Constructive Gravity

A New Approach to Modified Gravity Theories

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Standard approach to modified gravity:

Effective field theory approach: stipulate a modification of the Einstein-Hilbert action.

$\rightarrow$ What about the well-posedness of the initial value problem i.e. predictivity?

New approach discussed here:

Fundamental approach: derive gravity action such that the theory is predictive.

How to implement predictivity on general (e.g. non-metric) backgrounds? How to construct dynamics from kinematics?

$\rightarrow$ 'Constructive gravity' program

[Cf. Hojman, Kuchař & Teitelboim (1976); Rätzel, Rivera & Schuller (2011); Giesel, Schuller, Witte & Wohlfarth (2012); Düll, Schuller, Stritzelberger & Wolz (2017); Schuller & Werner (2017)]

Consider a smooth manifold M with chart (U, x) and some smooth tensor fields G for geometry and F for matter, of arbitrary order.

Spacetime geometry is probed by test matter, with linear field equations. The most general such test matter field PDE in (U, x) is

$$\left[\sum_{d=1}^{k} D_B^{A\mu_1\dots\mu_d}[G]\frac{\partial}{\partial x^{\mu_1}}\cdots\frac{\partial}{\partial x^{\mu_d}}\right] F_A = 0, \quad (*)$$

with some multi-index A, and highest derivative order k, assumed to be finite.

Principal polynomial

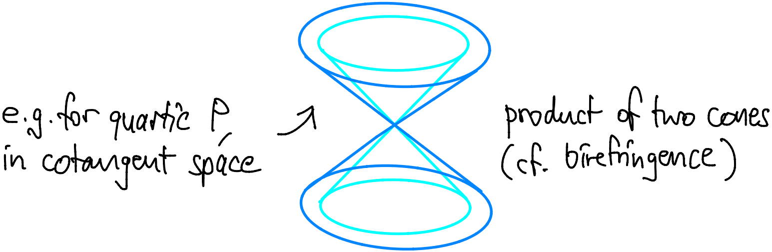
The (reduced) principal polynomial of $(*)$ is $P : T^*M \to \mathbb{R}$,

$$P \propto \det \left[D_B^{A\mu_1 \dots \mu_k}(x) p_{\nu_1} \dots p_{\mu_k} \right] = P^{\nu_1 \dots \nu_{\deg P}} p_{\nu_1} \dots p_{\nu_{\deg P}},$$

with totally symmetric principal polynomial tensor $P^{\nu_1 \dots \nu_{\deg P}}$.

Note: although $(*)$ was written in a chart, P is indeed tensorial.

Then the (generalized) null cone is $\{p \in T_x^*M \;:\; P(p) = 0\}$.



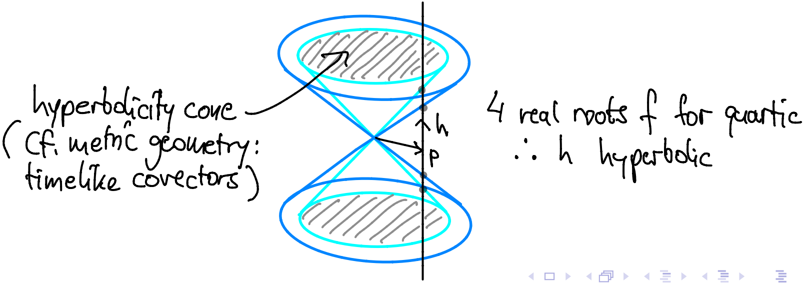
We are interested in causal kinematics of the generalized spacetime (M, G, F), which is determined by the Cauchy problem.

Given $(*)$ and initial data, the Cauchy problem is well-posed if

- $(*)$ has a unique solution in U
- which depends continuously on the initial data.

Then necessarily $(\Rightarrow)$, P is hyperbolic:

$\exists\ h \neq 0$ such that $\forall\ p: \ \ P(p + fh) = 0,$ only for f real.



So far, only covectors (momenta) have been considered. However, for predictivity, we also need time-orientation and hence dual vectors (trajectories). It turns out that:

If P is hyperbolic, then the dual polynomial $P^\sharp : TM \to \mathbb{R}$ exists, via the Gauss map $p \mapsto N$ with $P(p) = 0, P^\sharp(N) = 0$.

Note: hyperbolicity of P does not imply hyperbolicity of $P^\sharp$.

Now introduce a time-orientation vector field $T \in TM$ over U.

Denoting a null vector field by N, $P^\sharp(N) = 0$, then any vector field X can be decomposed as $X = N + tT$, for some $t : U \to \mathbb{R}$.

Thus, we obtain

$$\forall\ X :\ 0 = P^\sharp(N) = P^\sharp(X - tT),\ t \text{ real},$$

in other words, a hyperbolicity condition for $P^\sharp$!

Hence, a predictive kinematics for (M, G, F) implies that

- P be hyperbolic for causality; then also $P^\sharp$ exists;
- $P^\sharp$ be hyperbolic as well, for time-orientation.

This is called bihyperbolicity.

Note: this yields

- an energy-distinguishing property for observers, that is, $p(T) > 0$ or $p(T) < 0\ \forall$ hyperbolic T, and a
- unique Legendre map $\mathscr{L} : T^*M \to TM$ ('pulling indices').

Consider a hypersurface Σ embedded in spacetime, $\sigma : \Sigma \hookrightarrow M$, with 3 tangent (spacetime) vectors $e_i = \sigma^\mu{}_{,i}\partial_\mu$.

The conormal n, satisfying $n(e_i) = 0$, with normalization $P(n) = 1$ gives rise to a unique hypersurface normal vector field

$$T = \mathscr{L}(n).$$

Thus, one obtains a frame field $\{T, e_1, e_2, e_3\}$.

Now writing hypersurface deformations with lapse $\mathcal{N}$ and shift $\boldsymbol{\mathcal{N}} = \mathcal{N}^i \partial_i$

$$\dot{\sigma}^\mu = \mathcal{N} T^\mu + \mathcal{N}^i e^\mu_i,$$

yields a generalized ADM-split.

Now introducing normal and tangential deformation operators,

$$\mathscr{H}(\mathcal{N}) = \int_\Sigma \mathrm{d}^3x\ \mathcal{N}\ \underbrace{T^\mu \frac{\delta}{\delta\sigma^\mu}}_{\hat{\mathscr{H}}}, \quad \mathscr{D}(\boldsymbol{\mathcal{N}}) = \int_\Sigma \mathrm{d}^3x\ \mathcal{N}^i\ \underbrace{e^\mu{}_i \frac{\delta}{\delta\sigma^\mu}}_{\hat{\mathscr{D}}_i},$$

the change of a tensor field is $\dot{F}[\sigma] = (\mathscr{H}(\mathcal{N}) + \mathscr{D}(\boldsymbol{\mathcal{N}}))F[\sigma]$.

The spacetime kinematics is defined by the deformation algebra,

$$\begin{aligned}
[\mathscr{D}(\boldsymbol{\mathcal{N}}), \mathscr{D}(\boldsymbol{\mathcal{N}}')] &= -\mathscr{D}(\pounds_{\boldsymbol{\mathcal{N}}}\boldsymbol{\mathcal{N}}') \\
[\mathscr{D}(\boldsymbol{\mathcal{N}}), \mathscr{H}(\mathcal{N})] &= -\mathscr{H}(\pounds_{\boldsymbol{\mathcal{N}}}\mathcal{N}) \\
[\mathscr{H}(\mathcal{N}), \mathscr{H}(\mathcal{N}')] &= -\mathscr{D}((\deg P - 1)P^{ij}(\mathcal{N}'\partial_j\mathcal{N} - \mathcal{N}\partial_j\mathcal{N}')\partial_i),
\end{aligned}$$

where P^{ij} is constructed from the principal polynomial tensor.

Hypersurface deformation changes G according to

$$\dot{G}^A = \int_\Sigma \mathrm{d}^3x \left(\mathcal{N}\hat{\mathscr{H}} + \mathcal{N}^i\hat{\mathscr{D}}_i\right) G^A = \mathcal{N}K^A + \mathcal{N}_{;i}M^{Ai} + \pounds_{\boldsymbol{\mathcal{N}}}G^A.$$

Passing to canonical variables (G, π), the dynamics $\dot{G} = \{G, H\}$, $\dot{\pi} = \{\pi, H\}$ is obtained from an action of the form

$$S[G, \pi, \mathcal{N}, \mathcal{N}^i] = \int_\mathbb{R} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left(\dot{G}^A\pi_A - H\right),$$

$$\text{with} \quad H = \int_\Sigma \mathrm{d}^3x \left(\mathcal{N}\hat{\mathcal{H}} + \mathcal{N}^i\hat{\mathcal{D}}_i\right),$$

$\hat{\mathcal{H}}$ is called superhamiltonian, and $\hat{\mathcal{D}}$ is called supermomentum.

Now we stipulate that this dynamical hypersurface evolution coincide with the above hypersurface deformation, that is,

$$\mathscr{H} G = \{G, \hat{\mathcal{H}}\}, \qquad \mathscr{D}_i G = \{G, \hat{\mathcal{D}}_i\}.$$

These are called closure conditions. Hence, the kinematical deformation algebra gives rise to a dynamical evolution algebra,

$$\begin{aligned} \{\mathcal{D}(\boldsymbol{\mathcal{N}}), \mathcal{D}(\boldsymbol{\mathcal{N}}')\} &= \mathcal{D}(\pounds_{\boldsymbol{\mathcal{N}}} \boldsymbol{\mathcal{N}}') \\ \{\mathcal{D}(\boldsymbol{\mathcal{N}}), \mathcal{H}(\mathcal{N})\} &= \mathcal{H}(\pounds_{\boldsymbol{\mathcal{N}}} \mathcal{N}) \\ \{\mathcal{H}(\mathcal{N}), \mathcal{H}(\mathcal{N})\} &= \mathcal{D}((\deg P - 1) P^{ij} (N' \partial_j N - N \partial_j N') \partial_i). \end{aligned}$$

Solving these equations would yield the gravitational dynamics.

→ This is actually possible!

Supermomentum and superhamiltonian

The supermomentum obeys a subalgebra and is found explicitly,

$$\hat{\mathcal{D}}(\boldsymbol{\mathcal{N}}) = \int_\Sigma \mathrm{d}^3x \; \pi_A (\pounds_{\boldsymbol{\mathcal{N}}} G)^A.$$

The non-local superhamiltonian part is $\hat{\mathcal{H}}_{\mathrm{non-loc}} = -\partial_i (M^{Ai} \pi_A)$, leaving the local part $\hat{\mathcal{H}}_{\mathrm{loc}}$ such that overall

$$\hat{\mathcal{H}}[G, \pi] = \hat{\mathcal{H}}_{\mathrm{loc}}[G, \pi) + \hat{\mathcal{H}}_{\mathrm{non-loc}}[G, \pi].$$

It defines a canonical velocity of G, $K^A = \frac{\partial \hat{\mathcal{H}}_{\mathrm{loc}}}{\partial \pi_A}$, and a Lagrangian

$$\mathcal{L}[G, K) = \pi_A K^A - \hat{\mathcal{H}}_{\mathrm{loc}},$$

with $\pi_A = \frac{\partial \mathcal{L}}{\partial K^A}$ as required.

Thus one obtains a functional differential equation for the gravity Lagrangian $\mathcal{L}[G, K)$ from the evolution algebra and the closure conditions.

This can be converted to a set of partial differential equations, called the closure equations, with the ansatz

$$\mathcal{L}[G, K) = \sum_{k=0}^{\infty} C[G]_{A_1 \dots A_k} K^{A_1} \dots K^{A_k}.$$

For general G, the result is an infinite set of linear, homogeneous PDEs whose solution, if it exists, is $\mathcal{L}$.

Hence, predictive gravitational dynamics can be derived from the underlying spacetime kinematics.

One of those differential construction equations for the $C[G]_{A_1 \dots A_k}$ of the gravity Lagrangian reads thus,

$$0 = \frac{\partial C}{\partial \left(\frac{\partial^3 G^A}{\partial x^i \partial x^j \partial x^k} \right)} + \frac{\partial C_A}{\partial \left(\frac{\partial^2 G^B}{\partial x^{(i} \partial x^{j|}} \right)} M^{B|k)}.$$

Now suppose that $G = g$, a Lorentzian metric, then $M^{Ai} = 0$ and C can depend on at most second order derivatives of the metric.

The full analysis yields $C = -\frac{1}{2\kappa}\sqrt{-g}(R - 2\Lambda)$ with integration constants κ and Λ, i.e. GR! Cf. also Lovelock's theorem.

Applying this formalism to non-metric spacetime kinematics yields gravitational dynamics beyond GR.
First results obtained for area metric geometry.

Conclusions and outlook

- Predictive spacetime kinematics can be implemented mathematically with bihyperbolicity in general.
- The constructive gravity approach allows the derivation of gravitational dynamics from bihyperbolic kinematics.
- Application to non-metric kinematics yields dynamics beyond GR: the first derived, predictive modified gravity theories.
- There will be a *Constructive Gravity* parallel session at the upcoming Marcel Grossmann meeting in Rome in July.