Institute of
Space Sciences

Zeta Functions and Cosmological Applications

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GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD
CSIC
IEEC

[Closing Talk]

Zero point energy

QFT vacuum to vacuum transition: $\langle 0 | H | 0 \rangle$

Spectrum, normal ordering (harm oscill):

$$H = \left(n + \frac{1}{2} \right) \lambda_n \, a_n \, a_n^\dagger$$

$$\langle 0 | H | 0 \rangle = \frac{\hbar \, c}{2} \sum_n \lambda_n = \frac{1}{2} \, \mathrm{tr} \; H$$

gives ∞ physical meaning?

Regularization $+$ Renormalization (cut-off, dim, ζ)

Even then: Has the final value real sense ?

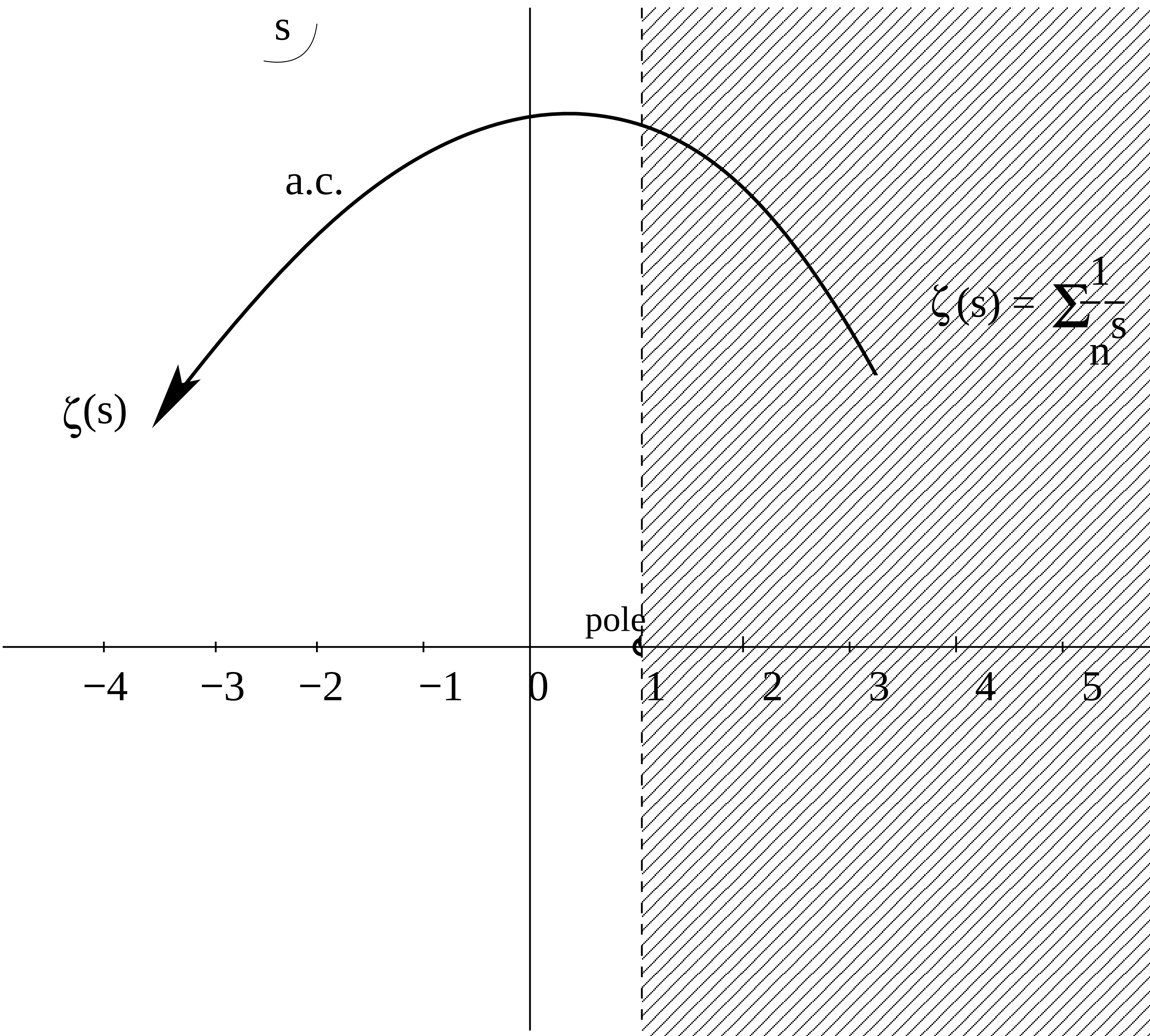
Effects of the Quantum Vacuum

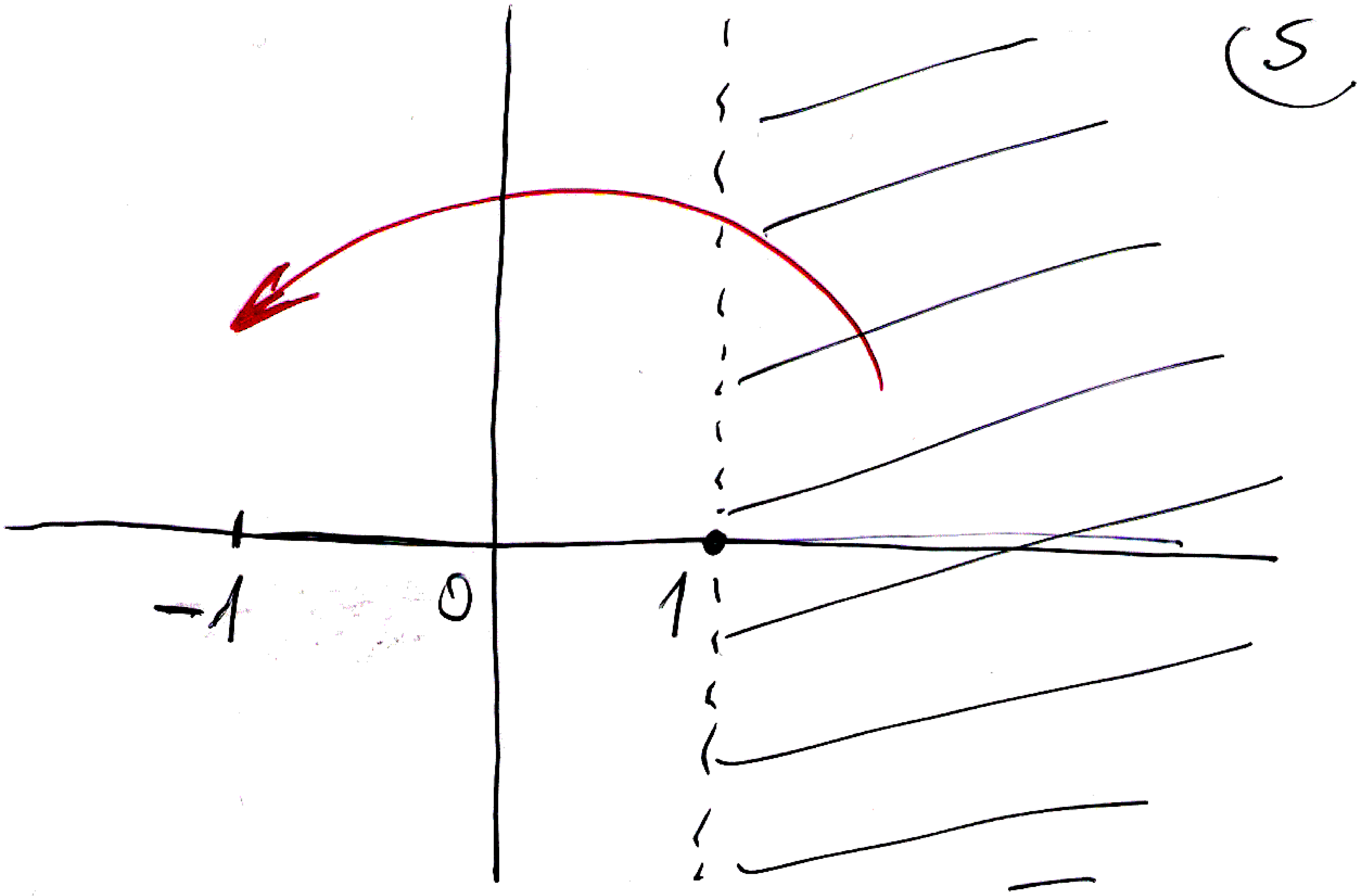
a) Negligible: Sonoluminiscence, Schwinger ~10^{-5}

b) Important: Wetting He3 – alcali ~30%

c) Incredibly big: Cosmological constant ~10^{120}

Riemann Zeta Function





$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

$$\zeta(0) = -\frac{1}{2} \quad \text{or} \quad 1 + 1 + 1 + \dots \overset{\zeta}{=} -\frac{1}{2}$$

$$\zeta(-1) = -\frac{1}{12} \quad \text{or} \quad 1 + 2 + 3 + \dots \overset{\zeta}{=} -\frac{1}{12}$$

⋮

F Yndurain, A Slavnov
"As everybody knows ..."

Operator Zeta F's in MΦ: Origins

- The Riemann zeta function $\zeta(s)$ is a function of a complex variable, s. To define it, one starts with the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$
which converges for all complex values of s with real Re $s > 1$, and then defines $\zeta(s)$ as the analytic continuation, to the whole complex $s-$plane, of the function given, Re $s > 1$, by the sum of the preceding series.

  Leonhard Euler already considered the above series in 1740, but for positive integer values of s, and later Chebyshev extended the definition to Re $s > 1$.

- Godfrey H Hardy and John E Littlewood, "Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes", Acta Math 41, 119 (1916)

  Did much of the earlier work, by establishing the convergence and equivalence of series regularized with the heat kernel and zeta function regularization methods

  G H Hardy, Divergent Series (Clarendon Press, Oxford, 1949)

  Srinivasa I Ramanujan had found for himself the functional equation of the zeta function

- Torsten Carleman, "Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes" (French), 8. Skand Mat-Kongr, 34-44 (1935)

  Zeta function encoding the eigenvalues of the Laplacian of a compact Riemannian manifold for the case of a compact region of the plane

- Robert T Seeley, “Complex powers of an elliptic operator. 1967 Singular Integrals" (Proc. Sympos. Pure Math., Chicago, Ill., 1966) pp. 288-307, Amer. Math. Soc., Providence, R.I.

  Extended this to elliptic pseudo-differential operators A on compact Riemannian manifolds. So for such operators one can define the determinant using zeta function regularization

- D B Ray, Isadore M Singer, “R-torsion and the Laplacian on Riemannian manifolds", Advances in Math 7, 145 (1971)

  Used this to define the determinant of a positive self-adjoint operator A (the Laplacian of a Riemannian manifold in their application) with eigenvalues $a_1, a_2, ...,$ and in this case the zeta function is formally the trace

$$\zeta_A(s) = \mathrm{Tr}\,(A)^{-s}$$

  the method defines the possibly divergent infinite product

$$\prod_{n=1}^{\infty} a_n = \exp[-\zeta_A{}'(0)]$$

- J. Stuart Dowker, Raymond Critchley

  “Effective Lagrangian and energy-momentum tensor in de Sitter space", Phys. Rev. D13, 3224 (1976)

  Abstract

  The effective Lagrangian and vacuum energy-momentum tensor $< T^{\mu\nu} >$ due to a scalar field in a de Sitter space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0,$ the renormalized $< T^{\mu\nu} >$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1},$ where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

Effective Lagrangian and energy-momentum tensor in de Sitter space

J. S. Dowker and Raymond Critchley
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(Received 29 October 1975)

The effective Lagrangian and vacuum energy-momentum tensor $\langle T^{\mu\nu}\rangle$ due to a scalar field in a de Sitter-space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2+\xi R+m^2)\varphi=0$. If $\xi=1/6$ and $m=0$, the renormalized $\langle T^{\mu\nu}\rangle$ equals $g^{\mu\nu}(960\pi^2a^4)^{-1}$, where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

I. INTRODUCTION

In a previous paper[1] (to be referred to as I) the effective Lagrangian $\mathfrak{L}^{(1)}$ due to single-loop diagrams of a scalar particle in de Sitter space was computed. It was shown to be real and was evaluated as a principal-part integral. The regularization method used was the proper-time one due to Schwinger[2] and others. We now wish to consider the same problem but using different techniques. In particular, we wish to make contact with the work of Candelas and Raine,[3] who first discussed the same problem using dimensional regularization.

Some properties of the various regularizations as applied to the calculation of the vacuum expectation value of the energy-momentum tensor have been contrasted by DeWitt.[4] We wish to pursue some of these questions within the context of a definite situation.

II. GENERAL FORMULAS: REGULARIZATION METHODS

We use exactly the notation of I, which is more or less standard, and begin with the expression for $\mathfrak{L}^{(1)}$ in terms of the quantum-mechanical propagator, $K(x'',x',\tau)$,

$$\mathfrak{L}^{(1)}(x')=-\tfrac{1}{2}i\lim_{x''\to x'}\int_0^\infty d\tau\,\tau^{-1}K(x'',x',\tau)e^{-im^2\tau}+X(x')\,. \tag{1}$$

There are two points regarding this expression which need some further discussion. Firstly, if we adopt the proper-time regularization method so that the infinities appear only when the τ integration, which is the final operation, is performed, then we can take the coincidence limit, $x''=x'$, through into the integrand. Further, since the regularized expression is continuous across the light cone, it does not matter how we let x'' approach x'. Secondly, the term X does not have to be a constant, but it should integrate to give a metric-independent contribution to the total action, $W^{(1)}$.

The Schwinger-DeWitt procedure is to derive an expression for K, either in closed form or as a general expansion to powers of τ, then to effect the coincidence limit in (1), and finally to perform the τ integration. This was the approach adopted in I. We proceed now to give a few more details.

We assume that we are working on a Riemannian space, $\mathfrak{M}$, of dimension d. The coincidence limit $K(x,x,\tau)$ can be expanded,[5]

$$K(x,x,\tau)=i(4\pi i\tau)^{-d/2}\sum_{n=0}^{\infty}a_n(x)(i\tau)^n\,, \tag{2}$$

where the a_n are scalars constructed from the curvature tensor on $\mathfrak{M}$ and whose functional form is independent of d. The manifold $\mathfrak{M}$ must not have boundaries, otherwise other terms appear in the expansion.

The expansion (2) is substituted into (1) to yield

$$\mathfrak{L}^{(1)}(x)=\tfrac{1}{2}i(4\pi)^{-d/2}\sum_n a_n(x)\int_0^\infty(i\tau)^{n-d/2-1}e^{-im^2\tau}d\tau\,. \tag{3}$$

The infinite terms are those for which $n\leq d/2$ (for d even) or $n\leq(d-1)/2$ (for d odd). For $d=4$, e.g. space-time, there are three infinite terms. These terms are removed by renormalization; the details are given by DeWitt.[4]

Another popular regularization technique is dimensional regularization.[6] In this method the dimension, d, is considered to be complex and all expressions are defined in a region of the d plane where they converge. The infinities appear when an analytic continuation to $d=4$ is performed to regain the physical quantities. This idea was originally developed for use in flat-space (i.e., Lorentz-invariant) situations for the momentum

- Stephen W Hawking, “Zeta function regularization of path integrals in curved spacetime", Commun Math Phys 55, 133 (1977)

  This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dims. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This EM tensor has an anomalous trace.

Commun. math. Phys. 55, 133—148 (1977)

Communications in
Mathematical
Physics

Zeta Function Regularization of Path Integrals in Curved Spacetime

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Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fith dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.

1. Introduction

The purpose of this paper is to describe a technique for obtaining finite values to path integrals for fields (including the gravitational field) on a curved spacetime background or, equivalently, for evaluating the determinants of differential operators such as the four-dimensional Laplacian or D'Alembertian. One forms a gemeralised zeta function from the eigenvalues λ_n of the operator

$$\zeta(s)=\sum_n \lambda_n^{-s}\ . \tag{1.1}$$

In four dimensions this converges for $\mathrm{Re}(s)>2$ and can be analytically extended to a meromorphic function with poles only at $s=2$ and $s=1$. It is regular at $s=0$. The derivative at $s=0$ is formally equal to $-\sum_n \log\lambda_n$. Thus one can define the determinant of the operator to be $\exp(-d\zeta/ds)|_{s=0}$.

Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \qquad z,t\in\mathbb{C},\ \text{Re}\,t>0$$

- Higher dimensions: Poisson summ formula (Riemann)

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \widetilde{f}(\vec{m})$$

$\widetilde{f}$ Fourier transform

[Gelbart + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]

- Truncated sums $\longrightarrow$ asymptotic series

ζ: EXPLICIT CALCULATIONS

Epstein zeta functions (quadratic)

$$\zeta_E = \sum_{\vec{n} \in \mathbb{Z}^d} Q(\vec{n})^{-s}$$

Q quadratic form

Barnes zeta functions (linear)

$$\zeta_B = \sum_{\vec{n} \in \mathbb{N}^d} L(\vec{n})^{-s}$$

L affine form (coeff's $\in \mathbb{Q}^+$)

Extensions:

$\zeta_E \rightarrow Q + L$ affine

$\rightarrow \displaystyle\sum_{\vec{n} \in \mathbb{N}^d}$ (truncation)

$\zeta_B \rightarrow \zeta_B'(0)$ (new formulas)

$\rightarrow \displaystyle{\sum_{\vec{n} \in \mathbb{Z}^d}}'$ (by analyt. cont.)

Ex ___

ζ-REGULARIZ: SPECTRUM KNOWN IMPLICITLY

- Example of the ball:
  - Operator

$$(-\Delta + m^2)$$

on the D-dim ball $B^D = \{x \in R^D; |x| \leq R\}$
with Dirichlet, Neumann or Robin BC

  - The zeta function

$$\zeta(s) = \sum_k \lambda_k^{-s}$$

  - Eigenvalues implicitly obtained from

$$(-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) \quad + \quad BC$$

  - In spherical coordinates:

$$\phi_{l,m,n}(r,\Omega) = r^{1-\frac{D}{2}} J_{l+\frac{D-2}{2}}(w_{l,n} r) Y_{l+\frac{D}{2}}(\Omega)$$

$J_{l+(D-2)/2}$ Bessel functions
$Y_{l+D/2}$ hyperspherical harmonics

  - Eigenvalues $w_{l,n}$ (>0) determined through BC

$$J_{l+\frac{D-2}{2}}(w_{l,n} R) = 0, \qquad \text{for Dirichlet BC}$$

$$\frac{u}{R} J_{l+\frac{D-2}{2}}(w_{l,n}R) + w_{l,n} J'_{l+\frac{D-2}{2}}(w_{l,n}r)\,|_{r=R} = 0,$$ for Robin BC

– Now, $\lambda_{l,n} = w_{l,n}^2 + m^2$

$$\zeta(s) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} d_l(D)(w_{l,n}^2 + m^2)^{-s}$$

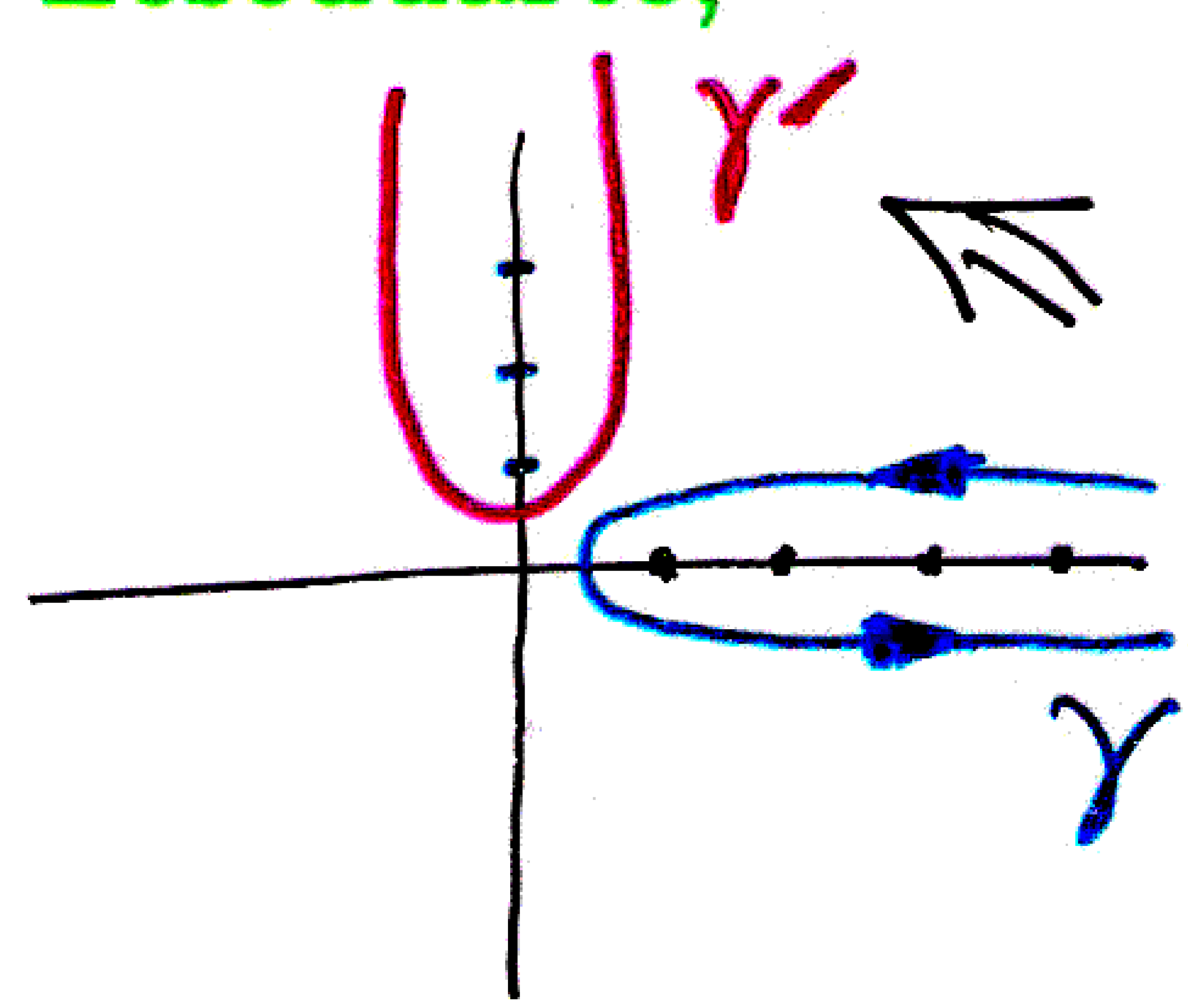
$w_{l,n}$ (> 0) is defined as the n-th root of the l-th equation, $d_l(D) = (2l + D - 2)\frac{(l+D-3)!}{l!\,(D-2)!}$

- **Procedure:**
  - Contour integral on the complex plane

$$\zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_\gamma \frac{dk}{2\pi i}\,(k^2+m^2)^{-s}\frac{\partial}{\partial k}\ln \Phi_{l+\frac{D-2}{2}}(kR)$$

γ runs counterclockwise and must enclose all the solutions [Ginzburg, Van Kampen, EE + I. Brevik]

- **Obtained:** [with Bordag, Kirsten, Leseduarte, Vassilievich,...]
  - Zeta functions
  - Determinants
  - Seeley [heat-kernel] coefficients



Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
4. M closed n-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of A, $s_0 = \dim M / \text{ord } A$ abscissa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut L_θ

(d) The only possible singularities of $\zeta^A(s)$ are poles at

$$s_j = (n - j)/m, \qquad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$

Definition of Determinant

H ΨDO operator $\{\varphi_i, \lambda_i\}$ spectral decomposition

$\prod_{i\in I} \lambda_i$?! $\ln \prod_{i\in I} \lambda_i = \sum_{i\in I} \ln \lambda_i$

Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \; Re\, s > 1$ (& analytic cont)

Def nition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr } H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} \, \text{tr } e^{-tH}, \; \text{Re} s > s_0$

Derivative: $\zeta'_H(0) = -\sum_{i\in I} \ln \lambda_i$

Determinant: Ray & Singer, '67

$$\det_{\zeta} H = \exp\left[-\zeta'_H(0)\right]$$

Weierstrass def: subtract leading behavior of λ_i in i, as $i \to \infty$,
until series $\sum_{i\in I} \ln \lambda_i$ converges $\Longrightarrow$ non-local counterterms !!

C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

Properties

- The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut
- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators ofcomplex order $m \in \mathbb{C}\backslash\mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]
- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum\nolimits'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum\nolimits_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum\nolimits_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \, \text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP } \zeta_A(-k) + \psi(k+1) \, \text{Res}_{s=-k} \zeta_A(s)\right], \quad s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP } \phi := \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p}\right]$$

“ Hi, Emilio. This is a question I have been trying to solve for years.
With a bit of luck you could maybe provide me with a hint or two.

- Imagine I’ve got a functional integral and I perform a point transformation (doesn’t involve derivatives). Its Jacobian is a kind of functional determinant, but of a non-elliptic operator (it is simply infinite times multiplication by a function.) Did anybody study this seriously?
- I do know, from at least one paper I did with Luis AG, that in some cases (T duality) one is bound to define something like
- det f(x) ~ det [f(x) O] / det O where O es an elliptic operator (e.g. the Laplacian)
- This is what Schwarz and Tseytlin did in order to obtain the dilaton transformation
- And LAG and I did also proceed in a basically similar way
- As I know, Konsevitch, too, uses a related method involving the multiplicative anomaly

Tell me what you know about, please. Thanks so much.- Hugs, Enrique “

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det\nolimits_\zeta(AB) \neq \det\nolimits_\zeta A \, \det\nolimits_\zeta B$$

$$\det_\zeta (AB) \overset{?}{=} \det_\zeta A \, \det_\zeta B$$

$$\log \det_\zeta = \mathrm{tr}_\zeta \log \;, \quad \det_\zeta = e^{\mathrm{tr}_\zeta \log}$$

$$\det_\zeta (AB) \overset{1}{=} e^{\mathrm{tr}_\zeta \log(AB)} \overset{2}{=} e^{\mathrm{tr}_\zeta (\log A + \log B)}$$

$$\overset{3}{=} e^{\mathrm{tr}_\zeta \log A + \mathrm{tr}_\zeta \log B} =$$

$$\overset{4}{=} e^{\mathrm{tr}_\zeta \log A} \, e^{\mathrm{tr}_\zeta \log B} =$$

$$\overset{5}{=} \det_\zeta A \cdot \det_\zeta B$$

$[A, B] = 0$ assumed!

Which step is wrong?

tr_ζ is <u>no</u> trace at all

$$\mathrm{tr}_\zeta (A_1 + A_2) \neq \mathrm{tr}_\zeta A_1 + \mathrm{tr}_\zeta A_2$$

recall

$$\mathrm{tr}_\zeta A = \zeta_A (s=-1) = \sum_n \lambda_n^{-s} \Big|_{s=-1}$$

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$$

- The multiplicative (or noncommutative) anomaly (defect) is def ned as

$$\delta(A,B) = \ln\left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B}\right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln\sigma(A,B)]^2\right\}}{2 \text{ ord } A \text{ ord } B \left(\text{ord } A + \text{ord } B\right)}$$

where $\sigma(A,B) = A^{\text{ord } B} B^{-\text{ord } A}$

The Dixmier Trace

- In order to write down an action in operator language one needs a functional that replaces integration
- For the Yang-Mills theory this is the Dixmier trace
- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

- Definition of the Dixmier trace of T:

$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\text{Dtr}\, T = \lim_{s\to 1^+} (s-1) \zeta_{T^{-1}}(s)$$

The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$
- Only trace one can define in the algebra of ΨDOs (up to multipl const)
- Definition: res $A = 2\, \text{Res}_{s=0}\, \text{tr}(A\Delta^{-s})$, Δ Laplacian
- Satisfies the trace condition: res $(AB) =$ res (BA)
- Important!: it can be expressed as an integral (local form)

$$\text{res } A = \int_{S^*M} \text{tr } a_{-n}(x,\xi)\, d\xi$$

with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue)

- If dim $M = n = -$ ord A (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and $\text{Res}_{s=1}\zeta_A(s) = \frac{1}{n}$ res A^{-1}
- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi] \; \exp\left\{-\int d^D x \left[\Phi^\dagger(x)(\quad)\Phi(x) + \cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det\,(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A & \\ & B \end{pmatrix}$$

$\det(AB)$ or $\det A \cdot \det B$?

- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$
- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

Recent Applications

arXiv:1707.03975

Effect of a magnetic field on Schwinger mechanism in de Sitter spacetime

Ehsan Bavarsad, Sang Pyo Kim, Clément Stahl, She-Sheng Xue

arXiv:1707.05485

Large gauge transformation, soft theorem, and infrared divergence in inflationary spacetime

Takahiro Tanaka, Yuko Urakawa

arXiv:1705.01525

A Laplace transform approach to linear equations with infinitely many derivatives and zeta-nonlocal field equations

Alan Chavez, Humberto Prado, Enrique G. Reyes

arXiv:1608.03133

Do Black Holes exist in a finite Universe having the yopology of a flat 3-torus?

Frank Steiner

Self-sustained traversable wormholes

Remo Garattini, Francisco S.N. Lobo. 2017. 25 pp.
Fundam. Theor. Phys. 189 (2017) 111-135

Can one hear the Riemann zeros in black hole ringing?

Rodrigo Aros, Fabrizzio Bugini, Danilo E. Diaz. 2016. 3 pp.
J. Phys. Conf. Ser. 720 (2016) no.1, 012009

Recent Applications

arXiv:1708.02627

Casimir Effect in the Rainbow Einstein's Universe

V. B. Bezerra, H. F. Mota, C. R. Muniz

arXiv:1605.09175

Quantum magnetic flux lines, BPS vortex zero modes, and one-loop string tension shifts

A. Alonso-Izquierdo, J. Mateos Guilarte, M. de la Torre Mayado

Journal-ref: Phys. Rev. D 94, 045008 (2016)

arXiv:1505.03276

Local zeta regularization and the scalar Casimir effect IV. The case of a rectangular box

Davide Fermi (Universita' di Milano), Livio Pizzocchero (Universita' di Milano)

arXiv:1505.01044

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Emilio Elizalde

Ten Physical Applications of Spectral Zeta Functions

Second Edition

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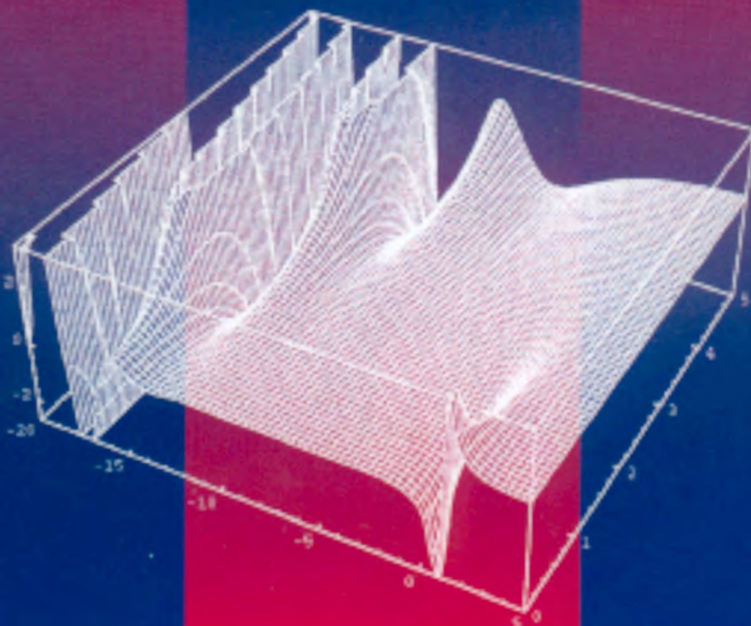

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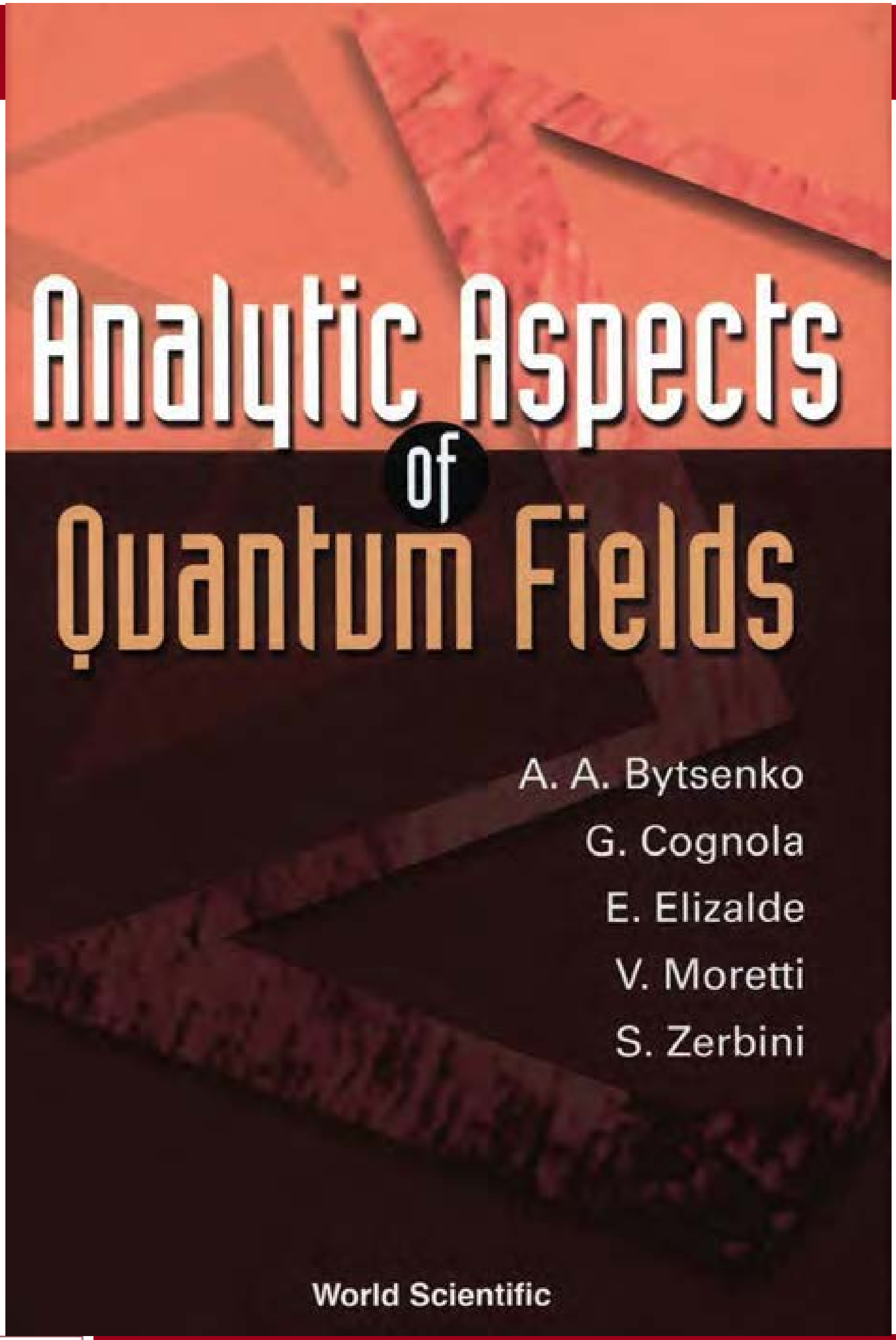
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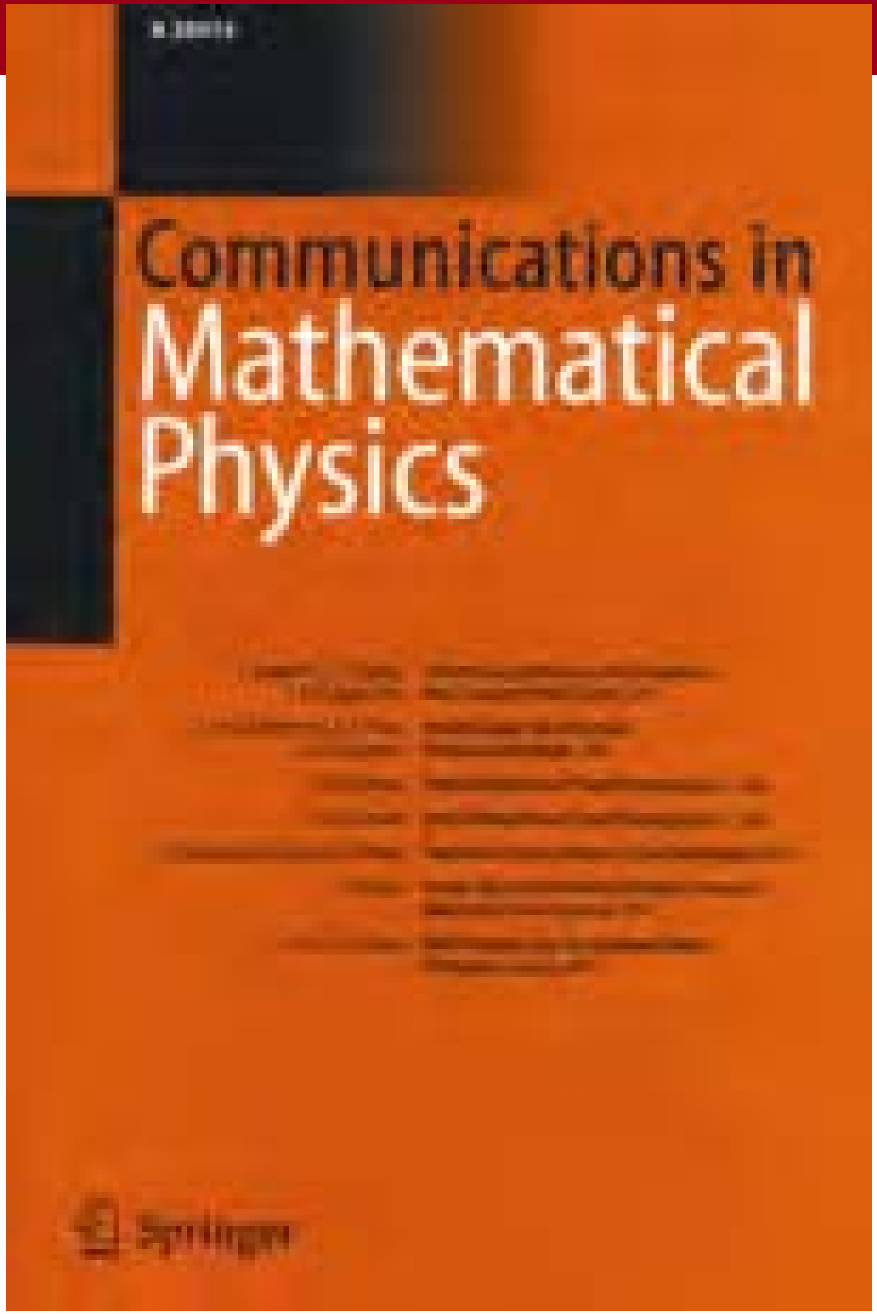
Communications in
Mathematical
Physics

Emilio Elizalde

ICE-CSIC & IEEC

Barcelona, Spain

For Misao Sasaki,
on his retiring from YITP,
with very best wishes
for a long and happy life.

The meaning of Big Bang*

The meaning of Big Bang

There are too many misconceptions about the original meaning of the expression *Big Bang*. Going back to the origins of the origin of everything, it turns out that, contrary to what we find in so many places, Fred Hoyle was not the very first to pronounce these two words together –in a cosmological context– in his famous BBC lecture of the year 1949. Actually, since the discovery during the late 1920s of the fact that distant spiral nebulae were speeding away from us at very high velocities –proportional to their distances (Hubble's law) – many cosmologists got convinced that, at some point back in the past, an explosion of some kind need have necessarily occurred. This would be responsible for having set out all those celestial objects in recession.

Cambridge astronomers and theoreticians, and this included the famous Arthur Eddington, often used the expressions *Bang* and *Big Bang* during the 1930s, in order to designate that cosmic explosion. To repeat, these terms had, therefore, the meaning of an original thrust, or big thrust, produced by some kind of cosmic explosion or instantaneous force of unknown nature, which was necessary in order to explain the high recession speeds of the galaxies. Vesto Slipher had reported this recession in 1914 already, in the year's meeting of the American Astronomical Society. Definitely, he was the first to discover that the beautiful model of the universe, eternal and static, was in serious trouble. This was a very remarkable observation and he was so convincing that his talk received from the audience, chronicles say, a long and standing applause.

It is thus clear that Hoyle did not invent the term *Big Bang*, in his acclaimed popular talk on the BBC of March 28th, 1949. However, he did give to it a radically different meaning, with scientific roots embedded in the deepest principles of Einstein's Theory of General Relativity. Even today, after 70 years, only specialists in this theory can actually understand the meaning of his extremely precise words. Thus spoke Hoyle:

[Lemaître's model implies that] *"... all matter in the universe was created in one Big Bang at a particular time ..."*

He pronounced these words, reports say, with an intonation clearly meaning that this fact was completely impossible, utterly absurd.

Fred Hoyle had been the first person to discover that all of us are stardust. This is, that most of the elements, of the atoms in our body could not have been formed in this initial stage of the cosmos, but only much later in its evolution, after galaxies appeared and stars evolved, in explosions of novae and supernovae (the now standard theory of stellar nucleosynthesis, which he pioneered) [Very recent work sets the formation of the first stars at about 180 million years and the first supernova explosions at about 80 million years later, in a colder than expected medium which seems to point to the presence of dark matter.] In his obituary *"Stardust memories*", written by John Gribbin and published in *The Independent* in 2005, there is a nice account of all that.

With Gold and Bondi, Hoyle had proposed the *Steady State Theory,* in an attempt at recomposing the static model of the universe that had reigned without rival until the already mentioned discoveries. To keep the cosmic energy density constant, in spite of the recession of the galaxies, matter and energy had to be created in their model, what was done by means of a creation field, out of nothing. This happened in faraway regions of the cosmos, through expansions of the fabric of space, in small proportions (e.g., many little Bangs); just in order to accurately compensate the decrease of the matter/energy density. Such creation seemed perfectly reasonable using GR. What was completely crazy (for Hoyle) was to imagine that "*all matter in the universe"* could have been created in only *"one Big Bang"* in a short instant of time in the past. Only a crazy mind would imagine this possibility. Such is the precise meaning of Hoyle's sentence, word by word (his particular intonation included).

By now it should be clear to the reader that Hoyle gave to the term *Big Bang* a completely different meaning from which it had had in Cambridge until that date. Indeed, from being an ordinary explosion, which simply set in motion the *pre-existing* masses of the cosmos, he converted it into a *creation* push, and incredibly huge expansion of the fabric of space, an enormous instantaneous negative pressure, which would allow for the possibility of the creation of the formidable amount of positive mass and energy of the whole universe. And all this starting out of nothing, in a unique creation blow, *one Big Bang*. In fact, Einstein's theory allows for this to happen (and many other things) without breaking at any stage the energy conservation principle (energy balance). However, the big question was, what *precise* mechanism could be invoked as responsible for such enormous blow up? No one, according to Hoyle.

But, alas, exactly thirty years later, an American PhD in theoretical physics appeared, with name Allan Guth, who was about to finish his last Post-Doc contract, and thus on the verge of being expelled from the American University system. Faced up with the imperious necessity to make some extraordinary discovery, and quickly, he managed to give birth to a brand new revolutionary theory, which he named *Cosmic Inflation*. With it, Guth was not just able to do what Hoyle considered as absolutely impossible, but, on top of it and in a single stroke, he solved all the endemic problems of the universe models with an origin, which had been accumulating during the preceding decades (like the horizon, causality, and absence of monopoles problems). The Physics upon which Guth grounded his theory was exactly the same that Hoyle and his colleagues had invoked in their *Steady State Theory*, namely (the reader will have surely guessed it by now) no other than General Relativity.

Until his death, Hoyle strongly defended this similarity, proclaiming on many occasions that inflation had not much more in it than his old theory of many years ago. This is not true, by any means. What is indeed true is the fact that the deep roots, the fundamental principles on which both theories stand are exactly the same. However, inflationary models, which we now count by the dozens, are much more elaborated and predictive.

Anyhow, the fact that there are so many models of inflation is not actually desirable. Allow me a last reflection. When we compare this with the situation of one hundred years ago, with the extraordinary beauty of Einstein's theory, we feel a bit disappointed. When he formulated it in 1915, General Relativity (as Special Relativity, before) was the result of pure human logic, with just the help of his 'most happy thought', namely the equivalence principle, and of the observational fact that the speed of light was constant. With the aid of a couple of extra (mathematical) considerations, the theory was *unique*, that is, the only possible theory for the universe, but for the simple addition (or subtraction) of a pure constant, the now so famous cosmological constant. He actually introduced it in 1917 in order to cope with the existing static model of the cosmos, discussed above.

Now, in comparison, the situation is much darker (to say it in modern terms). Even more, if we take into account that, additionally, Einstein's field equations have *just one* family of solutions describing a universe like ours, homogeneous and isotropic at large scale: the very famous Friedmann-Lemaître-Robertson-Walker model with cold dark matter and cosmological constant (this accounts for the dark energy component, which, as with dark matter, I have not discussed here). It is what we call nowadays the Standard Cosmological Model.

It is this model that has to be supplemented with a theory of inflation. Now, the question is: *with which one?*

Emilio Elizalde

For more details: arXiv:1801.09550 [physics.hist-ph]

For Misao Sasaki,
on his retiring from YITP,
with very best wishes
for a long and happy life

Thank You

有難う 御座います