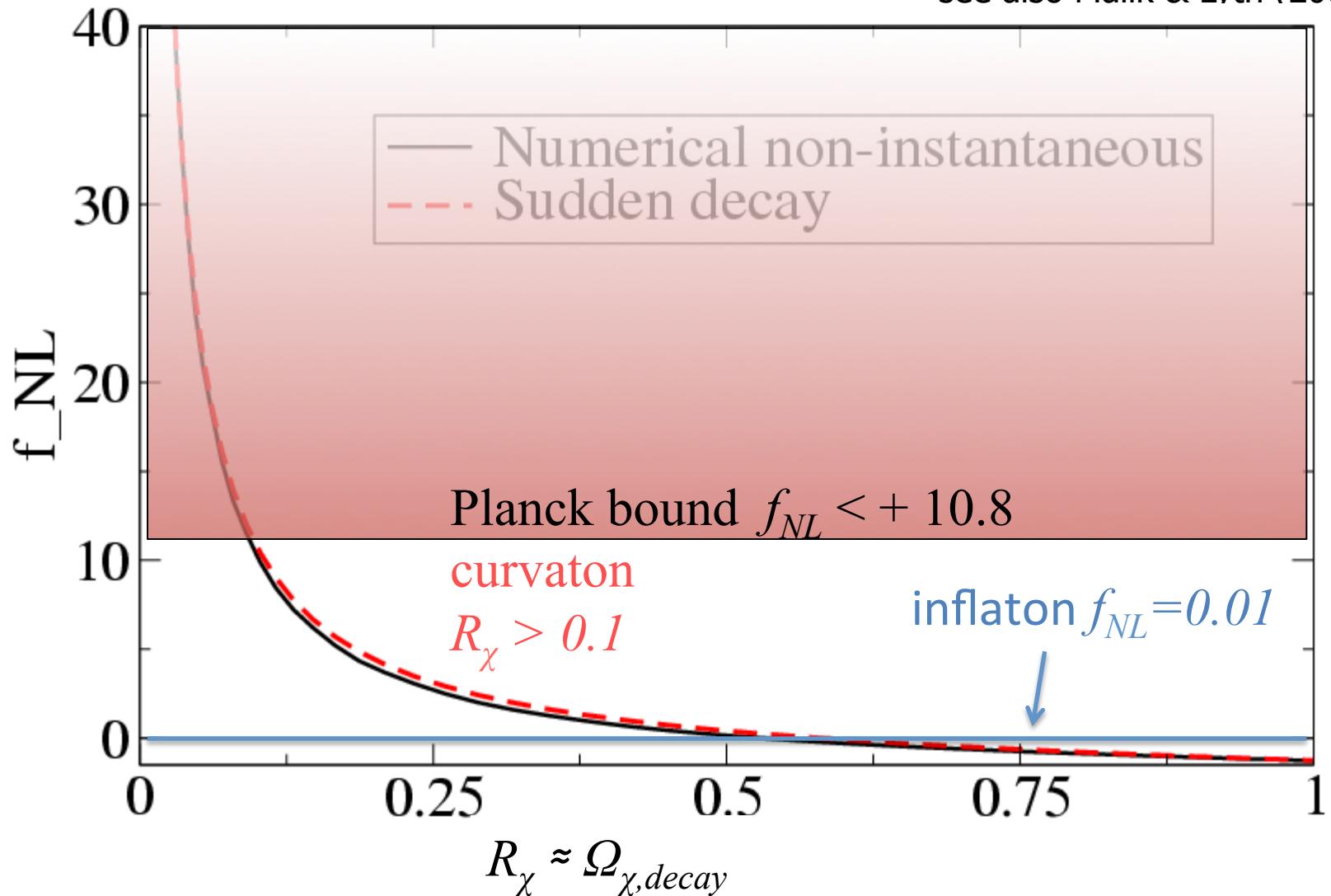
- moments obey an iterative relation (Vennin & Starobinsky 2015)

$$f_n \equiv \langle \mathcal{N}^n \rangle$$
$$\Rightarrow f_n'' - \frac{v'}{v^2} f_n' = -\frac{n}{v M_P^2} f_{n-1}$$

- ...but PBHs require full probability distribution function

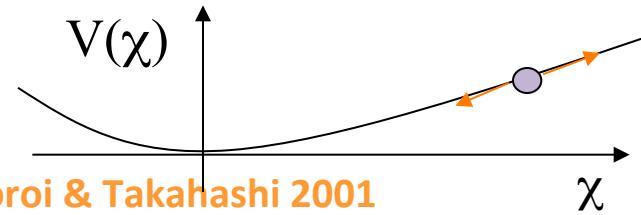
non-linearity parameter for quadratic curvaton

Sasaki, Valiviita & Wands (2006)
see also Malik & Lyth (2006)



curvaton scenario:

Linde & Mukhanov 1997; Enqvist & Sloth, Lyth & Wands, Moroi & Takahashi 2001



curvaton χ = weakly-coupled, late-decaying scalar field

- light field ($m < H$) during inflation acquires an almost scale-invariant, **Gaussian distribution of field fluctuations** on large scales
- **quadratic energy density** for free field, $\rho_\chi = m^2 \chi^2 / 2$
- spectrum of initially isocurvature density perturbations

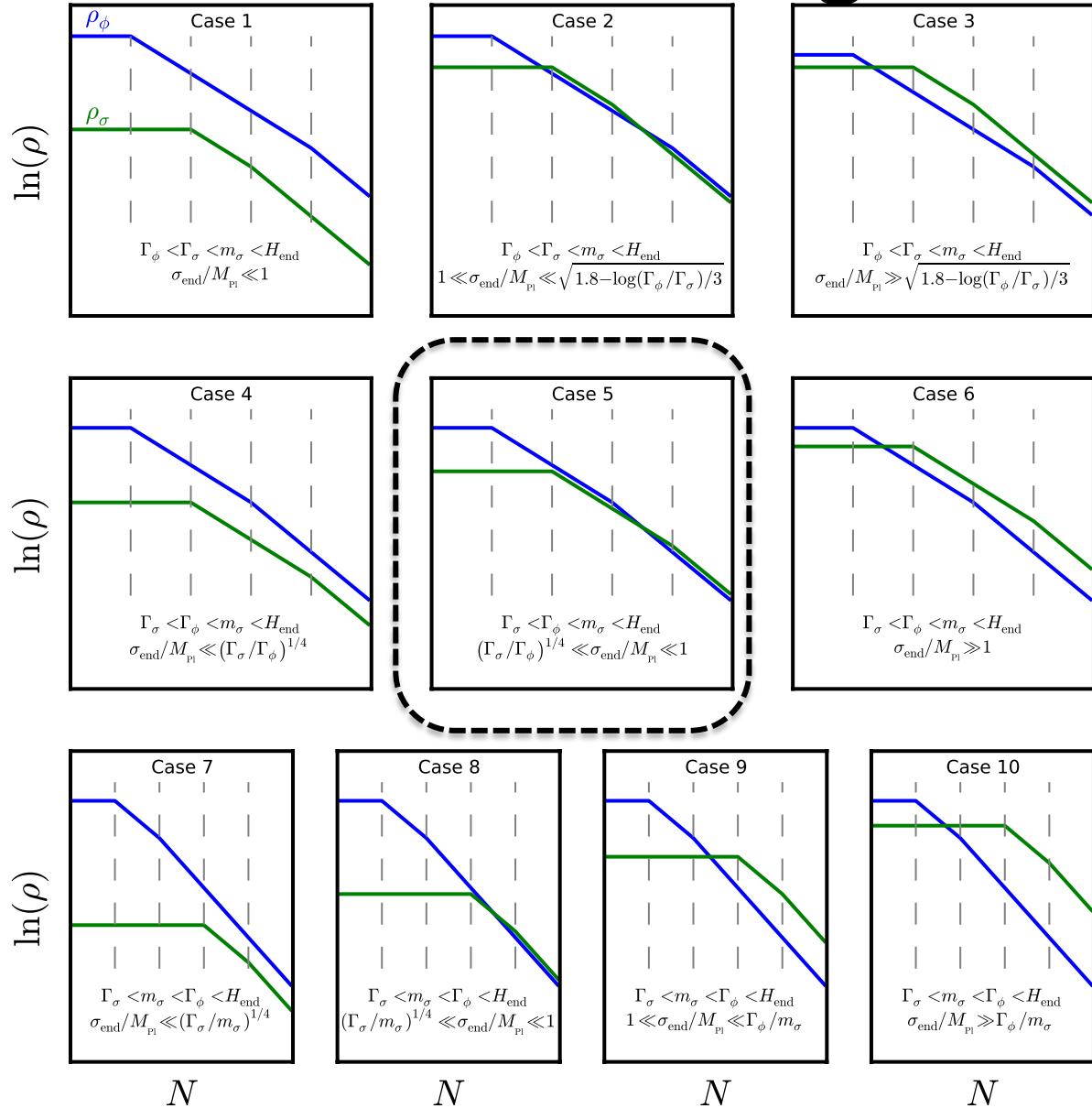
$$\zeta_\chi \approx \frac{1}{3} \frac{\delta \rho_\chi}{\rho_\chi} \approx \frac{1}{3} \left(\frac{2\chi \delta \chi + \delta \chi^2}{\chi^2} \right)$$

- **transferred to radiation when curvaton decays** after inflation with some **efficiency**, $0 < R_\chi < 1$, where $R_\chi \approx \Omega_{\chi, decay}$

$$\zeta = R_\chi \zeta_\chi \approx \frac{R_\chi}{3} \left(2 \frac{\delta \chi}{\chi} + \frac{\delta \chi^2}{\chi^2} \right)$$

$$= \zeta_G + \frac{3}{4R_\chi} \zeta_G^2 \quad \Rightarrow \quad f_{NL} = \frac{5}{4R_\chi}$$

10 curvaton reheating scenarios



Vennin, Koyama and Wands (2015)

after inflaton slow-roll ends

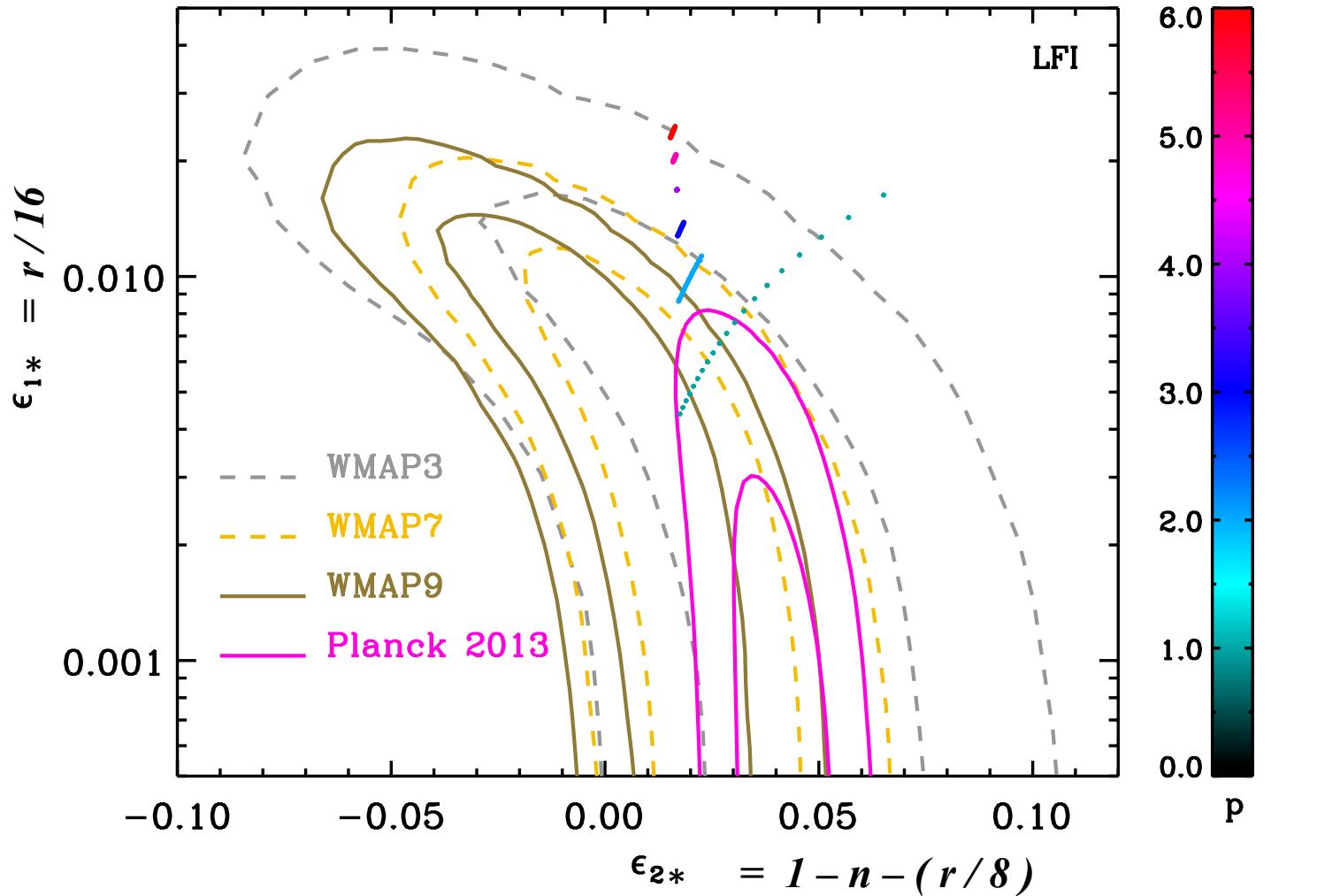
Case 5:

- Curvaton become heavy
- Inflaton decays
- Curvaton dominates
- Curvaton decays

Inflaton model predictions and Observations

Inflaton example: « large field inflation »

$$V(\phi) = M^4 \left(\frac{\phi}{M_{\text{Pl}}} \right)^p$$



LFI+curvaton vs Higgs inflation

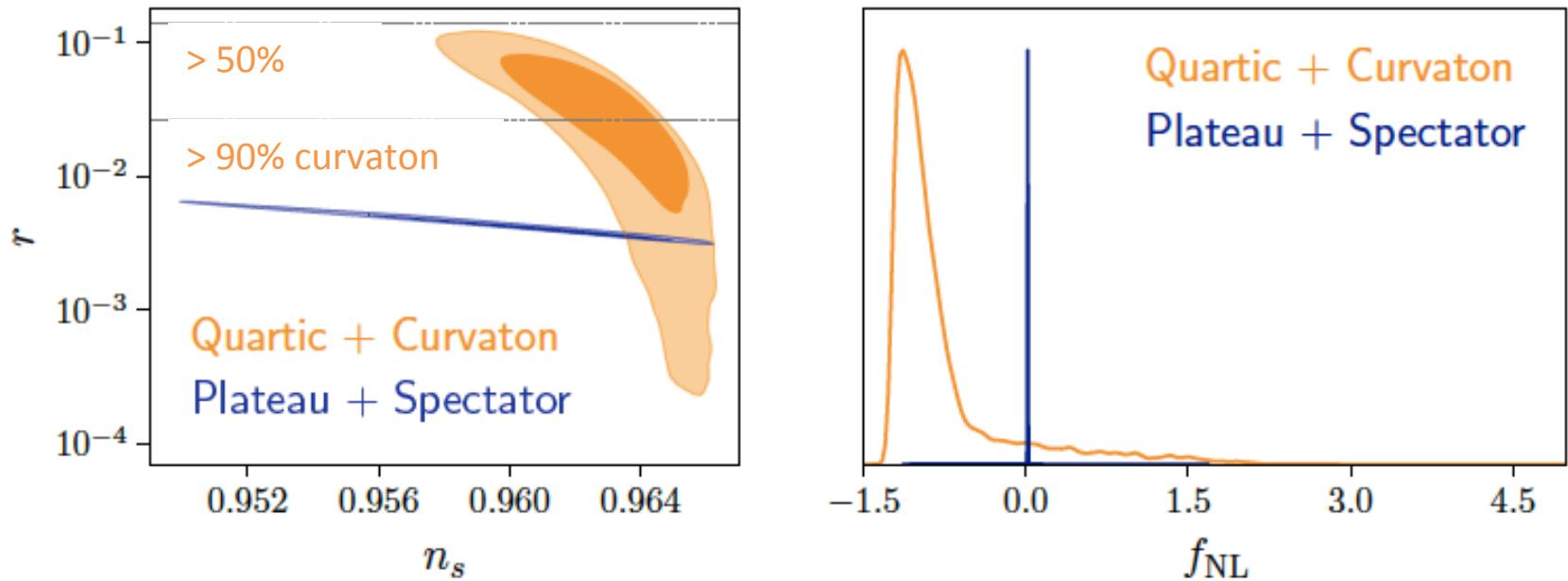


FIG. 2: Marginal posterior distributions over the key observables from inflation for plateau-like inflation (blue, darker) and quartic inflation (orange, clearer) with a spectator field. In the quartic case, the posterior fraction below the lower (upper) dotted line has more than 90% (50%) of primordial density perturbations generated by the curvaton field. Post-2020 CMB experiments would likely distinguish between or rule out both scenarios in terms of n_s and r . In combination with LSS data, the typical value of $f_{\text{NL}} = -5/4$ associated with the curvaton scenario could also be distinguished in the future from $f_{\text{NL}} \sim \mathcal{O}(10^{-2})$ in the inflaton scenario.