# Massive and Partially Massless (PM) graviton on curved space-times

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Cédric Deffayet
(IAP and IHÉS, CNRS Paris)

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1. **DOF counting for a Massive graviton (*à la* Fierz-Pauli) on an Einstein spacetime**

2. **Consistent massive graviton on arbitrary spacetime.**

   L. Bernard, C.D., M. von Strauss + A. Schmidt-May (2015-2016, PR**D**, JCAP)

3. **PM graviton on non Einstein spacetimes.**

   L. Bernard, C.D., K. Hinterbichler and M. von Strauss arXiv:1703.02538 (PR**D**)

## 1. DOF counting for a massive graviton on an Einstein space-time (Fierz-Pauli linear theory).

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with

$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) + \frac{m^2}{2} \left( h_{\mu\nu} - g_{\mu\nu} h \right)$$

Kinetic operator

Cosmological constant

Mass term

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Comes from expanding the Einstein-Hilbert action $\int d^4x \sqrt{-g}\,(R - 2\Lambda)$

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Kinetic operator

Cosmological constant

Mass term

Only consistent mass term

$$\left( \text{among} \quad \frac{m^2}{2} \left( h_{\mu\nu} - \# \, g_{\mu\nu} h \right) \right)$$

## DOF counting

The fierz Pauli theory for a massive graviton of mass $m$ propagates

- 2 DOF if $m = 0$

Massless graviton

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- 2 DOF if $m = 0$

Generic massive graviton

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Partially Massless graviton

- 4 DOF if $m^2 = 2\,\Lambda/3$

# How to count DOF ?

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$$E_{\mu\nu} \equiv \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) + \frac{m^2}{2} \left( h_{\mu\nu} - g_{\mu\nu} h \right)$$

Kinetic operator

Cosmological constant

Mass term

Comes from expanding the

Einstein-Hilbert action $\int d^4x \sqrt{-g} \left( R - 2\Lambda \right)$

This implies the (Bianchi) offshell identities

$$\nabla^\mu \left[ \mathcal{D}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \right) \right] = 0$$

Results in the off-shell identity

$$\nabla^\mu E_{\mu\nu} = \frac{m^2}{2}\left(\nabla^\mu h_{\mu\nu} - g^{\rho\sigma}\nabla_\nu h_{\rho\sigma}\right)$$

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$$\nabla^\mu E_{\mu\nu} = \frac{m^2}{2}\left(\nabla^\mu h_{\mu\nu} - g^{\rho\sigma}\nabla_\nu h_{\rho\sigma}\right)$$

$$\left(\text{i.e.}\quad \nabla^\mu E_{\mu\nu} \sim 0\right)$$

NB: in this talk

$$E_{\mu\nu} \simeq 0$$

<u>on shell</u>

$$\nabla^{\mu} E_{\mu\nu} \sim 0$$

Equals <u>off-shell</u> up to undifferentiated or (only) once differentiated $h_{\mu\nu}$
but <u>NOT</u> twice differentiated $h_{\mu\nu}$

Results in an the off-shell identity

$$\nabla^\mu E_{\mu\nu} = \frac{m^2}{2}(\nabla^\mu h_{\mu\nu} - g^{\rho\sigma}\nabla_\nu h_{\rho\sigma})$$

$$\left(\text{i.e.} \quad \nabla^\mu E_{\mu\nu} \sim 0\right)$$

And the on-shell relation

$$\nabla^\mu h_{\mu\nu} - \nabla_\nu h \simeq 0$$

4 vector constraints

Kills 4 out of 10 DOF of $h_{\mu\nu}$

Taking an extra derivative of the field equation operator yields (off shell)

$$\nabla^\mu \nabla^\nu E_{\mu\nu} = \frac{m^2}{2} \left( \nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^2 h \right)$$

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$$\nabla^\mu \nabla^\nu E_{\mu\nu} = \frac{m^2}{2}\left(\nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^2 h\right)$$

While tracing it with the metric gives

$$g^{\mu\nu} E_{\mu\nu} = \nabla^2 h - \nabla^\mu \nabla^\nu h_{\mu\nu} + \left(\Lambda - \frac{3m^2}{2}\right) h$$

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$$g^{\mu\nu} E_{\mu\nu} = \nabla^2 h - \nabla^\mu \nabla^\nu h_{\mu\nu} + \left(\Lambda - \frac{3m^2}{2}\right) h$$

Hence we have the identity

$$2\nabla^\mu \nabla^\nu E_{\mu\nu} + m^2 g^{\mu\nu} E_{\mu\nu} = \frac{m^2}{2}\left(2\Lambda - 3m^2\right) h$$

Yielding on shell $\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$

$$\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$$

Generically: yields $h \simeq 0$

i.e. a “scalar constraint” $\mathcal{C} \equiv h \simeq 0$

reducing from 6 to 5 the

number of propagating DOF

$$\boxed{\left(2\Lambda - 3m^2\right) h \simeq 0}$$

However, if $2\Lambda = 3m^2$

Then this vanishes identically

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However, if $2\Lambda = 3m^2$

Then this vanishes identically

As is $2\nabla^\mu \nabla^\nu E_{\mu\nu} + m^2 g^{\mu\nu} E_{\mu\nu}$

Shows the existence of a gauge symmetry

$$\Delta h_{\mu\nu} = \left(\nabla_\mu \nabla_\nu + \frac{m^2}{2} g_{\mu\nu}\right) \xi(x) = \left(\nabla_\mu \nabla_\nu + \frac{\Lambda}{3} g_{\mu\nu}\right) \xi(x)$$

Hence, if $2\Lambda = 3m^2$

one has 6 - 2 = 4 DOF (and a gauge symmetry)

The massive graviton is
said to be
“Partially massless” (PM)

Two questions:

Can a fully non linear PM theory exist ?

Can a PM graviton exist on non Einstein space-times ?

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Can a PM graviton exist on non Einstein space-times ?

Here we address the second one…

First, we need to introduce the theory of a massive graviton on arbitrary backgrounds

**2. Consistent massive graviton $h_{\mu\nu}$ on an arbitrary background specified by some given (non dynamical) metric $g_{\mu\nu}$**

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$$S^{(2)} = -\frac{1}{2} M_g^2 \int \mathrm{d}^4 x \sqrt{|g|} h_{\mu\nu} \left( \mathcal{E}^{\mu\nu\rho\sigma} + m^2 \mathcal{M}^{\mu\nu\rho\sigma} \right) h_{\rho\sigma}$$

Einstein-Hilbert kinetic operator

Mass term

The theory has been obtained in

L.Bernard, CD, M. von Strauss
1410.8302 + 1504.04382
+ 1512.03620 (with A. Schmidt-May)

out of the dRGT theory

de Rahm, Gabadadze;
de Rham, Gababadze, Tolley, 2010, 2011

Our massive graviton theory is defined by

Our massive graviton theory is defined by

1. A symmetric tensor $S_{\mu\nu}$ obtained from the background curvature solving

$$R^{\mu}{}_{\nu} = m^2 \left[ \left( \beta_0 + \frac{1}{2} e_1 \beta_1 \right) \delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu} \right]$$

with $\beta_0$, $\beta_1$ and $\beta_2$ dimensionless parameters and *m* the graviton mass, $e_n$ the symmetric polynomials

$$\begin{cases} e_0 = 1\,, \\ e_1 = S^{\rho}{}_{\rho}\,, \\ e_2 = \frac{1}{2}\left(S^{\rho}{}_{\rho} S^{\nu}{}_{\nu} - S^{\rho}{}_{\nu} S^{\nu}{}_{\rho}\right)\,, \\ e_3 = \frac{1}{6}\left(S^{\rho}{}_{\rho} S^{\nu}{}_{\nu} S^{\mu}{}_{\mu} - 3 S^{\mu}{}_{\mu} S^{\rho}{}_{\nu} S^{\nu}{}_{\rho} + 2 S^{\rho}{}_{\nu} S^{\nu}{}_{\mu} S^{\mu}{}_{\rho}\right)\,, \\ e_4 = \det(S)\,. \end{cases}$$

# 2. The following (linear) field equations

$$
\begin{aligned}
E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{m^2}{2} \Big[ & 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right)\left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right) \\
& - \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma} \\
& - \left(\beta_1 + \beta_2 e_1\right)\left(g_{\mu\rho} \delta S^{\rho}{}_{\nu} + g_{\nu\rho} \delta S^{\rho}{}_{\mu}\right) \Big] \simeq 0\,,
\end{aligned}
$$

## 2. The following (linear) field equations

$$E_{\mu\nu} \equiv \boxed{\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma}} + \frac{m^2}{2} \Big[ 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right) \left( h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu} \right)$$
$$- \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma}$$
$$- \left(\beta_1 + \beta_2 e_1\right) \left( g_{\mu\rho} \delta S^{\rho}{}_{\nu} + g_{\nu\rho} \delta S^{\rho}{}_{\mu} \right) \Big] \simeq 0 \,,$$

Linearized Einstein operator

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \equiv -\frac{1}{2} \Big[ \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} \nabla^2 + g^{\rho\sigma} \nabla_{\mu} \nabla_{\nu} - \delta^{\rho}_{\mu} \nabla^{\sigma} \nabla_{\nu} - \delta^{\rho}_{\nu} \nabla^{\sigma} \nabla_{\mu} - g_{\mu\nu} g^{\rho\sigma} \nabla^2 + g_{\mu\nu} \nabla^{\rho} \nabla^{\sigma}$$
$$+ \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} R - g_{\mu\nu} R^{\rho\sigma} \Big] h_{\rho\sigma} \,,$$

# 2. The following (linear) field equations

$$
\begin{aligned}
E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{m^2}{2}\Big[ & 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right)\left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right) \\
& - \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma} \\
& - \left(\beta_1 + \beta_2 e_1\right)\left(g_{\mu\rho} \boxed{\delta S^{\rho}{}_{\nu}} + g_{\nu\rho} \delta S^{\rho}{}_{\mu}\right) \Big] \simeq 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\lambda}{}_{\mu} = \frac{1}{2} g^{\nu\lambda} \Big[ & e_4\, c_1 \left(\delta^{\rho}_{\nu}\delta^{\sigma}_{\mu} + \delta^{\sigma}_{\nu}\delta^{\rho}_{\mu} - g_{\mu\nu} g^{\rho\sigma}\right) + e_4\, c_2 \left(S^{\rho}_{\nu}\delta^{\sigma}_{\mu} + S^{\sigma}_{\nu}\delta^{\rho}_{\mu} - S_{\mu\nu} g^{\rho\sigma} - g_{\mu\nu} S^{\rho\sigma}\right) \\
& - e_3\, c_1 \left(\delta^{\rho}_{\nu} S^{\sigma}_{\mu} + \delta^{\sigma}_{\nu} S^{\rho}_{\mu}\right) + \left(e_2\, c_1 - e_4\, c_3 + e_3\, c_2\right) S_{\mu\nu} S^{\rho\sigma} \\
& + e_4\, c_3 \left[\delta^{\sigma}_{\mu}[S^2]^{\rho}_{\nu} + \delta^{\rho}_{\mu}[S^2]^{\sigma}_{\nu} - g^{\rho\sigma}[S^2]_{\mu\nu} + \delta^{\rho}_{\nu}[S^2]^{\sigma}_{\mu} + \delta^{\sigma}_{\nu}[S^2]^{\rho}_{\mu} - g_{\mu\nu}[S^2]^{\rho\sigma}\right] \\
& - e_3\, c_2 \left(S^{\rho}_{\nu} S^{\sigma}_{\mu} + S^{\sigma}_{\nu} S^{\rho}_{\mu}\right) - e_3\, c_3 \left(S^{\sigma}_{\mu}[S^2]^{\rho}_{\nu} + S^{\rho}_{\mu}[S^2]^{\sigma}_{\nu} + S^{\rho}_{\nu}[S^2]^{\sigma}_{\mu} + S^{\sigma}_{\nu}[S^2]^{\rho}_{\mu}\right) \\
& + \left(e_3\, c_3 - e_1\, c_1\right) \left(S^{\rho\sigma}[S^2]_{\mu\nu} + S_{\mu\nu}[S^2]^{\rho\sigma}\right) - \left(c_1 - e_2\, c_3\right) \left([S^2]^{\rho}_{\nu}[S^2]^{\sigma}_{\mu} + [S^2]^{\sigma}_{\nu}[S^2]^{\rho}_{\mu}\right) \\
& + c_4\, [S^2]_{\mu\nu}[S^2]^{\rho\sigma} + c_1 \left([S^3]_{\mu\nu} S^{\rho\sigma} + S_{\mu\nu}[S^3]^{\rho\sigma}\right) + c_2 \left([S^3]_{\mu\nu}[S^2]^{\rho\sigma} + [S^2]_{\mu\nu}[S^3]^{\rho\sigma}\right) \\
& + c_3\, [S^3]_{\mu\nu}[S^3]^{\rho\sigma} \Big] h_{\rho\sigma}\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta S^{\lambda}{}_{\mu} = \frac{1}{2}\, g^{\nu\lambda} \Big[\, & e_4\, c_1 \left(\delta^{\rho}_{\nu}\delta^{\sigma}_{\mu} + \delta^{\sigma}_{\nu}\delta^{\rho}_{\mu} - g_{\mu\nu} g^{\rho\sigma}\right) + e_4\, c_2 \left(S^{\rho}_{\nu}\delta^{\sigma}_{\mu} + S^{\sigma}_{\nu}\delta^{\rho}_{\mu} - S_{\mu\nu} g^{\rho\sigma} - g_{\mu\nu} S^{\rho\sigma}\right) \\
& - e_3\, c_1 \left(\delta^{\rho}_{\nu} S^{\sigma}_{\mu} + \delta^{\sigma}_{\nu} S^{\rho}_{\mu}\right) + \left(e_2\, c_1 - e_4\, c_3 + e_3\, c_2\right) S_{\mu\nu} S^{\rho\sigma} \\
& + e_4\, c_3 \left[\delta^{\sigma}_{\mu} [S^2]^{\rho}_{\nu} + \delta^{\rho}_{\mu} [S^2]^{\sigma}_{\nu} - g^{\rho\sigma} [S^2]_{\mu\nu} + \delta^{\rho}_{\nu} [S^2]^{\sigma}_{\mu} + \delta^{\sigma}_{\nu} [S^2]^{\rho}_{\mu} - g_{\mu\nu} [S^2]^{\rho\sigma}\right] \\
& - e_3\, c_2 \left(S^{\rho}_{\nu} S^{\sigma}_{\mu} + S^{\sigma}_{\nu} S^{\rho}_{\mu}\right) - e_3\, c_3 \left(S^{\sigma}_{\mu} [S^2]^{\rho}_{\nu} + S^{\rho}_{\mu} [S^2]^{\sigma}_{\nu} + S^{\rho}_{\nu} [S^2]^{\sigma}_{\mu} + S^{\sigma}_{\nu} [S^2]^{\rho}_{\mu}\right) \\
& + \left(e_3\, c_3 - e_1\, c_1\right) \left(S^{\rho\sigma} [S^2]_{\mu\nu} + S_{\mu\nu} [S^2]^{\rho\sigma}\right) - \left(c_1 - e_2\, c_3\right) \left([S^2]^{\rho}_{\nu} [S^2]^{\sigma}_{\mu} + [S^2]^{\sigma}_{\nu} [S^2]^{\rho}_{\mu}\right) \\
& + c_4\, [S^2]_{\mu\nu} [S^2]^{\rho\sigma} + c_1 \left([S^3]_{\mu\nu} S^{\rho\sigma} + S_{\mu\nu} [S^3]^{\rho\sigma}\right) + c_2 \left([S^3]_{\mu\nu} [S^2]^{\rho\sigma} + [S^2]_{\mu\nu} [S^3]^{\rho\sigma}\right) \\
& + c_3\, [S^3]_{\mu\nu} [S^3]^{\rho\sigma} \,\Big]\, h_{\rho\sigma}\,,
\end{aligned}
$$

$$
\text{With}\quad \left\{
\begin{aligned}
c_1 &= \frac{e_3 - e_1 e_2}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}\,, \quad c_2 = \frac{e_1^2}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}\,, \\
c_3 &= \frac{-e_1}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}\,, \quad c_4 = \frac{e_3 - e_1^3}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}\,.
\end{aligned}
\right.
$$

# How we got it out of dRGT theory ?

$$S_{g,m} = M_g^2 \int \mathrm{d}^4x \sqrt{|g|} \left[ R(g) - 2m^2 V\left(S;\beta_n\right)\right]$$

Uses **two** metrics:
$g$ and $f$

Where $V(S;\beta_n) = \sum_{n=0}^{3} \beta_n e_n(S)$

with $$S^{\mu}{}_{\sigma} S^{\sigma}{}_{\nu} = g^{\mu\sigma} f_{\sigma\nu} = \mathfrak{F}^{\mu}{}_{\nu}$$

Has been shown to propagate 5 DOF in a fully non linear way (dRGT, Hassan, Rosen )…

… evading Boulware-Deser no-go « theorem »

Idea: expand dRGT theory around

arbitrary backgrounds

However $$S^{\mu}{}_{\sigma} S^{\sigma}{}_{\nu} = g^{\mu\sigma} f_{\sigma\nu} = \mathfrak{F}^{\mu}{}_{\nu}$$

- The « square root » tensor $S^{\mu}{}_{\nu}$ makes things unpleasant !
- Two metrics in the game !

**Linearize around some background geometry (e.g. here for the « $\beta_1$ » models)**

$$\mathcal{G}_{\mu\nu} + m^2 \left[\beta_0\, g_{\mu\nu} + \beta_1\, g_{\mu\rho}\left(e_1(S)\delta^\rho_\nu - S^\rho{}_\nu\right)\right] = 0$$

Expa nd

arbitr

Easy to get in terms of the linear perturbation $h_{\mu\nu}$ of $g_{\mu\nu}$

A technical difficulty:

expand the matrix squa oot $S^\mu{}_\nu$

from $$S^\mu{}_\sigma S^\sigma{}_\nu = g^{\mu\sigma} f_{\sigma\nu} = \mathfrak{F}^\mu{}_\nu$$

One has $$\underbrace{S^\mu{}_\nu\,(\delta S)^\nu{}_\sigma + (\delta S)^\mu{}_\nu\, S^\nu{}_\sigma = \delta\mathfrak{F}^\mu{}_\sigma}$$

Sylvester (Matrix) equation (A X + X B = C)

Solving the Sylvester equation

(which is possible iff the spectra of $S$ and -$S$ do not intersect)

we get

$$\delta S = \mathbb{X}^{-1}\Big[\delta S^2[S^2 - e_1 S] + [S^2 + e_2 \mathbb{1}]\delta S^2 - \frac{1}{2}\sum_{m=1}^{4}\sum_{k=1}^{m}(-1)^{m+k} e_{m-k} S^{4-m}[S^{k-2}\delta S^2]\Big]$$

with

$$\begin{cases} \mathbb{X} \equiv e_3 \mathbb{1} + e_1 S^2 \\ \mathbb{X}^{-1} = -\dfrac{1}{e_4(\mathbb{X})}\displaystyle\sum_{n=0}^{3}(-1)^n \mathbb{X}^{3-n} e_n(\mathbb{X}) \end{cases}$$

One thus obtains the linearized field equations

$$\delta E_{\mu\nu} = \delta \mathcal{G}_{\mu\nu} + m^2\, \mathcal{M}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma}$$

Linearized Einstein tensor

(curvature dependent) Mass term

I.e. the field equations shown previously (where previous $E_{\mu\nu}$ is denoted above as $\delta E_{\mu\nu}$):

$$\begin{aligned} E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} + \frac{m^2}{2}\Big[ & 2\left(\beta_0 + \beta_1 e_1 + \beta_2 e_2\right) h_{\mu\nu} - \left(\beta_1 + \beta_2 e_1\right)\left(h_{\mu\rho} S^{\rho}{}_{\nu} + h_{\nu\rho} S^{\rho}{}_{\mu}\right) \\ & - \left(\beta_1 g_{\mu\nu} + \beta_2 e_1 g_{\mu\nu} - \beta_2 S_{\mu\nu}\right) h_{\rho\sigma} S^{\rho\sigma} + \beta_2 g_{\mu\nu} h_{\rho\sigma} (S^2)^{\rho\sigma} \\ & - \left(\beta_1 + \beta_2 e_1\right)\left(g_{\mu\rho}\delta S^{\rho}{}_{\nu} + g_{\nu\rho}\delta S^{\rho}{}_{\mu}\right)\Big] \simeq 0\,, \end{aligned}$$

In these field equations …

- Using the background equations of motion:

$$R^{\mu}{}_{\nu} = m^2 \left[ \left( \beta_0 + \frac{1}{2} e_1 \beta_1 \right) \delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu} \right]$$

We **got rid** of the second metric $f_{\mu\nu}$ and …

**… expressed everything in terms of $g_{\mu\nu}$ and its curvature**

In these field equations …

- Expressions have been simplified using Cayley-Hamilton theorem stating that $S^{\mu}{}_{\nu}$ obeys

$$S^4 - e_1 S^3 + e_2 S^2 - e_3 S + e_4 \mathbb{1} = 0$$

4th power of *S* in the matricial sense

Allowing to replace any power of S

$$\left[S^i\right]^{\mu}{}_{\nu} = S^{\mu}{}_{\sigma_1} S^{\sigma_1}{}_{\sigma_2} \cdots S^{\sigma_{i-1}}{}_{\nu}$$

with $i \geq 4$ by linear combinations of lower powers of S

This provides consistent (as we now show) field equations (and action) for

a **massive graviton** on a background simply defined by just one **arbitrary metric** $g_{\mu\nu}$

NB: We cheked that (in the special cases of diagonal metrics g and f) our equations (before getting rid of the non dynamical metric f) match those obtained by Guarato & Durrer 2014 (which are not fully general and not explicitly covariant)

## DOF counting (consistency check)

Start from the (linear) field equations

$$E_{\mu\nu} = \delta\mathcal{G}_{\mu\nu} + m^2\,\mathcal{M}_{\mu\nu}{}^{\rho\sigma}\,h_{\rho\sigma}$$

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Start from the (linear) field equations

$$E_{\mu\nu} = \delta\mathcal{G}_{\mu\nu} + m^2\,\mathcal{M}_{\mu\nu}{}^{\rho\sigma}\,h_{\rho\sigma}$$

The linearized Bianchi identities $\delta\left(\nabla^\mu\,\mathcal{G}_{\mu\nu}\right) = 0$

Reading $\nabla^\mu\delta\mathcal{G}_{\mu\nu} \sim 0$ i.e.

$$\nabla^\mu\,\delta\mathcal{G}_{\mu\nu} - h^{\mu\rho}\,\nabla_\rho\,\mathcal{G}_{\mu\nu} - \mathcal{G}_{\sigma\nu}\,\nabla_\rho\,h^{\sigma\rho} + \frac{1}{2}\,\mathcal{G}_{\sigma\nu}\,\nabla^\sigma\,h - \frac{1}{2}\,\mathcal{G}_{\mu\sigma}\,\nabla_\nu\,h^{\mu\sigma} = 0$$

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Start from the (linear) field equations

$$E_{\mu\nu} = \delta \mathcal{G}_{\mu\nu} + m^2 \, \mathcal{M}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma}$$

The linearized Bianchi identities $\delta \left( \nabla^\mu \mathcal{G}_{\mu\nu} \right) = 0$

$$\left( \begin{array}{l} \text{Reading} \quad \nabla^\mu \delta \mathcal{G}_{\mu\nu} \sim 0 \quad \text{i.e.} \\ \nabla^\mu \delta \mathcal{G}_{\mu\nu} - h^{\mu\rho} \nabla_\rho \mathcal{G}_{\mu\nu} - \mathcal{G}_{\sigma\nu} \nabla_\rho h^{\sigma\rho} + \frac{1}{2} \mathcal{G}_{\sigma\nu} \nabla^\sigma h - \frac{1}{2} \mathcal{G}_{\mu\sigma} \nabla_\nu h^{\mu\sigma} = 0 \end{array} \right)$$

Yield the (off-shell) identities $\nabla^\mu E_{\mu\nu} \sim 0$

## DOF counting (consistency check)

Start from the (linear) field equations

$$E_{\mu\nu} = \delta\mathcal{G}_{\mu\nu} + m^2\,\mathcal{M}_{\mu\nu}{}^{\rho\sigma}\,h_{\rho\sigma}$$

The linearized Bianchi identities $\delta\left(\nabla^\mu\,\mathcal{G}_{\mu\nu}\right) = 0$

Reading $\nabla^\mu \delta\mathcal{G}_{\mu\nu} \sim 0$ i.e.

$$\nabla^\mu\,\delta\mathcal{G}_{\mu\nu} - h^{\mu\rho}\,\nabla_\rho\,\mathcal{G}_{\mu\nu} - \mathcal{G}_{\sigma\nu}\,\nabla_\rho\,h^{\sigma\rho} + \frac{1}{2}\mathcal{G}_{\sigma\nu}\,\nabla^\sigma\,h - \frac{1}{2}\,\mathcal{G}_{\mu\sigma}\,\nabla_\nu\,h^{\mu\sigma} = 0$$

Yield the (off-shell) identities $\nabla^\mu E_{\mu\nu} \sim 0$

Yielding the **four on shell vector constraints**

$$\nabla^\mu E_{\mu\nu} \simeq 0$$

Can one get an **extra scalar constraint** in a way analogous to Fierz-Pauli on Einstein space-time ?

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However, we can trace both with the metric $g^{\mu\nu}$ and with its Ricci curvature $R^{\mu\nu}$ (or equivalently $S^{\mu\nu}$)

So we look for linear combinations of the scalars

$$\begin{cases} \Phi_i \equiv \left[S^i\right]^{\mu\nu} E_{\mu\nu} \\ \Psi_i \equiv \dfrac{1}{2}\left[S^i\right]^{\mu\nu} \nabla_\nu \nabla^\lambda E_{\lambda\mu} \end{cases}$$

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Thanks again to Cayley Hamilton theorem, only 8 of them are independent …

… I.e. we look for 4 $u_i$ and 4 $v_i$ such that

$$\sum_{i=0}^{3} \left(u_i \, \Phi_i + v_i \, \Psi_i\right) \sim 0$$

After some algebra we find

$$\sum_{i=0}^{3} (u_i \, \Phi_i + v_i \, \Psi_i) \sim \sum_{i=0}^{26} \alpha_i \aleph_i$$

$\aleph_i$: 26 « irreducible » scalars made by contracting $\nabla_\mu \nabla_\nu h_{\rho\sigma}$ with powers of $S$, e.g.

$$\left\{ \begin{array}{ll} \Box\, h \,, & \nabla_\rho \nabla_\sigma \, h^{\rho\sigma} \,, \\ S^{\rho\sigma} \, \nabla_\rho \nabla_\sigma \, h \,, & S^{\rho\sigma} \, \nabla_\rho \nabla_\lambda \, h^\lambda{}_\sigma \,, \\ [S^2]^{\rho\sigma} \, \nabla_\rho \nabla_\sigma \, h \,, & [S^2]^{\rho\sigma} \, \nabla_\rho \nabla_\lambda \, h^\lambda{}_\sigma \,, \\ S^{\rho\sigma} \, S^{\mu\nu} \, \nabla_\rho \nabla_\sigma \, h_{\mu\nu} \,, & S^{\rho\sigma} \, S^{\mu\nu} \, \nabla_\rho \nabla_\mu \, h_{\sigma\nu} \,, \\ [S^3]^{\rho\sigma} \, \nabla_\rho \nabla_\sigma \, h \,, & [S^3]^{\rho\sigma} \, \nabla_\rho \nabla_\lambda \, h^\lambda{}_\sigma \quad \ldots \end{array} \right.$$

After some algebra we find

$$\sum_{i=0}^{3} \left( u_i \, \Phi_i + v_i \, \Psi_i \right) \sim \sum_{i=0}^{26} \alpha_i \aleph_i$$

$\alpha_i$

Are scalar functions of $u_i \, , v_i \, , S_{\mu\nu} \, , g_{\mu\nu}$

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$$\left( \begin{aligned} \alpha_1 = \quad & - u_0 - \tfrac{1}{2} e_1 \, u_1 - \tfrac{1}{2} \left( e_1^2 - 2 \, e_2 \right) u_2 - \tfrac{1}{2} \left( e_1^3 - 3 e_1 e_2 + 3 e_3 \right) u_3 \\ & + \frac{m^2 \beta_1}{2} \left[ - c_1 e_4 \, v_0 + c_3 e_4^2 \, v_2 - \left( c_1 e_3 e_4 + c_2 e_4^2 \right) v_3 \right] \end{aligned} \right)$$

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That should all vanish …

….i.e. 26 equations

for the 8 unknown $u_i, v_i$

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Are scalar functions of $u_i\,,\, v_i\,,\, S_{\mu\nu}\,,\, g_{\mu\nu}$

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That should all vanish …

….i.e. 26 equations
for the 8 unknown $u_i, v_i$

However !

The $\aleph_i$ scalars are not all independent from each other thanks to identities (Syzygies) …

… that we derived using again the Cayley-Hamilton Theorem (or the « second fundamental theorem » of the theory of invariants) i.e. we define a matrix *M* by

$$M = x_0 A + x_1 B + x_2 C + x_3 D$$

With
$$\begin{cases} A^{\mu}_{\ \nu} = h^{\mu}_{\ \nu} & C^{\mu}_{\ \nu} = \left[S^2\right]^{\mu}_{\ \nu} \\ B^{\mu}_{\ \nu} = S^{\mu}_{\ \nu} & D^{\mu}_{\ \nu} = \left[S^3\right]^{\mu}_{\ \nu} \end{cases}$$

i.e. we define a matrix *M* by

$$M = x_0 A + x_1 B + x_2 C + x_3 D$$

With $\left\{ \begin{array}{ll} A^\mu_\nu = h^\mu_\nu & C^\mu_\nu = [S^2]^\mu_\nu \\ B^\mu_\nu = S^\mu_\nu & D^\mu_\nu = [S^3]^\mu_\nu \end{array} \right.$

Then each term with a given homogeneity in the $\{x_i\}$

In the Cayley Hamilton identity:

$$M^4 = e_1(M)M^3 - e_2(M)M^2 + e_3(M)M - e_4(M)\mathbb{1}$$

Yields some identity betwen the matrices A,B,C,D

Picking out the identities linear in $h_{\mu\nu}$ and acting on it with two derivatives we get non trivial identities between the $\aleph_i$

e.g.

$$A_1 - A_2 - e_2\left(D_1 - D_2\right) - e_3\left(2F_1 - F_2 - F_3\right) - \left(e_4 - e_1 e_3\right)\left(G_1 - G_2\right)$$
$$- \left(e_4 - e_1 e_3\right)\left(2H_1 - H_2 - H_3\right) - e_2 e_3\left(2I_1 - I_2 - I_3\right) + e_3^2\left(J_1 - J_2\right) \sim 0$$

with

$$A_1 = [S^3]^{\rho\sigma}\,[S^3]^{\mu\nu}\,\nabla_\rho \nabla_\mu\, h_{\sigma\nu}$$
$$A_2 = [S^3]^{\rho\sigma}\,[S^3]^{\mu\nu}\,\nabla_\rho \nabla_\sigma\, h_{\mu\nu}$$
$$D_1 = [S^2]^{\rho\sigma}\,[S^2]^{\mu\nu}\,\nabla_\rho \nabla_\mu\, h_{\sigma\nu}$$
$$\vdots$$

At the end of the day, demanding that

$$\sum_{i=0}^{3} \left( u_i \, \Phi_i + v_i \, \Psi_i \right) \sim 0$$

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We get a unique solution for $u_i$ and $v_i$ yielding the off-shell identity

$$\frac{m^2 \beta_1}{4} e_4 \Phi_0 + e_3 \Psi_0 - e_2 \Psi_1 + e_1 \Psi_2 - \Psi_3 \sim 0$$

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$$\frac{m^2 \beta_1}{4} e_4 \Phi_0 + e_3 \Psi_0 - e_2 \Psi_1 + e_1 \Psi_2 - \Psi_3 \sim 0$$

and the (on-shell scalar) constraint

$$\boxed{\frac{m^2 \beta_1}{4} e_4 \Phi_0 + e_3 \Psi_0 - e_2 \Psi_1 + e_1 \Psi_2 - \Psi_3 \simeq 0}$$

## To conclude part 2.

- We have obtained a theory for a massive graviton with 5 (or less) d.o.f. on an arbitrary background

  (agrees at first order in curvature with Buchbinder, Gitman, Krykhtin 1999)

- Can be used independently of dRGT as a well behaved theory of a massive graviton in just one background metric

- Various applications, e.g. : the correct theory of a massive graviton in FLRW space time

  (does not agree with Guarato-Durrer 2014)

- See also the recent Mazuet-Volkov (2017) for the same game played with vierbeins

## 3. PM graviton on non Einstein space-times ?

We use the theory introduced in part 2.

Hence we look for (non Einstein) space-times where the scalar constraint (here written with $\beta_1$ and $\beta_2$ non vanishing)

$$\mathcal{C} \equiv (S^{-1})^{\nu}{}_{\rho} \nabla^{\rho} \nabla^{\mu} E_{\mu\nu} + \frac{m^2 \beta_1}{2} g^{\mu\nu} E_{\mu\nu} + m^2 \beta_2 S^{\mu\nu} E_{\mu\nu} \simeq 0$$

Identically vanishes

Yielding the gauge symmetry $h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu}$ with

$$\Delta h_{\mu\nu} = \left[ (S^{-1})_{\mu}{}^{\rho} \nabla_{\rho} \nabla_{\nu} + (S^{-1})_{\nu}{}^{\rho} \nabla_{\rho} \nabla_{\mu} + m^2 \beta_1 g_{\mu\nu} + 2 m^2 \beta_2 S_{\mu\nu} \right] \xi(x)$$

Hence we look for (non Einstein) space-times where the scalar constraint (here written with $\beta_1$ and $\beta_2$ non vanishing)

$$\mathcal{C} \equiv (S^{-1})^{\nu}{}_{\rho} \nabla^{\rho} \nabla^{\mu} E_{\mu\nu} + \frac{m^2 \beta_1}{2} g^{\mu\nu} E_{\mu\nu} + m^2 \beta_2 S^{\mu\nu} E_{\mu\nu} \simeq 0$$

Identically vanishes

Yielding the gauge symmetry $h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu}$ with

$$\Delta h_{\mu\nu} = \left[ (S^{-1})_{\mu}{}^{\rho} \nabla_{\rho} \nabla_{\nu} + (S^{-1})_{\nu}{}^{\rho} \nabla_{\rho} \nabla_{\mu} + m^2 \beta_1 g_{\mu\nu} + 2m^2 \beta_2 S_{\mu\nu} \right] \xi(x)$$

We need to look in detail at the structure of the constraint

The scalar constraint reads

$$\mathcal{C} = m^2 \left[ \left( A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \right) \tilde{h}_{\beta\lambda} + B_\rho^{\beta\lambda} \, \nabla^\rho \tilde{h}_{\beta\lambda} \right]$$

With $h_{\mu\nu} = \tilde{h}_{\mu\lambda} S^\lambda_{\ \nu} + S_\mu^{\ \lambda} \tilde{h}_{\lambda\nu}$

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With $h_{\mu\nu} = \tilde{h}_{\mu\lambda} S^{\lambda}_{\ \nu} + S_{\mu}^{\ \lambda} \tilde{h}_{\lambda\nu}$

$$A^{\beta\lambda} \equiv m^2 \, S^{\beta}_{\ \rho} \Big[ \left( \beta_0\beta_1 + \beta_0\beta_2 e_1 + \frac{1}{2}\beta_1^2 e_1 \right) g^{\rho\lambda} + \left( -2\beta_0\beta_2 - \frac{1}{2}\beta_1^2 - 2\beta_2^2 e_2 + \beta_2^2 e_1^2 \right) S^{\rho\lambda} - \beta_2^2 e_1 [S^2]^{\rho\lambda} \Big] ,$$

The scalar constraint reads

$$\mathcal{C} = m^2 \left[ \left( A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \right) \tilde{h}_{\beta\lambda} + B_{\rho}^{\beta\lambda} \nabla^{\rho} \tilde{h}_{\beta\lambda} \right]$$

With $h_{\mu\nu} = \tilde{h}_{\mu\lambda} S^{\lambda}_{\ \nu} + S_{\mu}^{\ \lambda} \tilde{h}_{\lambda\nu}$

$$\begin{aligned}
\tilde{A}^{\beta\lambda} \equiv \frac{1}{2} (\beta_1 + \beta_2 e_1) [S^{-1}]^{\nu}_{\gamma} \Big[ & -\nabla^{\gamma} S^{\rho\lambda} \nabla_{\nu} S^{\beta}_{\rho} + \nabla^{\gamma} S^{\beta}_{\rho} \nabla^{\lambda} S^{\rho}_{\nu} + \nabla^{\gamma} S^{\rho}_{\nu} \nabla^{\lambda} S^{\beta}_{\rho} - \nabla^{\gamma} S_{\rho\nu} \nabla^{\rho} S^{\beta\lambda} \\
& - S^{\rho\lambda} \nabla^{\gamma} \nabla_{\nu} S^{\beta}_{\rho} + S^{\beta}_{\rho} \nabla^{\gamma} \nabla^{\lambda} S^{\rho}_{\nu} \Big] + \beta_2 [S^{-1}]^{\nu}_{\ \gamma} \Big[ S^{\beta}_{\rho} \nabla^{\lambda} S^{\rho}_{\nu} \nabla^{\gamma} e_1 - S^{\beta}_{\rho} \nabla_{\nu} S^{\rho\lambda} \nabla^{\gamma} e_1 \\
& + S^{\lambda}_{\rho} \nabla^{\gamma} S^{\beta}_{\mu} \nabla_{\nu} S^{\rho\mu} + S^{\beta}_{\mu} \nabla^{\gamma} S^{\lambda}_{\rho} \nabla_{\nu} S^{\rho\mu} + S^{\lambda}_{\mu} \nabla^{\gamma} S^{\mu\rho} \nabla_{\nu} S^{\beta}_{\rho} + S^{\mu\rho} \nabla^{\gamma} S^{\lambda}_{\mu} \nabla_{\nu} S^{\beta}_{\rho} \\
& - 2 S^{\beta}_{\mu} \nabla^{\gamma} S^{\mu\lambda} \nabla_{\nu} e_1 - S^{\rho}_{\mu} \nabla^{\gamma} S^{\mu}_{\nu} \nabla^{\beta} S^{\lambda}_{\rho} - S^{\beta}_{\mu} \nabla^{\gamma} S^{\mu}_{\rho} \nabla^{\lambda} S^{\rho}_{\nu} - S^{\mu}_{\rho} \nabla^{\gamma} S^{\beta}_{\mu} \nabla^{\lambda} S^{\rho}_{\nu} \\
& - S^{\beta}_{\rho} \nabla^{\gamma} S^{\mu}_{\nu} \nabla^{\lambda} S^{\rho}_{\mu} + S^{\beta}_{\mu} \nabla^{\gamma} S^{\mu}_{\nu} \nabla^{\lambda} e_1 + S^{\mu}_{\rho} \nabla^{\gamma} S_{\mu\nu} \nabla^{\rho} S^{\beta\lambda} - S^{\lambda}_{\mu} \nabla^{\gamma} S^{\mu}_{\nu} \nabla^{\rho} S^{\beta}_{\rho} \\
& - S^{\lambda}_{\mu} \nabla^{\gamma} S^{\beta}_{\rho} \nabla^{\mu} S^{\rho}_{\nu} - S^{\beta}_{\rho} \nabla^{\gamma} S^{\lambda}_{\mu} \nabla^{\mu} S^{\rho}_{\nu} - S^{\lambda}_{\mu} \nabla^{\gamma} S^{\rho}_{\nu} \nabla^{\mu} S^{\beta}_{\rho} + 2 S^{\beta}_{\mu} \nabla^{\gamma} S^{\mu\lambda} \nabla^{\rho} S_{\rho\nu} \\
& + 2 S^{\beta}_{\rho} \nabla^{\gamma} S_{\mu\nu} \nabla^{\mu} S^{\rho\lambda} + S^{\lambda}_{\rho} S^{\beta}_{\mu} \nabla^{\gamma} \nabla_{\nu} S^{\rho\mu} + [S^2]^{\lambda\rho} \nabla^{\gamma} \nabla_{\nu} S^{\beta}_{\rho} - [S^2]^{\beta\lambda} \nabla^{\gamma} \nabla_{\nu} e_1 \\
& - [S^2]^{\beta}_{\rho} \nabla^{\gamma} \nabla^{\lambda} S^{\rho}_{\nu} - S^{\lambda}_{\mu} S^{\beta}_{\rho} \nabla^{\gamma} \nabla^{\mu} S^{\rho}_{\nu} + [S^2]^{\beta\lambda} \nabla^{\gamma} \nabla^{\rho} S_{\rho\nu} \Big] + \beta_2 \Big[ + \nabla^{\beta} S^{\lambda}_{\gamma} \nabla^{\gamma} e_1 \\
& - \nabla_{\gamma} S^{\beta\lambda} \nabla^{\gamma} e_1 - \nabla^{\mu} S^{\rho}_{\mu} \nabla^{\beta} S^{\lambda}_{\rho} - \nabla^{\mu} S^{\beta}_{\rho} \nabla^{\lambda} S^{\rho}_{\mu} + \nabla^{\mu} S^{\beta}_{\mu} \nabla^{\lambda} e_1 + \nabla_{\mu} S^{\mu}_{\rho} \nabla^{\rho} S^{\beta\lambda} \\
& - \nabla^{\mu} S^{\lambda}_{\mu} \nabla^{\rho} S^{\beta}_{\rho} - \nabla^{\rho} S^{\lambda}_{\mu} \nabla^{\mu} S^{\beta}_{\rho} + 2 \nabla_{\mu} S^{\beta}_{\rho} \nabla^{\mu} S^{\rho\lambda} - S^{\beta}_{\rho} \nabla^{\lambda} \nabla^{\mu} S^{\rho}_{\mu} + S^{\beta}_{\gamma} \nabla^{\gamma} \nabla^{\lambda} e_1 \\
& - S^{\lambda}_{\gamma} \nabla^{\gamma} \nabla^{\rho} S^{\beta}_{\rho} + S^{\beta}_{\rho} \nabla^{\gamma} \nabla_{\gamma} S^{\rho\lambda} \Big] + (\beta \leftrightarrow \lambda) \, ,
\end{aligned}$$

The scalar constraint reads

$$\mathcal{C} = m^2\left[\left(A^{\beta\lambda} + \tilde{A}^{\beta\lambda}\right)\tilde{h}_{\beta\lambda} + B_\rho^{\beta\lambda}\nabla^\rho\tilde{h}_{\beta\lambda}\right]$$

With $h_{\mu\nu} = \tilde{h}_{\mu\lambda}S^\lambda_{\ \nu} + S_\mu^{\ \lambda}\tilde{h}_{\lambda\nu}$

$$\begin{aligned}
B_\rho^{\beta\lambda} \equiv\ & \frac{1}{2}\left(\beta_1+\beta_2 e_1\right)[S^{-1}]^\nu_\gamma\ \left[-S^{\sigma\lambda}\delta^\gamma_\rho\nabla_\nu S^\beta_\sigma + \delta^\gamma_\rho S^\beta_\sigma\nabla^\lambda S^\sigma_\nu + \delta^\lambda_\rho S^\beta_\sigma\nabla^\gamma S^\sigma_\nu - S^{\beta\lambda}\nabla^\gamma S_{\nu\rho}\right] \\
& + \beta_2\,[S^{-1}]^\nu_{\ \gamma}\left[\delta^\gamma_\rho S^\lambda_\delta S^\beta_\mu\nabla_\nu S^{\delta\mu} + \delta^\gamma_\rho[S^2]^{\lambda\mu}\nabla_\nu S^\beta_\mu - \delta^\gamma_\rho[S^2]^{\beta\lambda}\nabla_\nu e_1 - \delta^\gamma_\rho[S^2]^\beta_\mu\nabla^\lambda S^\mu_\nu\right. \\
& - \delta^\gamma_\rho S^\lambda_\mu S^\beta_\delta\nabla^\mu S^\delta_\nu + \delta^\gamma_\rho[S^2]^{\beta\lambda}\nabla^\mu S_{\mu\nu} + S^{\beta\lambda}S^\mu_\rho\nabla^\gamma S_{\mu\nu} + [S^2]^{\beta\lambda}\nabla^\gamma S_{\rho\nu} - \delta^\beta_\rho[S^2]^\lambda_\mu\nabla^\gamma S^\mu_\nu \\
& \left.- S^\beta_\rho S^\lambda_\mu\nabla^\gamma S^\mu_\nu\right] + \beta_2\left[-S^\beta_\delta\nabla^\lambda S^\delta_\rho + S^\beta_\rho\nabla^\lambda e_1 - S^\lambda_\mu\nabla^\mu S^\beta_\rho + 2S^\beta_\delta\nabla_\rho S^{\delta\lambda} + \delta^\beta_\rho S^\lambda_\gamma\nabla^\gamma e_1\right. \\
& \left.- S^{\beta\lambda}\nabla_\rho e_1 + S^{\beta\lambda}\nabla_\mu S^\mu_\rho - \delta^\beta_\rho S^\lambda_\delta\nabla^\mu S^\delta_\mu - S^\beta_\rho\nabla^\mu S^\lambda_\mu\right] + (\beta\leftrightarrow\lambda)\,.
\end{aligned}$$

To get a PM theory we need to look for space-times

where $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$ vanish identically.

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Most general solution ?

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where $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$ vanish identically.

Most general solution ?

Assume $\boxed{\nabla_\rho S_{\mu\nu} = 0}$

(i.e. $S_{\mu\nu}$ covariantly constant)

makes $\tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$ vanish.

Space-times possessing a covariantly constant tensor $H_{\mu\nu}$ are severely restricted…

$$\nabla_\rho H_{\mu\nu} = 0$$

$$\underbrace{\nabla_{[\mu}\nabla_{\nu]} H_{\rho\lambda} = R_{\mu\nu\rho}{}^{\sigma} H_{\sigma\lambda} + R_{\mu\nu\lambda}{}^{\sigma} H_{\rho\sigma}}$$

$$\Downarrow$$

$$R_{\mu\nu\rho}{}^{\sigma} H_{\sigma\lambda} + R_{\mu\nu\lambda}{}^{\sigma} H_{\rho\sigma} = 0$$

Non trivial integrability conditions

Space-times possessing a covariantly constant tensor are classified as (provided $H_{\mu\nu}$ is not proportional to the metric)

1. Spacetime is $2 \otimes 2$ decomposable

   $$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = g_{ab}(x^c)\mathrm{d}x^a\mathrm{d}x^b + g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j$$

   and $H_{\mu\rho}H^{\rho}{}_{\nu} = H_{\mu\nu}$

2. The spacetime admits a covariantly constant vector $N^{\mu}$ and $H_{\mu\nu} = N_{\mu}N_{\nu}$

Note further that the definition

$$R^{\mu}{}_{\nu} = m^2 \left[ \left( \beta_0 + \frac{1}{2} e_1 \beta_1 \right) \delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu} \right]$$

Imposes that $\nabla_\rho S_{\mu\nu} = 0$ $\Rightarrow$ $\nabla_\rho R_{\mu\nu} = 0$

Hence the space-time must be "Ricci Symmetric"

…i.e. the Ricci tensor is covariantly constant …

The spacetimes of interest for us here will all have

$$(R^2)^{\rho}{}_{\nu} = r_1\, R^{\rho}{}_{\nu} + r_2\, \delta^{\rho}_{\nu} \quad \text{with } r_1 \text{ and } r_2 \text{ constant}$$

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\begin{cases} \delta^{\beta}_{\lambda}\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2}e_1\right) + S^{\beta}_{\lambda}\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2}\right) - (S^2)^{\beta}_{\lambda}(e_1\beta_2^2) = 0 \\ R^{\mu}{}_{\nu} = m^2\left[\left(\beta_0 + \frac{1}{2}e_1\beta_1\right)\delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu}\right] \end{cases}$$

The spacetimes of interest for us here will all have

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And have to solve (in order to get a PM graviton)

$$\begin{cases} \boxed{\delta^{\beta}_{\lambda}\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2}e_1\right) + S^{\beta}_{\lambda}\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2}\right) - (S^2)^{\beta}_{\lambda}(e_1\beta_2^2) = 0} \\ R^{\mu}{}_{\nu} = m^2\left[\left(\beta_0 + \frac{1}{2}e_1\beta_1\right)\delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu}\right] \end{cases}$$

In order to get a vanishing $\mathcal{C} = m^2 A^{\beta\lambda}\tilde{h}_{\beta\lambda}$

The spacetimes of interest for us here will all have

$(R^2)^{\rho}{}_{\nu} = r_1\, R^{\rho}{}_{\nu} + r_2\, \delta^{\rho}_{\nu}$ with $r_1$ and $r_2$ constant

as a consequence of the integrability conditions

And have to solve (in order to get a PM graviton)

$$\begin{cases} \delta^{\beta}_{\lambda}\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2}e_1\right) + S^{\beta}_{\lambda}\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2}\right) - (S^2)^{\beta}_{\lambda}(e_1\beta_2^2) = 0 \\ \boxed{R^{\mu}{}_{\nu} = m^2\left[\left(\beta_0 + \frac{1}{2}e_1\beta_1\right)\delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu}\right]} \end{cases}$$

From the definition of $S^{\rho}{}_{\nu}$

The spacetimes of interest for us here will all have

$$(R^2)^{\rho}{}_{\nu} = r_1\, R^{\rho}{}_{\nu} + r_2\, \delta^{\rho}_{\nu}$$ with $r_1$ and $r_2$ constant

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And have to solve (in order to get a PM graviton)

$$\begin{cases} \delta^{\beta}_{\lambda}\left(\beta_2\beta_0 e_1 + \beta_0\beta_1 + \frac{\beta_1^2}{2} e_1\right) + S^{\beta}_{\lambda}\left(-2\beta_2\beta_0 + \beta_2^2 e_1^2 - 2\beta_2^2 e_2 - \frac{\beta_1^2}{2}\right) - (S^2)^{\beta}_{\lambda}(e_1\beta_2^2) = 0 \\ R^{\mu}{}_{\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2} e_1 \beta_1\right)\delta^{\mu}_{\nu} + (\beta_1 + \beta_2 e_1)\, S^{\mu}{}_{\nu} - \beta_2 (S^2)^{\mu}{}_{\nu}\right] \end{cases}$$

NB: this implies that the Ricci tensor obeyes indeed the required relation

# Explicit solutions (I)

with
$$\begin{cases} (R^2)^\rho{}_\nu = r_1\, R^\rho{}_\nu + r_2\, \delta^\rho_\nu \\ u \equiv \dfrac{\beta_0 \beta_2}{\beta_1^2}, \quad v \equiv m^2 \beta_0 \dfrac{16u^2 - 24u - 3}{u\,(4u+3)} \end{cases}$$

dimensionless

dimensionful

$$2 \otimes 2 \begin{cases} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \dfrac{2\mathrm{d}x^0 \mathrm{d}x^1}{(1 + (R-E)x^0 x^1/8)^2} - \dfrac{2\mathrm{d}x^2 \mathrm{d}x^3}{(1-(R+E)x^2 x^3/8)^2} \\ \text{type D}: \quad r_1 = \dfrac{R}{2}\,, \qquad r_2 = -\dfrac{1}{16}\left(R^2 - E^2\right) \\ E^2 = \dfrac{72\left(9 + 60u + 144u^2 + 64u^3\right)}{\left(-9 - 132u - 48u^2 + 64u^3\right)^2} R^2 \\ v = \left(2 + \dfrac{96u}{-9 - 132u - 48u^2 + 64u^3}\right) R \end{cases}$$

Analogous to $2\Lambda = 3m^2$

# Explicit solutions (II)

with
$$\left\{\begin{array}{l} (R^2)^{\rho}{}_{\nu} = r_1\, R^{\rho}{}_{\nu} + r_2\, \delta^{\rho}_{\nu} \\ \boxed{u \equiv \dfrac{\beta_0 \beta_2}{\beta_1^2}}, \quad \boxed{v \equiv m^2 \beta_0 \dfrac{16u^2 - 24u - 3}{u\,(4u+3)}} \end{array}\right.$$

dimensionless dimensionful

$$1 \otimes 3 \left\{\begin{array}{l} g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} = \dfrac{\mathrm{d}x^2 + \mathrm{d}y^2 - \epsilon\, \mathrm{d}w^2}{\left[1 + R\,(x^2 + y^2 - \epsilon\, w^2)/24\right]^2} + \epsilon\, \mathrm{d}z^2 \\ \text{type O}: \quad r_1 = \dfrac{R}{3}, \qquad r_2 = 0 \\ 256u^4 + 1280u^3 + 4320u^2 + 144u - 135 = 0 \\ \boxed{v = \dfrac{6\,(32u^3 + 42u + 9)}{128u^3 + 144u^2 + 144u + 27} R} \end{array}\right.$$

Analogous to $2\Lambda = 3m^2$

One simple example of the last kind is Einstein static Universe !

$$\begin{cases} \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2\,\mathrm{d}\Sigma^2 \\ \text{with } \mathrm{d}\Sigma^2 = \gamma_{ij}\,\mathrm{d}x^i \mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-k\,r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \end{cases}$$

$$\Rightarrow \quad R = \frac{6k}{a^2}$$

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$$\Rightarrow \quad R = \frac{6k}{a^2}$$

The gauge symmetry reads

$$\begin{cases} \Delta h_{tt} = \left[-\frac{(4u+3)(v-2R)}{v}\nabla_t\partial_t + \frac{u(6R+8Ru-3v)v}{(-3-24u+16u^2)(-2R+vv)}\right]\xi(x)\,, \\ \Delta h_{ti} = -\frac{(4u+3)(v-2R)\left(\frac{8R}{3}-2v\right)}{2v\left(\frac{8R}{3}-v\right)}\nabla_{(t}\partial_{i)}\,\xi(x)\,, \\ \Delta h_{ij} = \left[-\frac{(4u+3)(v-2R)}{\frac{8R}{3}-v}\nabla_i\partial_j + \frac{uv(-18R+8Ru+9v)}{3(-3-24u+16u^2)(-2R+v)}\,a^2\,\gamma_{ij}\right]\xi(x)\,, \end{cases}$$

With e.g. one solution being $(u,\,v) = (-0.2006,\,0.6622R)$

# Conclusions

PM exists on non Einstein spacetimes !

(in contrast with previous no-go claim by Deser, Joung, Waldron
In 1208.1307 [hep-th]…
… in particular some of our solution have a vanishing Bach tensor)

Solution for the vanishing
Of $A^{\beta\lambda} + \tilde{A}^{\beta\lambda}$ and $B_{\rho}^{\beta\lambda}$
are not known in full generality !

Thank you ! and best wishes to Misao !