Higher Derivative Field Theories

Marco Crisostomi

Université Paris Saclay – LPT, IPhT, DAp

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Based on [arXiv:1703.01623] and [arXiv:1710.04531]

in collaboration with:

Remko Klein, Diederik Roest – Karim Noui, Christos Charmousis, David Langlois

Historical Overview

- 1971 – **Lovelock**

  the most general metric theory of gravity yielding conserved second order equations of motion in arbitrary number of dimensions

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Second order field equations

What's wrong with higher order field equations?

Ostrogradsky theorem – 1850

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$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \qquad \overset{\text{nd}}{\Longrightarrow} \qquad \ddot{q} = F(q,\, \dot{q})$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} \qquad \text{nd} \Longrightarrow \dot{q} = f(q,\, p) \qquad H(q,\, p) = p\, f - L(q,\, f)$$

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- Higher derivative $L = L(q,\, \dot{q},\, \ddot{q})$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0 \qquad \overset{\text{nd}}{\Longrightarrow} \qquad \ddddot{q} = F(q,\, \dot{q},\, \ddot{q},\, \dddot{q})$$

$$Q \equiv \dot{q}\,, \quad p \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\,, \quad P \equiv \frac{\partial L}{\partial \ddot{q}}\,, \qquad \text{nd} \Longrightarrow \ddot{q} = f(q,\, Q,\, P)$$

$$H(q,\, Q,\, p,\, P) = p\, Q + P f - L(q,\, Q,\, f)$$

H linear in p $\Rightarrow$ unbounded energy

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WAY OUT: break the assumptions

Evading the Ostrogradsky instability

Assumptions: 1) two variables 2) degenerate Langlois & Noui [arXiv:1510.06930]

- $L = L(q_1, \dot{q}_1, \ddot{q}_1, q_2, \dot{q}_2) \longrightarrow L(q_1, Q, \dot{Q}, q_2, \dot{q}_2) + \lambda(\dot{q}_1 - Q)$

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Generalization: Motohashi et al. [arXiv:1603.09355]; Klein & Roest [arXiv:1604.01719]

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Primary constraints $\iff 0 = P_{(mn)} \equiv v_m^A L_{\dot{\psi}_A \dot{\psi}_B} v_n^B$

Secondary constraints $\iff 0 = S_{[mn]} \equiv 2\, v_m^A L_{\dot{\psi}_{[A} \psi_{B]}} v_n^B$

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Field theories: [arXiv:1703.01623]

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1 primary constraint $\implies$ $\exists$ **1** secondary constraint

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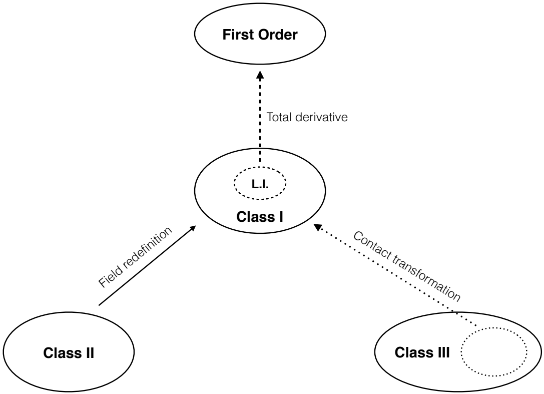

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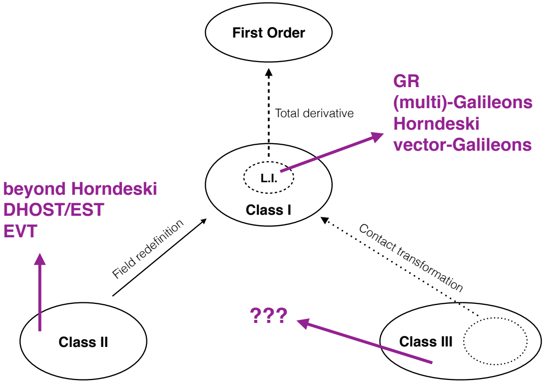

Pure Metric Theories

$$S[g_{\mu\nu}] = \int d^4x \sqrt{-g}\, L\left(g_{\mu\nu},\, \partial_\rho g_{\mu\nu},\, \partial_\rho\partial_\sigma g_{\mu\nu}\right) \qquad \supseteq \qquad \dot{K}_{ij} \qquad \supseteq \qquad \ddot{\gamma}_{ij}$$

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FULLY DEGENERATE THEORIES

6 primary constraints $\iff$ $\mathcal{A}^{ij,\lambda m}(x,y) \equiv \dfrac{\partial^2 L}{\partial \ddot{\gamma}_{ij}(x) \partial \ddot{\gamma}_{\lambda m}(y)} = 0$

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6 secondary constraints $\implies$ EH

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C has NO secondary constraints $\longrightarrow$ 5 dof (3 Ostrogradsky modes)

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PARTIALLY DEGENERATE THEORIES

5 primary constraints $\iff$ Rank $\left[\mathcal{A}^{ij,\lambda m}(x,y)\right] = 1$

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$f(R)$ $f(GB)$ $f(P) \longrightarrow$ Chern-Simons gravity Jackiw & Pi [gr-qc/0308071]

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CS is conformal invariant $\longrightarrow$ 4 dof

4 secondary constraints missing :(

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$$S[g_{\mu\nu}] = \int d^4x \sqrt{-g}\, L\left(g_{\mu\nu},\, \partial_\rho g_{\mu\nu},\, \partial_\rho \partial_\sigma g_{\mu\nu}\right) \qquad \supseteq \qquad \dot{K}_{ij} \qquad \supseteq \qquad \ddot{\gamma}_{ij}$$

PARTIALLY DEGENERATE THEORIES

5 primary constraints $\iff$ Rank $[\mathcal{A}^{ij,\lambda m}(x,y)] = 1$

$f(R)$ $f(GB)$ $f(P) \longrightarrow$ Chern-Simons gravity Jackiw & Pi [gr-qc/0308071]

5 secondary constraints $\implies$ f(R) , f(GB) + X

CS is conformal invariant $\longrightarrow$ 4 dof

4 secondary constraints missing :(

In the Unitary gauge is OK ! :)

Chiral Scalar-Tensor Theories

One new ingredient: Levi-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$

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First derivatives of the scalar field only:

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Including second derivatives of the scalar field:

$$L_1 = \varepsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi^\rho \phi_\mu \phi^\sigma_\nu$$

$$L_2 = \varepsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi^\rho_\mu \phi^\sigma_\nu \,, \qquad L_3 = \varepsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi^\sigma \phi^\rho_\mu \phi^\lambda_\nu \phi_\lambda \,,$$

$$L_4 = \varepsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \phi_\nu \phi^\rho_\mu \phi^\sigma_\lambda \phi^\lambda \,, \qquad L_5 = \varepsilon^{\mu\nu\alpha\beta} R_{\alpha\rho\sigma\lambda} \phi^\rho \phi_\beta \phi^\sigma_\mu \phi^\lambda_\nu \,,$$

$$L_6 = \varepsilon^{\mu\nu\alpha\beta} R_{\beta\gamma} \phi_\alpha \phi^\gamma_\mu \phi^\lambda_\nu \phi_\lambda \,, \qquad L_7 = (\Box\phi) L_1$$

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6 (+ 1) primary constraints $\implies$ ✓ tuning the free functions

6 (+ 1) secondary constraints $\implies$ X no way

Chiral Scalar-Tensor Theories

Unitary gauge:

First derivatives of the scalar field only

$$\begin{aligned} S_{UG} &= \frac{2\,\dot{\phi}^2\,\epsilon^{ij\lambda}}{N}\Big[2\,(2a_1 + a_2 + 4a_4)\Big(K K_{mi} D_\lambda K^m_j + {}^{(3)}R_{mi} D_\lambda K^m_j - K_{mi} K^{mn} D_\lambda K_{jn}\Big) \\ &- (a_2 + 4a_4)\Big(2\,K_{mi} K^n_j D_n K^m_\lambda + {}^{(3)}R_{j\lambda m}{}^n D_n K^m_i\Big)\Big] \end{aligned}$$

Including second derivatives of the scalar field

$$\begin{aligned} S_{UG} &= \frac{\dot{\phi}^3}{N^4}\,\epsilon^{ij\lambda}\Big\{2\,N\Big[b_1 N K_{mi} D_\lambda K^m_j + (b_4 + b_5 - b_3)\,\dot{\phi} K_{mi} K^n_j D_n K^m_\lambda\Big] \\ &+ \dot{\phi}\Big[b_3\ {}^{(3)}R_{j\lambda m}{}^n K^m_i D_n N - 2\,(b_4 + b_5)\ {}^{(3)}R_{m\lambda} K^m_j D_i N\Big]\Big\} \end{aligned}$$

Summary

- It is possible to avoid Ostrogradsky instabilities in higher order field theories in a non-trivial way
- Brand new theories
- New phenomenology – GW in parity breaking scalar-tensor theories

Thank you