GC2024, 29th January 2024
YITP, Kyoto

Adiabatic and non-adiabatic evolution during inflation

David Wands

Institute of Cosmology and Gravitation, University of Portsmouth

arXiv:2311:03281

with Joe Jackson, Hooshyar Assadullahi, Andrew Gow, Kazuya Koyama, Vincent Vennin

and previous work with Chris Pattison, Laura Iacconi, Matteo Fasiello
Sam Leach, Andrew Liddle and Misao Sasaki

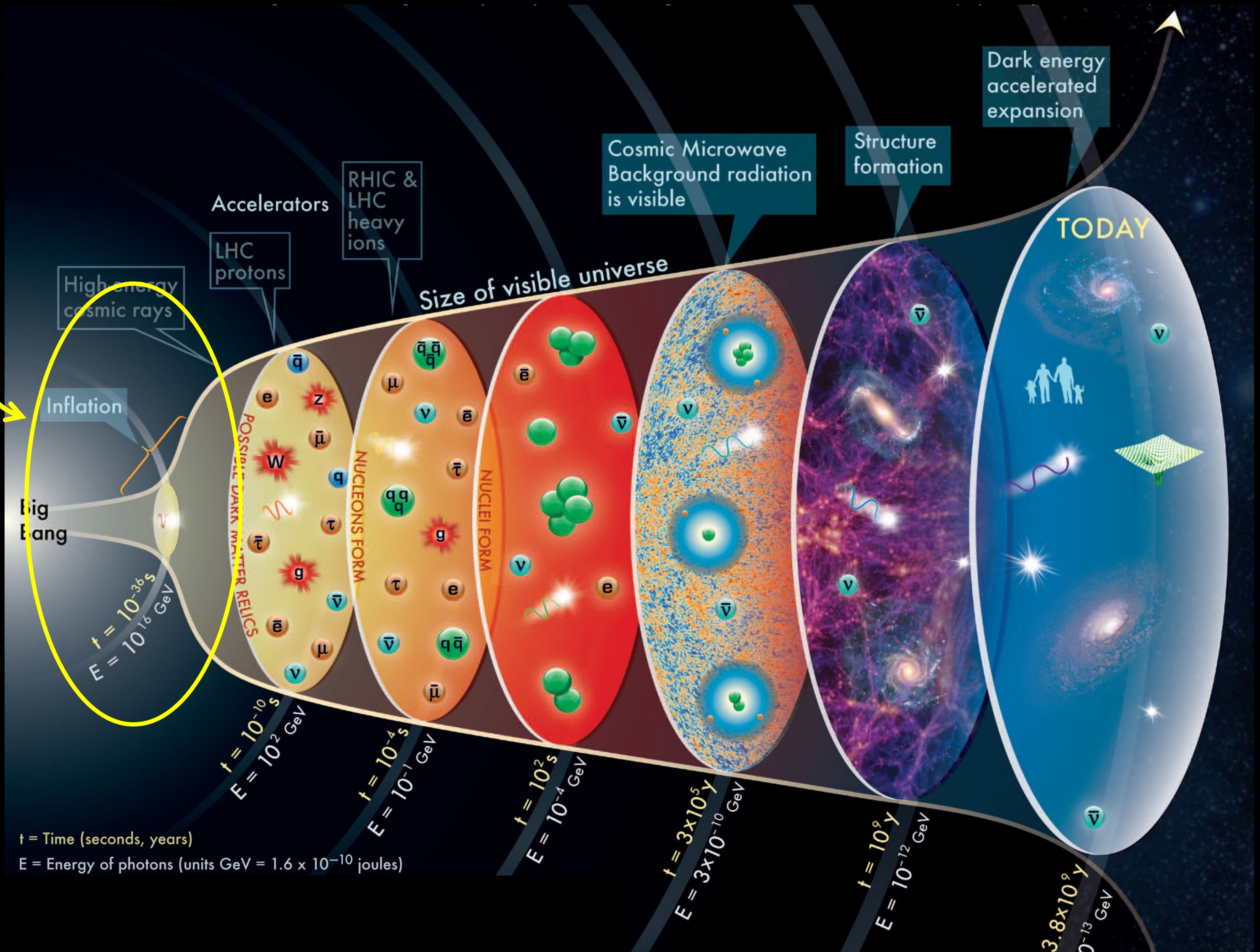
outline

- Slow-roll inflation
  - weakly scale-dependent, adiabatic density perturbations
  - as seen in the CMB anisotropies and LSS
- Ultra-slow roll inflation
  - enhanced density perturbations on small scales
  - could be seen in gravitational relics (SGWB, PBH)
- Sudden transition
  - need to kick slow roll into ultra-slow roll
  - leads to particle production and non-adiabatic perturbations
- Challenging to study nonlinearity
  - separate universe approach breaks down on some scales

Inflation

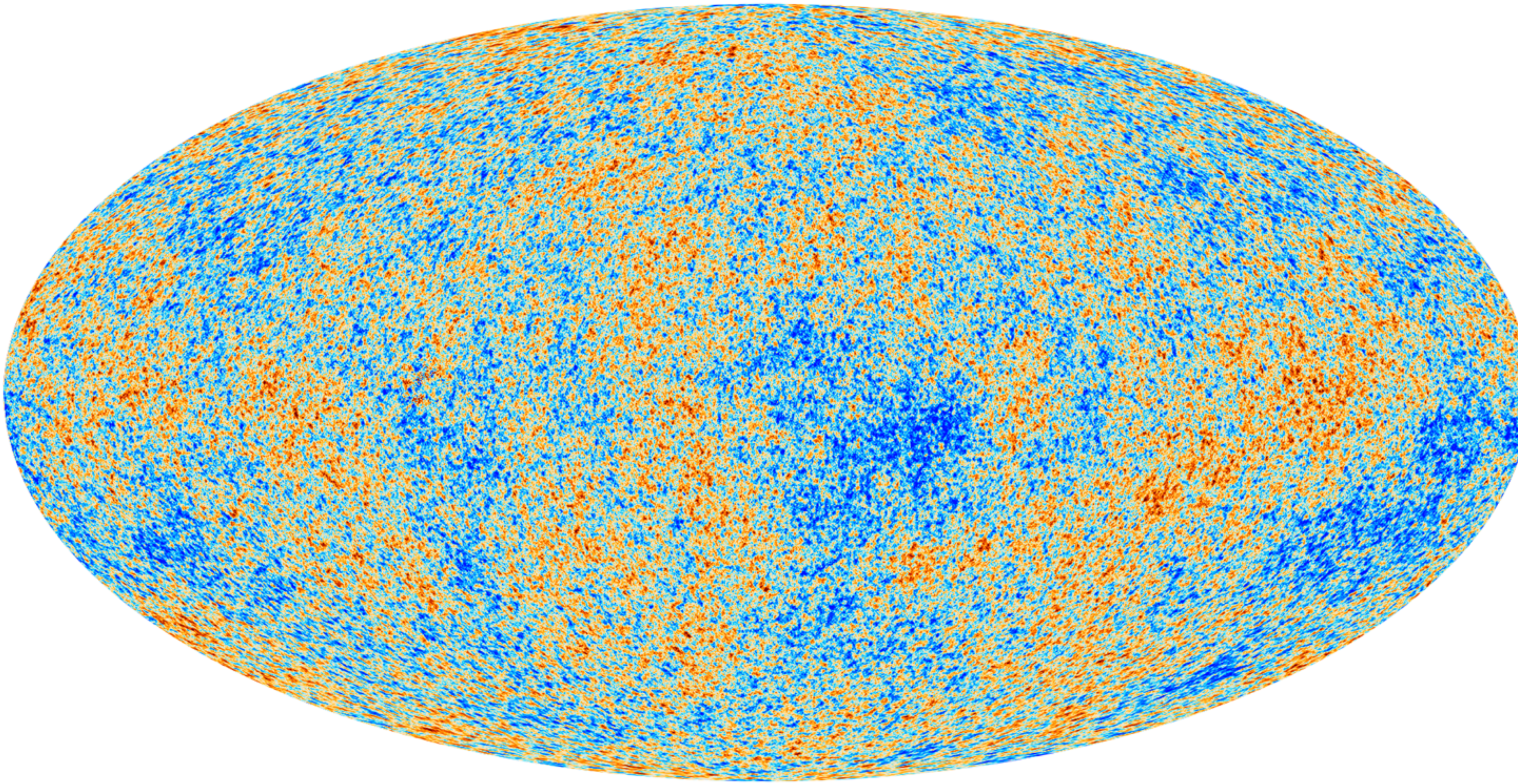
= accelerated expansion in the very early universe

- classical expansion smoothes, isotropises and flattens
- *quantum fluctuations create inhomogeneous structure*



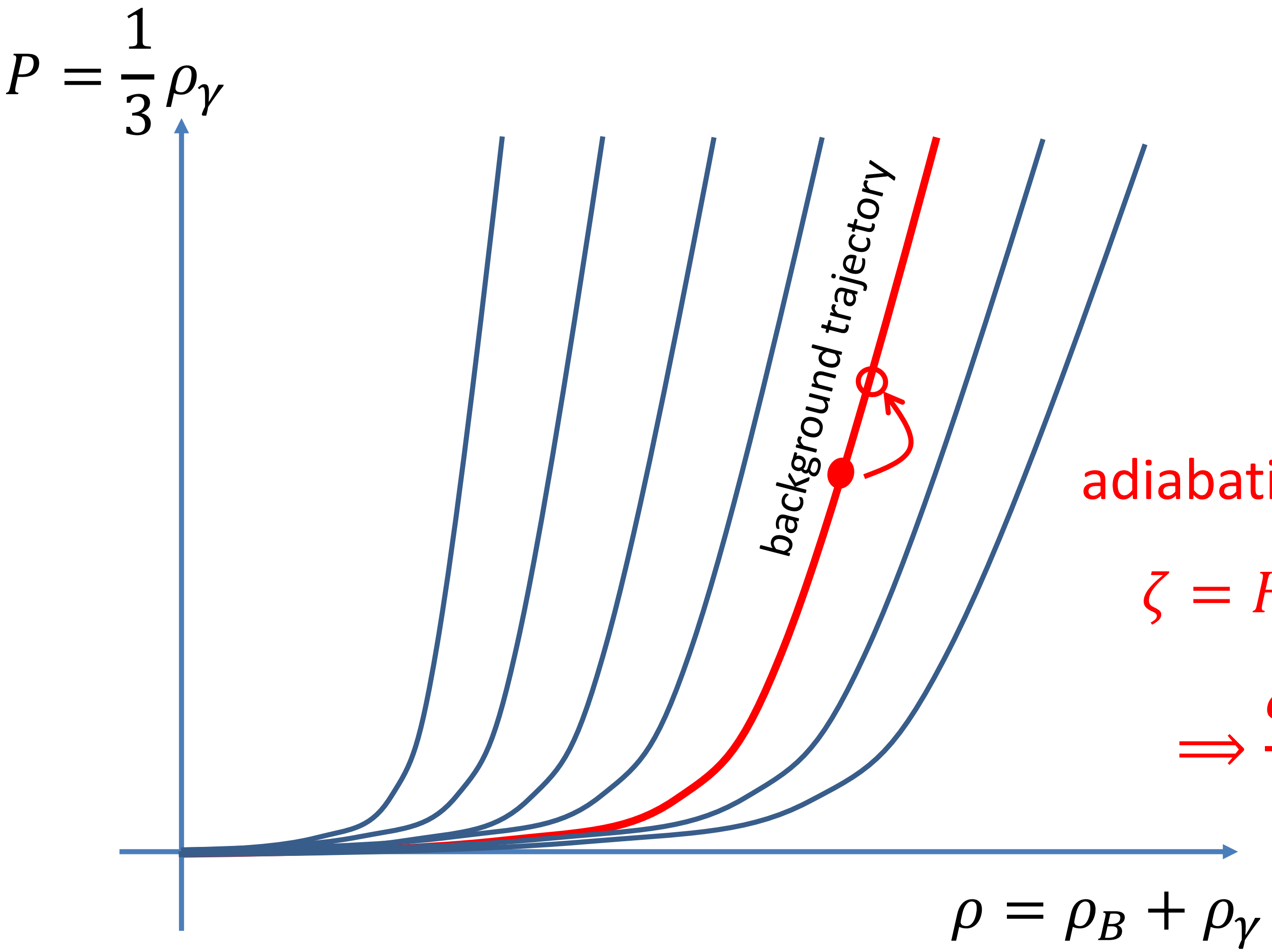
primordial density perturbations

cosmic microwave background temperature anisotropies observed by Planck satellite



$$\langle \zeta^2 \rangle \sim \left\langle \frac{\delta T^2}{T^2} \right\rangle \approx 10^{-10}$$

Baryon-photon plasma pressure vs density

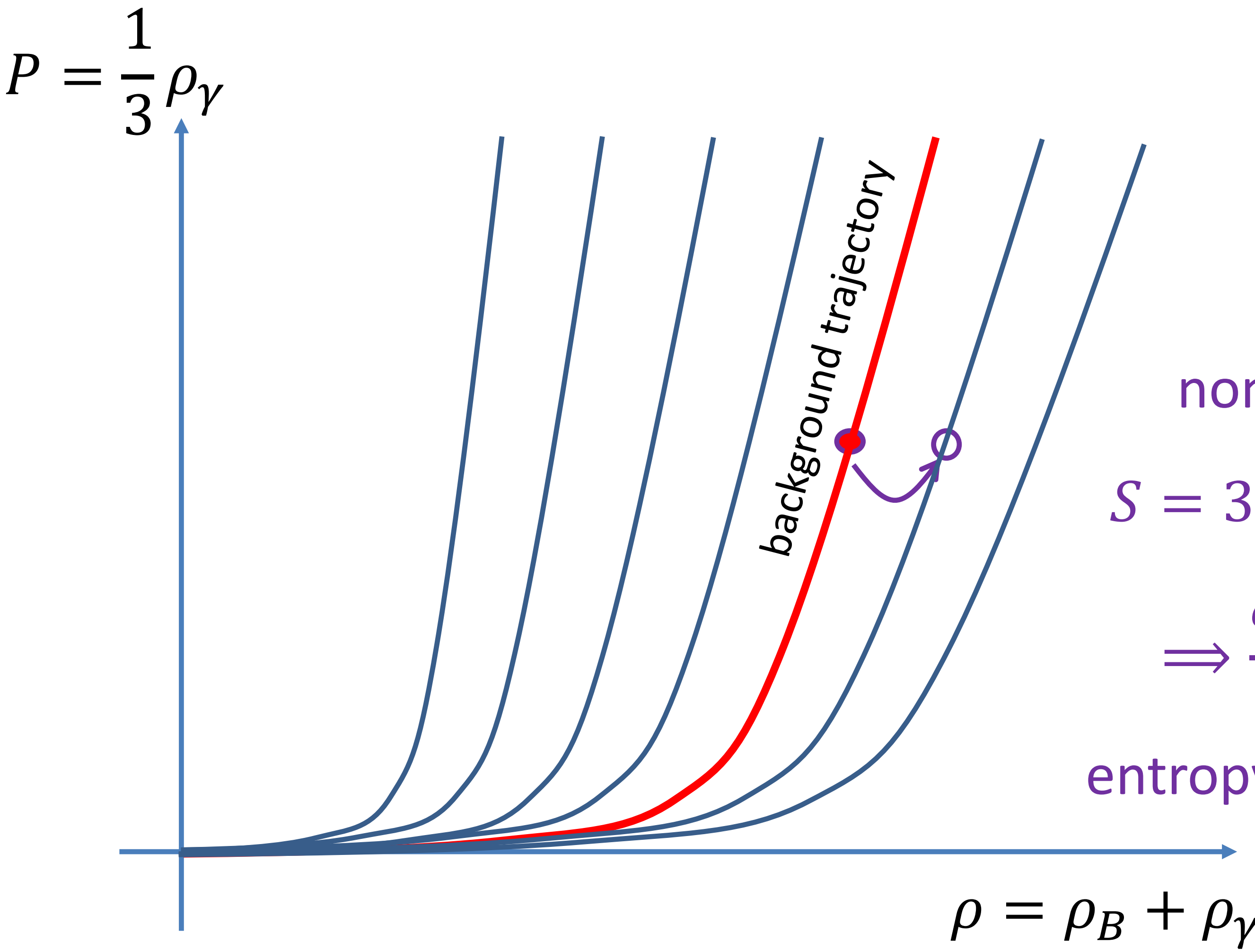


adiabatic perturbation

$$\zeta = H\frac{\delta\rho}{\dot{\rho}} = H\frac{\delta P}{\dot{P}}$$

$$\Longrightarrow \frac{\delta n_\gamma}{n_\gamma} = \frac{\delta n_B}{n_B}$$

Baryon-photon plasma pressure vs density



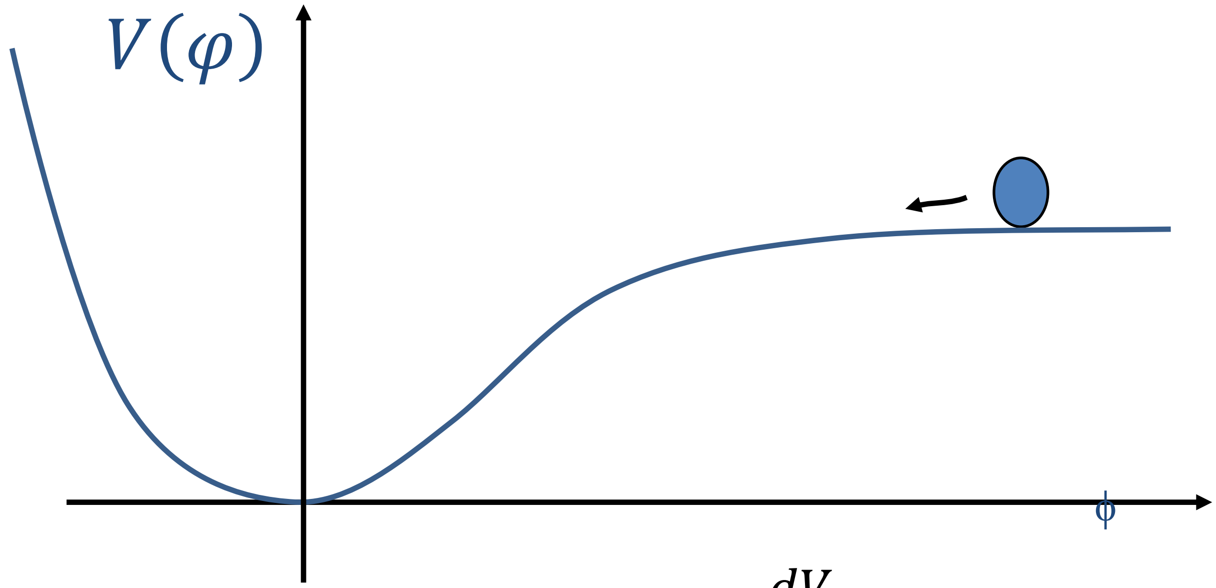
non-adiabatic

$$S = 3H\left(\frac{\delta P}{\dot{P}} - \frac{\delta \rho}{\dot{\rho}}\right)$$

$$\Longrightarrow \frac{\delta n_\gamma}{n_\gamma} \neq \frac{\delta n_B}{n_B}$$

entropy perturbation

Classical inflation:



$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0$$

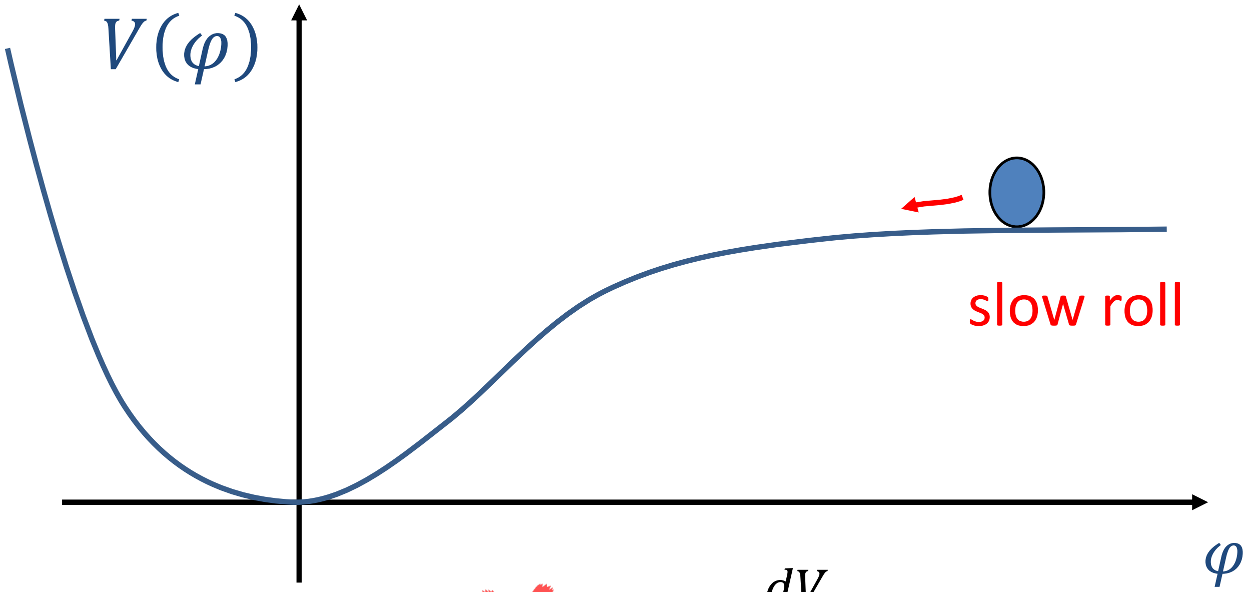
$$H^2 = \frac{8\pi}{3M_P^2}\left(V(\varphi) + \frac{1}{2}\dot{\varphi}^2\right)$$

spatially homogeneous scalar field, $\varphi(t)$, in FLRW background

scale factor $a = e^N$, **adiabatic** Hubble expansion, $H \equiv \dot{a}/a$

($\Delta Q = 0$ for isolated system)

Scalar field inflation: $V(\varphi) > \dot{\varphi}^2$



$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0$$

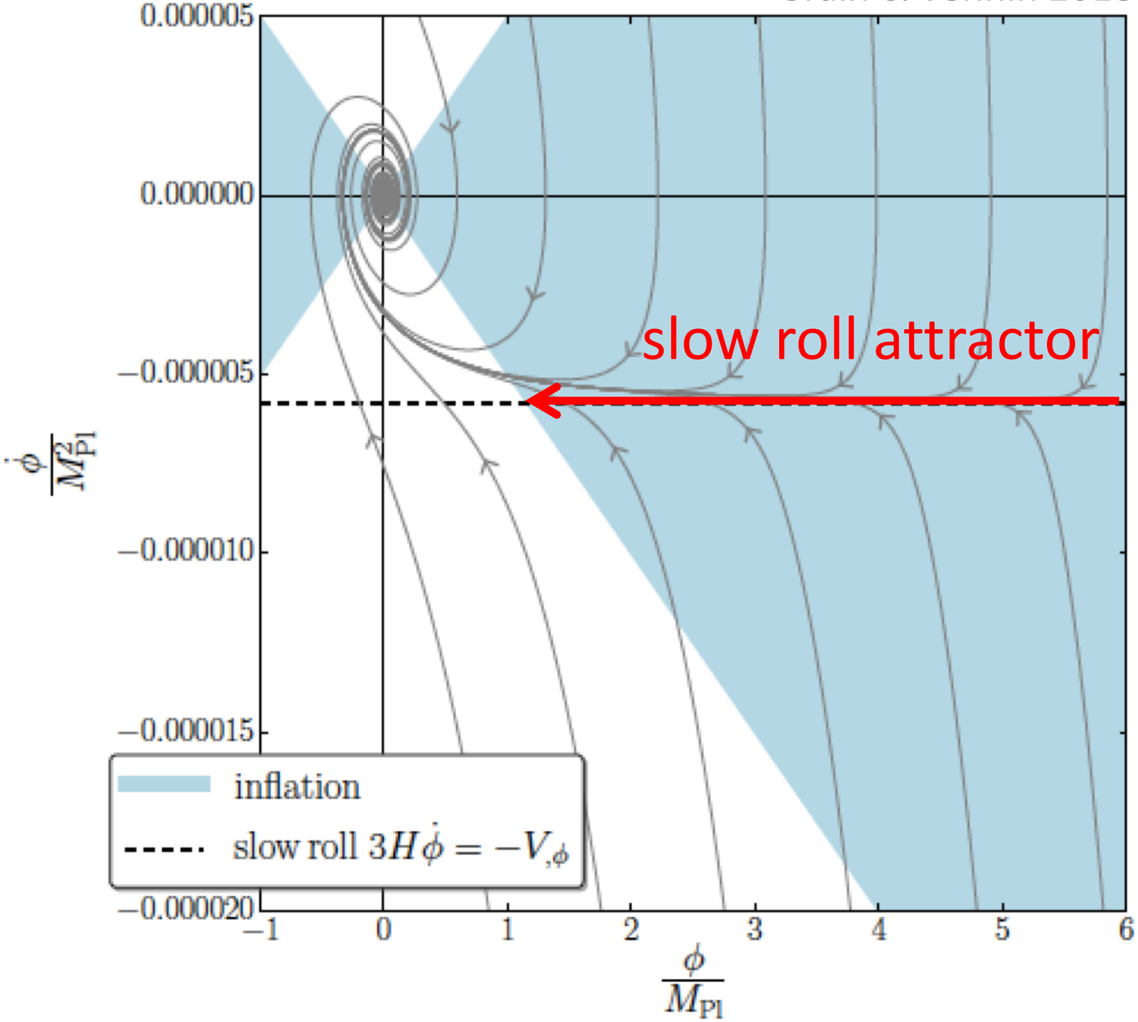
$$H^2 = \frac{8\pi}{3M_P^2}\left(V(\varphi) + \frac{1}{2}\dot{\varphi}^2\right)$$

quasi-de Sitter ($\epsilon \equiv -\dot{H}/H^2 \ll 1$)
damping ≈ driving ($\epsilon_2 \equiv \dot{\epsilon}/H\epsilon \ll 1$)

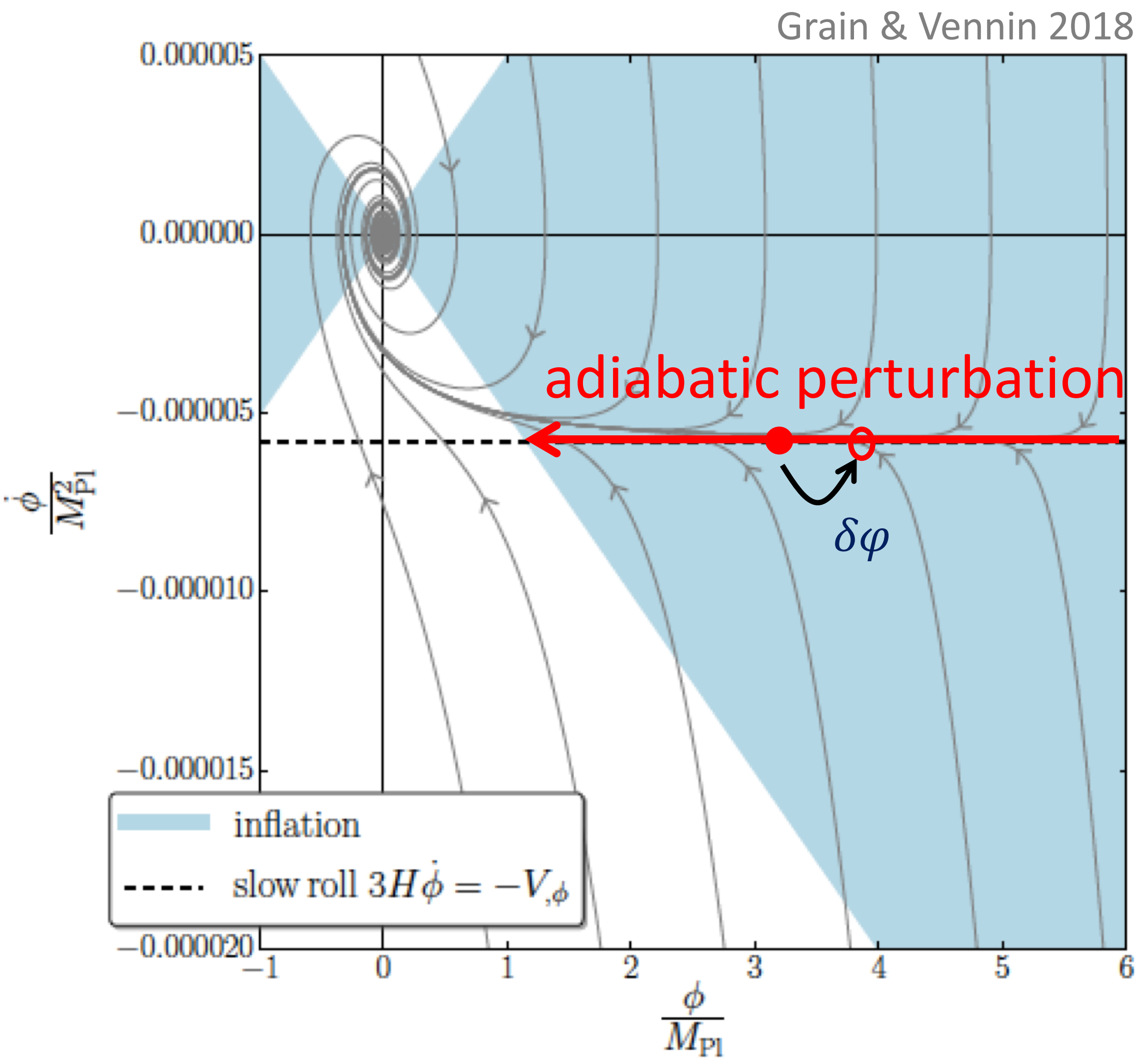
slow roll is a stable attractor whenever it exists

Scalar field inflation: $V(\varphi) = \frac{1}{2} m^2 \varphi^2$

Grain & Vennin 2018



Scalar field inflation: $V(\varphi) = \frac{1}{2}m^2\varphi^2$



$$\zeta = H\frac{\delta\rho}{\dot\rho} = H\frac{\delta P}{\dot P}$$

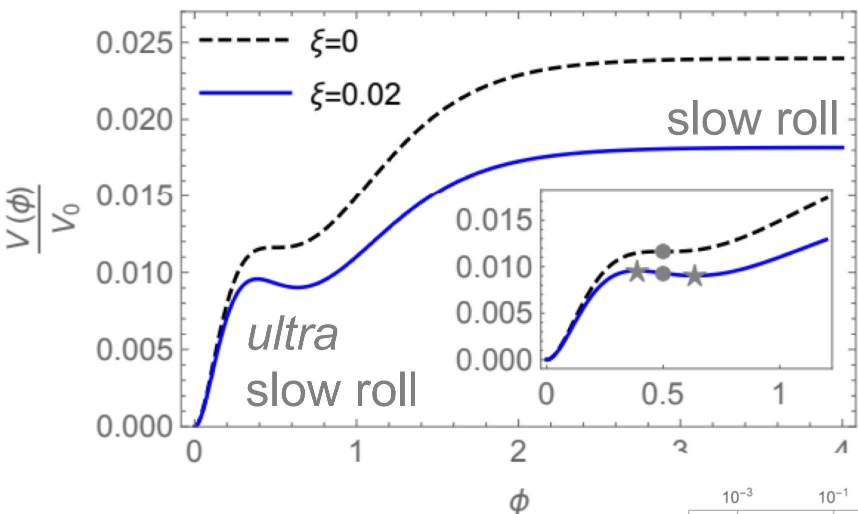
$$\Rightarrow \frac{\delta\varphi}{\dot\varphi} = \frac{\delta\dot\varphi}{\ddot\varphi}$$

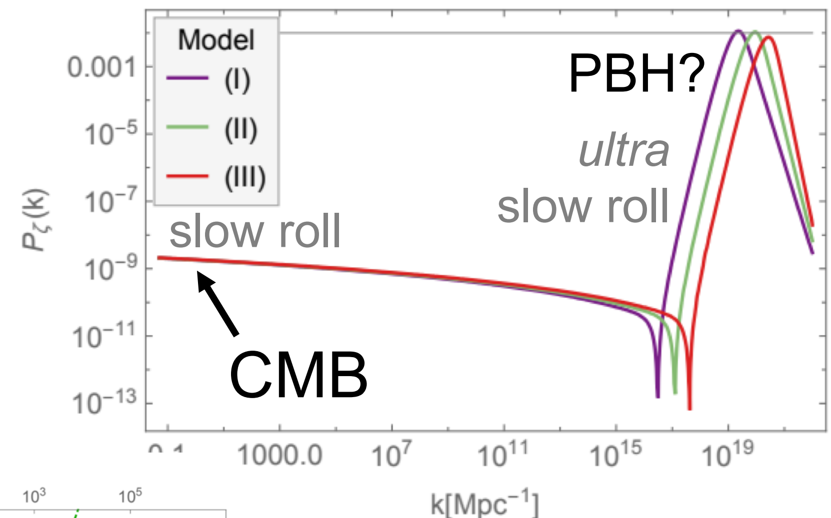
for example

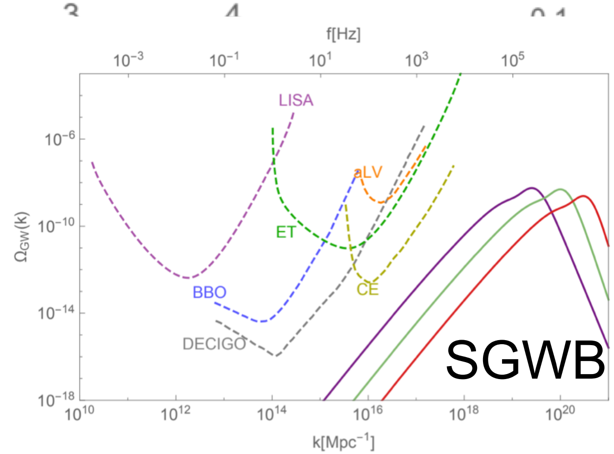
inflection points in α-attractor inflation

Dalianis, Kehagias & Tringas (2019); Iacconi, Assadullahi, Fasiello & Wands (2022)

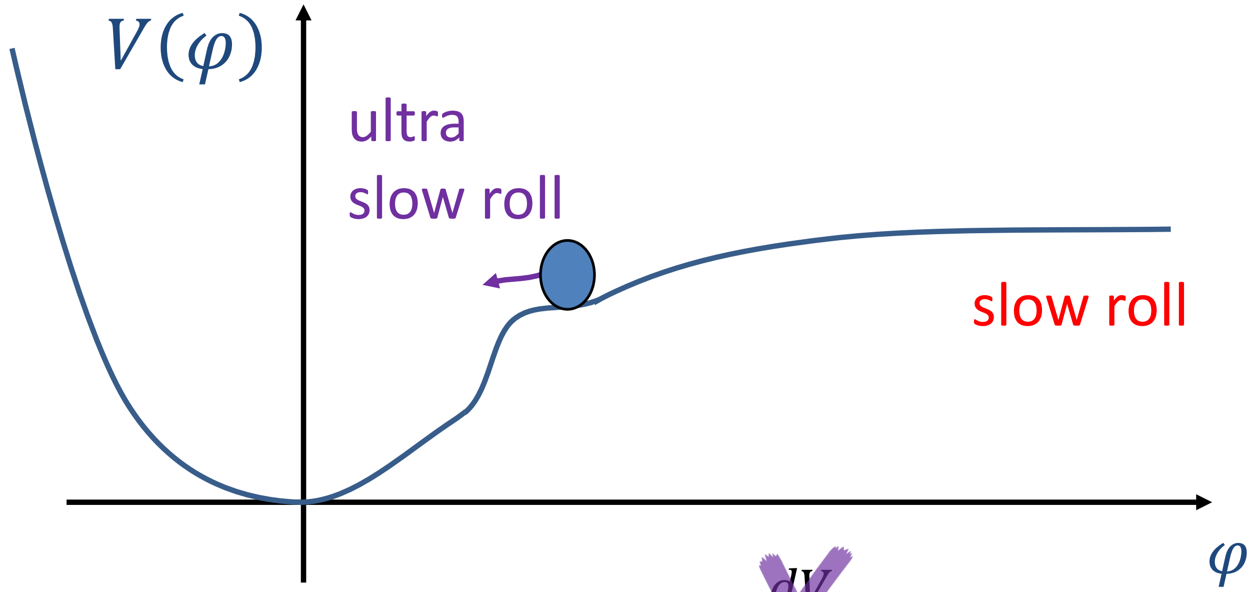
$$V(\phi) = V_0 \left\{ (r_{\text{infl}} - \xi) \tanh\left(\frac{\phi}{\sqrt{6\alpha}}\right) - \tanh^2\left(\frac{\phi}{\sqrt{6\alpha}}\right) + \frac{1}{3r_{\text{infl}}} \tanh^3\left(\frac{\phi}{\sqrt{6\alpha}}\right) \right\}^2$$







Scalar field inflation: $V(\varphi) > \dot{\varphi}^2$



$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0$$

$$H^2 = \frac{8\pi}{3M_P^2}\left(V(\varphi) + \frac{1}{2}\dot{\varphi}^2\right)$$

quasi-de Sitter ($\epsilon \equiv -\dot{H}/H^2 \ll 1$)

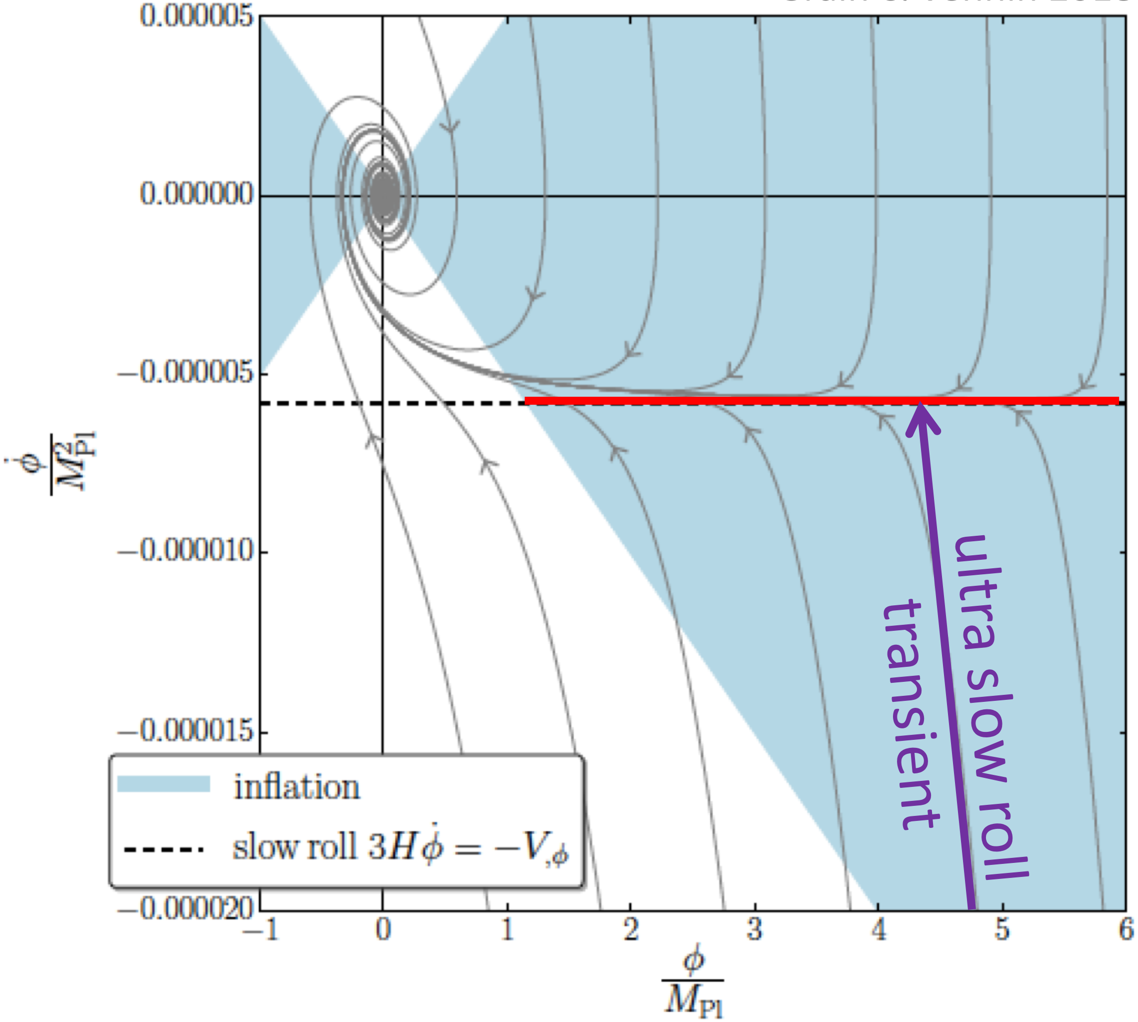
damping >> driving ($\epsilon_2 \equiv \dot{\epsilon}/H\epsilon \approx -6$)

usually a transient phase that relaxes back to slow-roll attractor

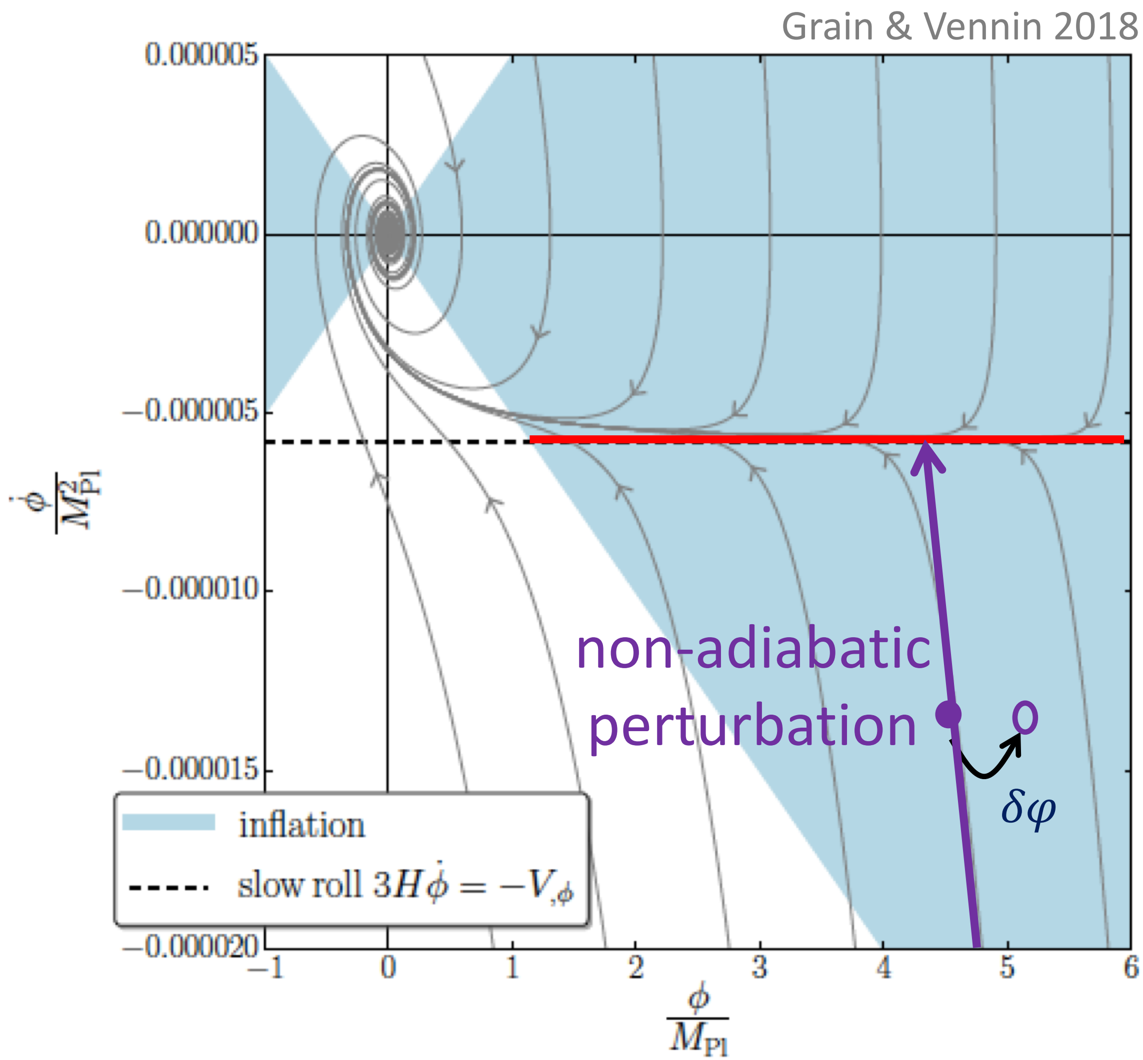
(see Pattison et al 2019)

Scalar field inflation: $V(\varphi) = \frac{1}{2}m^2\varphi^2$

Grain & Vennin 2018



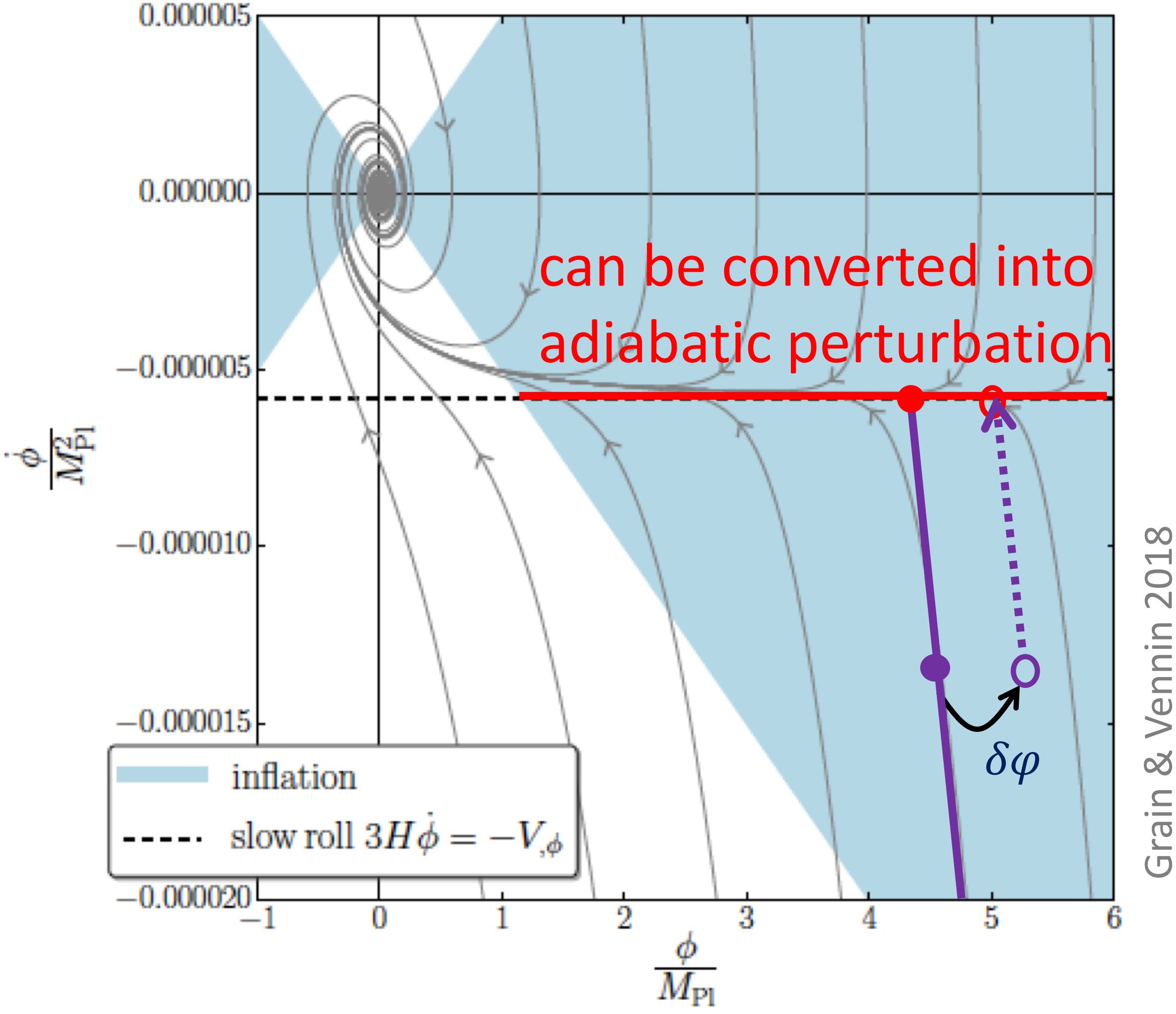
Scalar field inflation: $V(\varphi) = \frac{1}{2}m^2\varphi^2$



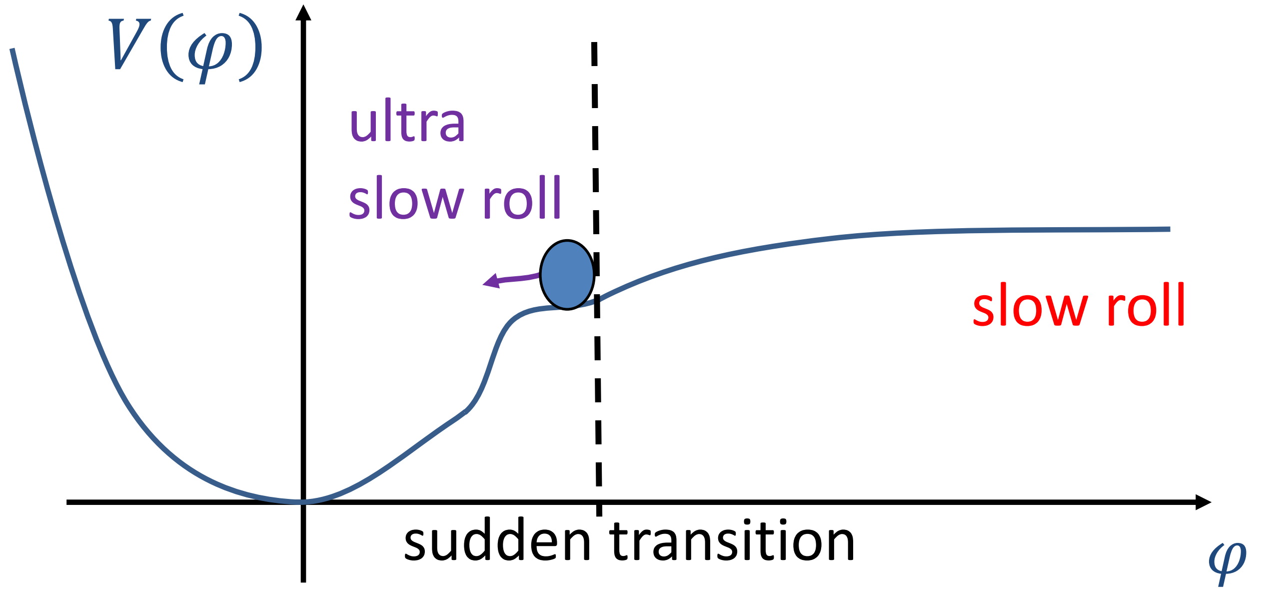
$$S \propto H\left(\frac{\delta\dot\varphi}{\ddot\varphi} - \frac{\delta\varphi}{\dot\varphi}\right)$$

$$\Rightarrow \frac{\delta\rho}{\dot\rho} \neq \frac{\delta P}{\dot P}$$

Scalar field inflation: $V(\varphi) = \frac{1}{2}m^2\varphi^2$



Scalar field inflation: $V(\varphi) > \dot{\varphi}^2$



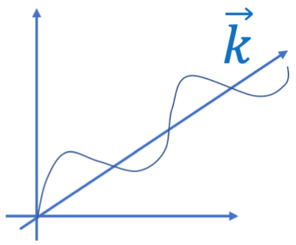
$$[V']^{+}_{-} \neq 0$$

feature in potential, e.g., sudden change in slope

$\rightarrow$ **non-adiabatic** change in quantum fluctuations

(non-Bunch-Davies state)

quantum fluctuations:



- $\boldsymbol{\varphi(t) \to \varphi(t) + \delta\varphi(t, \vec{x})}$ in perturbed FLRW spacetime
- Sasaki-Mukhanov variable, $\boldsymbol{v = a\delta\varphi}$, in spatially-flat gauge

  Fourier modes obey oscillator equation

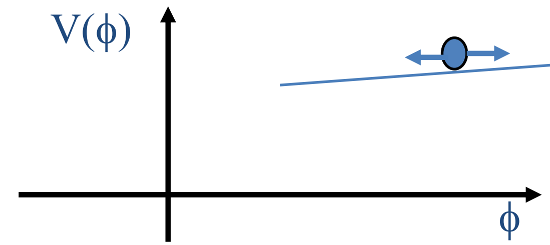
$$\boldsymbol{v_k'' + \left(k^2 + \mu^2(\eta)\right) v_k = 0}$$

  where $' \equiv d/d\eta = a\, d/dt$ and time-dependent mass

$$\boldsymbol{\mu^2 = -\frac{z''}{z}} \quad \text{and} \quad \boldsymbol{z \equiv \frac{a\dot{\varphi}}{H}}$$

- **adiabatic vacuum state**
  - frequency real $\omega_k^2(\eta) = k^2 + \mu^2(\eta) > 0$
  - slowly-varying $|\dot{\omega}_k| \ll \omega_k^2$
  - defined on sub-Hubble scales ($k > aH$) during slow-roll

massless field in de Sitter:



- mode equation

$$v_k'' + \left(k^2 + \mu^2(\eta)\right)v_k = 0$$

where $z \propto a = -1/H\eta$

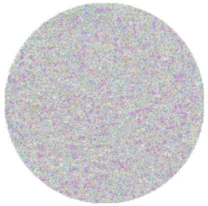
$$\mu^2 = -\frac{z''}{z} = -\frac{2}{\eta^2}$$

- Bunch-Davies vacuum state

$$v_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

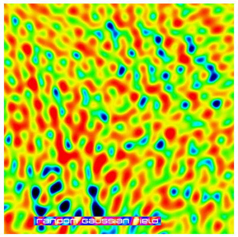
  – $k > aH$: small-scale/underdamped oscillations at early times

$$v_k = \frac{1}{\sqrt{2k}}e^{-ik\eta}$$

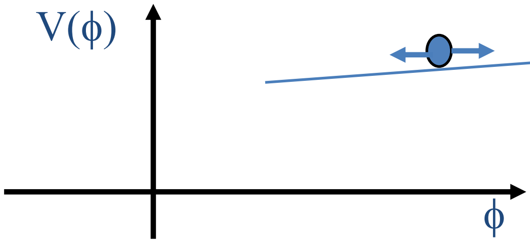


  – $k < aH$: large-scale/overdamped perturbations ``frozen-in''

$$\delta\varphi_k = \frac{v_k}{a} = \frac{-iH}{\sqrt{2k^3}}$$



massless field in de Sitter:



- mode equation

$$v_k'' + \left(k^2 + \mu^2(\eta)\right)v_k = 0$$

where $z \propto a = -1/H\eta$

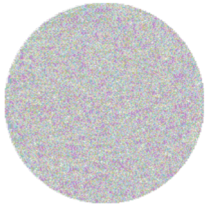
$$\mu^2 = -\frac{z''}{z} = -\frac{2}{\eta^2}$$

- Bunch-Davies vacuum state

$$v_k = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

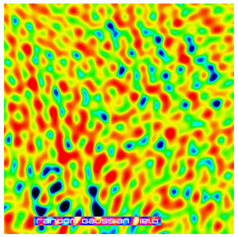
  – $k > aH$: small-scale/underdamped oscillations at early times

$$v_k = \frac{1}{\sqrt{2k}}e^{-ik\eta}$$



  – $k < aH$: large-scale/overdamped perturbations ``frozen-in''

$$\mathcal{P}_{\delta\varphi} = \frac{4\pi}{(2\pi)^3}\left|\frac{v_k}{a}\right|^2 = \left(\frac{H}{2\pi}\right)^2$$



separate universe approach:

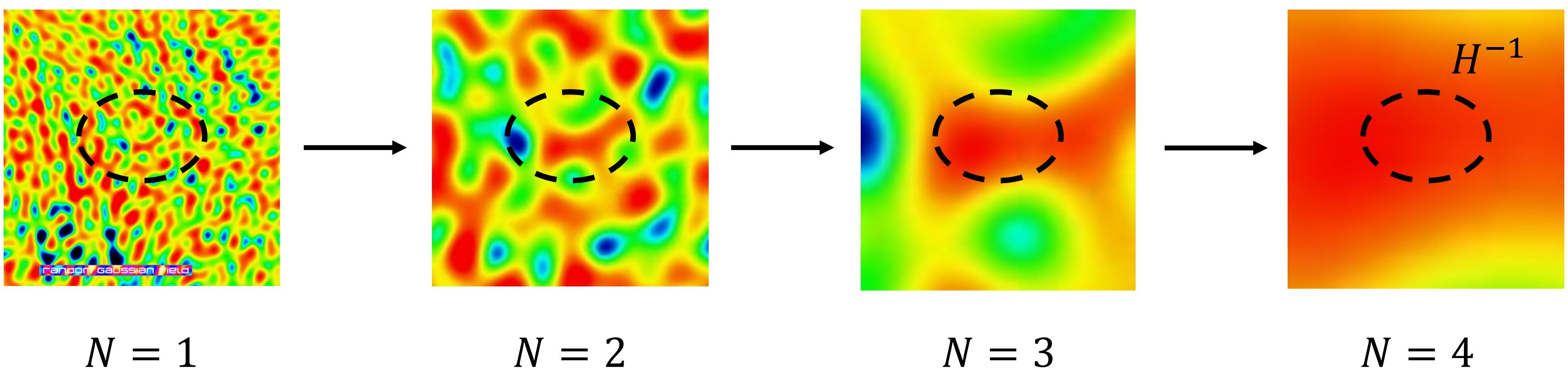
Salopek & Bond (1990); Sasaki & Tanaka (1998);
Wands, Malik, Lyth & Liddle (2001); Rigopoulos & Shellard (2003)

- spatially-homogeneous limit ($k \to 0$) of mode equation

$$v_0'' + \left(\cancel{k^2} - \frac{z''}{z}\right) v_0 = 0$$

good approx on super-Hubble scales ($k < aH$) during inflation

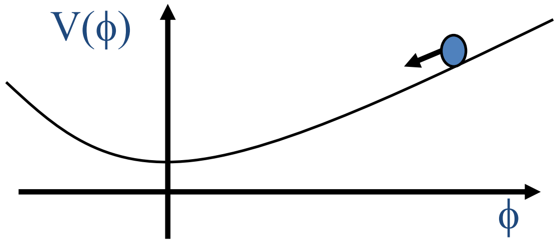
$$a \equiv e^N = e^{\int H\,dt}$$



the coarse-grained field on large scales obeys the same nonlinear equations of motion locally as unperturbed background universe

(neglecting spatial gradients)

separate universes:



- *local* scalar field $\boldsymbol{\varphi_{\vec{x}}(t) \to \varphi(t) + \delta\varphi(t)}$ in FLRW spacetime, scale factor $\boldsymbol{e^{N+\delta N}}$, Hubble expansion, $\boldsymbol{H \to H + \delta H}$

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + 3\dot{\varphi}\delta H + \frac{d^2V}{d\varphi^2}\delta\varphi = 0$$

$$2H\delta H = \frac{8\pi}{3M_P^2}\left(\frac{dV}{d\varphi}\delta\varphi + \dot{\varphi}\delta\dot{\varphi}\right)$$

where dots denote derivatives with respect to *local* proper time

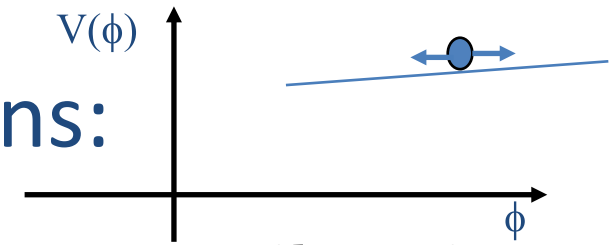
- e.g., massless field in de Sitter ($\delta H = 0$)

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} = 0$$

general solution

$$\delta\varphi = C + Da^{-3}$$

homogeneous perturbations:



- mode equation in spatially-homogeneous limit ($k \to 0$)

$$v_0'' + \left(\cancel{k^2} - \frac{z''}{z}\right) v_0 = 0$$

- general solution

$$v_0 = \tilde{C} z + \widetilde{D} z \int \frac{d\eta}{z^2}$$

  - for massless field in de Sitter $z \propto a = -1/H\eta$

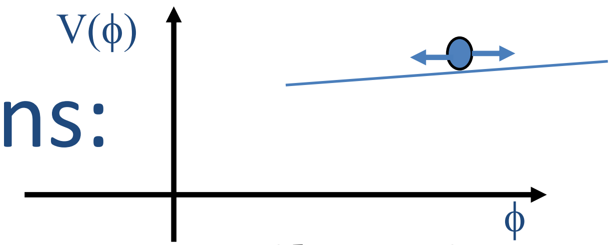
$$\delta\varphi_0 = \frac{v_0}{a} = C + D\eta^3$$

  - identifying particular solution for adiabatic vacuum state

$C = \frac{-iH}{\sqrt{2k^3}}$, scale-invariant growing mode on super-Hubble scales

$D\eta^3 = \frac{H}{3\sqrt{2k^3}} \left(\frac{k}{aH}\right)^3$, decaying on super-Hubble scales

homogeneous perturbations:



- mode equation in spatially-homogeneous limit ($k \to 0$)

$$v_0'' + \left(\cancel{k^2} - \frac{z''}{z}\right) v_0 = 0$$

- general solution

$$v_0 = Cz + D \int \frac{d\eta}{z^2}$$

  – for massless field in de Sitter $z \propto a = -1/H\eta$

$$\delta\varphi_0 = \frac{v_0}{a} = C + D\eta^3$$

  – identifying particular solution for adiabatic vacuum state

$\mathcal{P}_C = \left(\frac{H}{2\pi}\right)^2$, scale-invariant spectrum on super-Hubble scales

$\mathcal{P}_D = \mathcal{O}\left(\frac{k}{aH}\right)^6$, suppressed on super-Hubble scales

the power of separate universes

- ordinary differential equations rather than partial DEs, $\varphi_{\vec{x}}(t)$

- describes nonlinear evolution on large scales

- conserved curvature perturbation for adiabatic perturbations

$$\dot{\zeta} \propto \delta P - \frac{\dot{P}}{\dot{\rho}} \delta\rho$$

- used in stochastic inflation (Starobinsky) to describe nonlinear quantum diffusion of coarse-grained fields during inflation

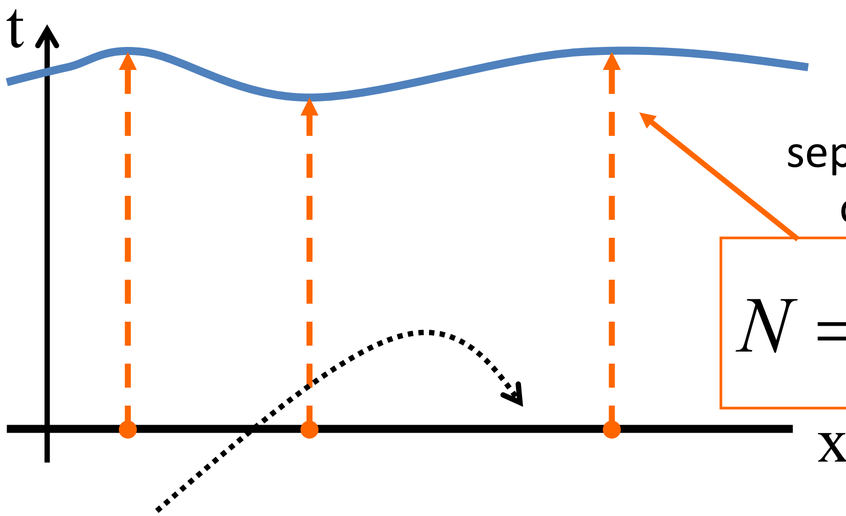
- calculate curvature perturbation in terms of the local integrated Hubble expansion

$$\zeta = \delta N = \delta \left(\int H \, dt \right)$$

nonlinear δN for primordial density perturbations

Starobinsky '85; Sasaki & Stewart '96; Lyth & Rodriguez '05

after inflation: **curvature perturbation ζ constant on uniform-density hypersurface**



$$N = \int_{initial}^{final} H\, dt$$

during inflation: **field perturbations $\phi_I(x,t_i)$ on initial spatially-flat hypersurface**

$$\zeta = N(\varphi_I) - \bar{N} = \sum_I \frac{\partial N}{\partial \varphi_I}\delta\varphi_I + \frac{1}{2}\sum_{I,J}\frac{\partial^2 N}{\partial\varphi_I \partial\varphi_J}\delta\varphi_I\delta\varphi_J + \cdots$$

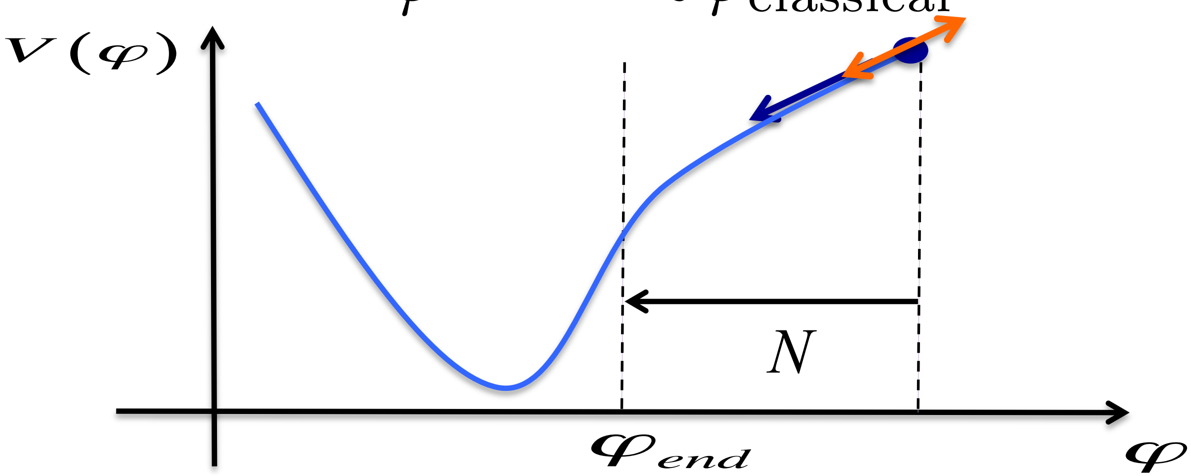
Classical vs quantum inflation

- **Classical** inflation

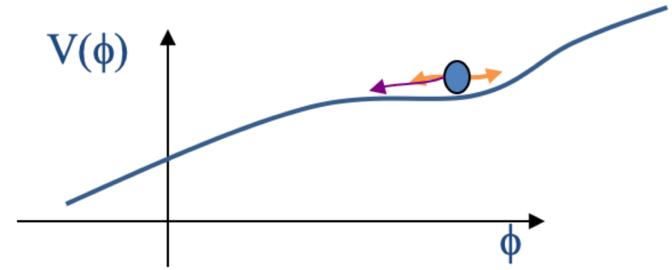
$$N = \int H \, dt = \int \frac{H}{\dot{\varphi}} \, d\varphi$$

- **Quantum** field fluctuations about fixed FLRW background lead to primordial metric perturbations in limit $k \ll aH$

$$\zeta \equiv \delta N = \frac{H}{\dot{\varphi}} \, \delta\varphi = \frac{\delta\varphi_{\text{quantum}}}{\delta\varphi_{\text{classical}}}$$



Ultra-slow-roll inflation:



- 2D phase space for field and momentum, $\pi = \dot{\varphi}/H$

$$\frac{d\varphi}{dN} = \pi$$

$$\frac{d\pi}{dN} = -(3-\epsilon)\pi - \frac{V'}{H^2}$$

- classical solution

  – $\pi \propto e^{-3N}$, $\varphi = \varphi_e + \frac{\pi}{3}(e^{-3N} - 1)$

- non-perturbative δN

$$N(\varphi, \pi) = -\frac{1}{3}\ln\left(1 - \frac{3(\varphi_e - \varphi)}{\pi}\right)$$

$$\zeta = \delta N = -\left(\pi + 3(\varphi_e - \varphi)\right)^{-1}\delta\varphi + \cdots$$

Namjoo, Firouzjahi & Sasaki, arXiv:1210.3692

Cai et al, arXiv:1711.09998

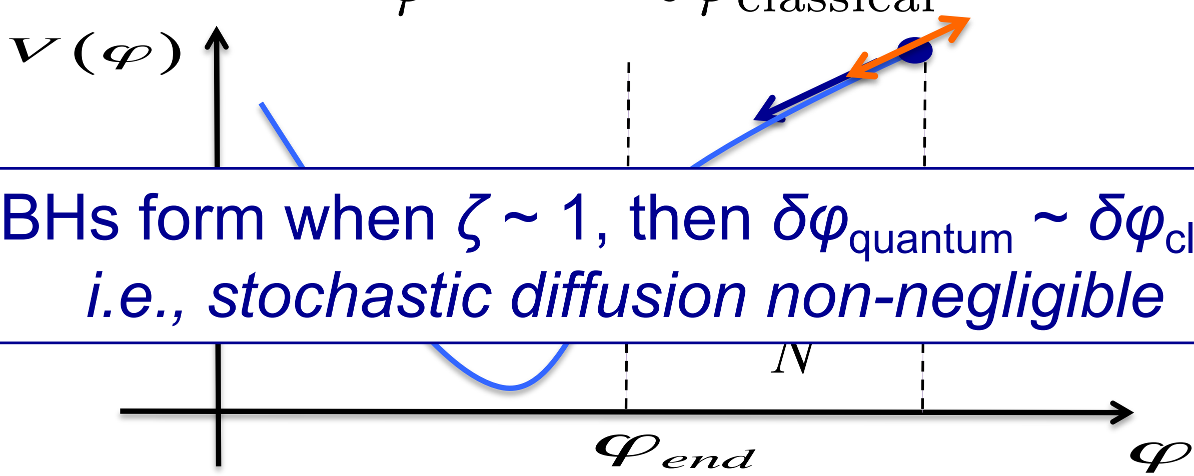
Classical vs quantum inflation

- **Classical** inflation

$$N = \int H\,dt = \int \frac{H}{\dot{\varphi}}\,d\varphi$$

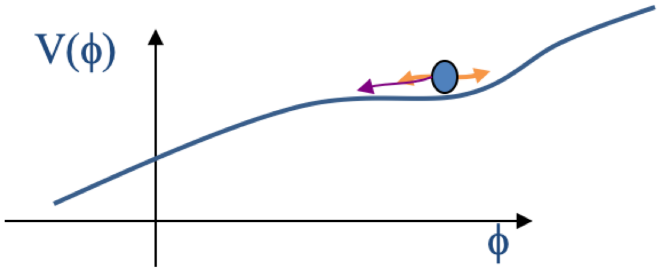
- **Quantum** field fluctuations about fixed FLRW background lead to primordial metric perturbations

$$\zeta \equiv \delta N = \frac{H}{\dot{\varphi}}\,\delta\varphi = \frac{\delta\varphi_{\rm quantum}}{\delta\varphi_{\rm classical}}$$



if PBHs form when $\zeta \sim 1$, then $\delta\varphi_{\rm quantum} \sim \delta\varphi_{\rm classical}$,
i.e., stochastic diffusion non-negligible

Stochastic ultra-slow-roll:



- Langevin evolution of long-wavelength field + momentum, (φ,π), with respect to e-folds, N:

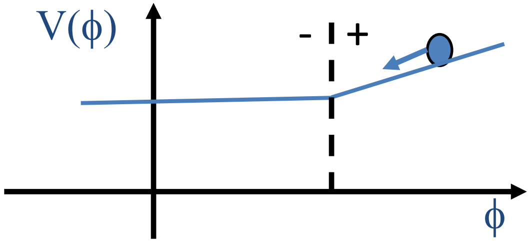
$$\frac{d\varphi}{dN} = \pi \quad + \hat{\xi}_\varphi$$
$$\frac{d\pi}{dN} = -(3-\epsilon)\pi - \frac{V'}{H^2} \quad + \hat{\xi}_\pi$$

- **ultra-slow roll**, Bunch-Davies state on super-Hubble ($k_{cg} << aH$):

$$\langle \hat{\xi}_\varphi(N_1)\hat{\xi}_\varphi(N_2)\rangle = \left(\frac{H}{2\pi}\right)^2 \delta(N_1 - N_2)\,, \quad \langle \hat{\xi}_\pi(N_1)\hat{\xi}_\pi(N_2)\rangle = 0$$

Firouzjahi, Nassiri-Rad & Noorbala (2018)
Assadullahi, Pattison, Vennin & Wands, 2101.05741

piecewise linear potential:



- Starobinsky (1992):

$$V(\varphi) = \begin{cases} A_+(\varphi - \varphi_T) & \text{for} \quad \varphi > \varphi_T \\ A_-(\varphi - \varphi_T) & \text{for} \quad \varphi < \varphi_T \end{cases}$$

- mode equation

$$v_k'' + \left(k^2 + \mu^2(\eta)\right) v_k = 0$$

where

$$\mu^2 = -\frac{2}{\eta^2} + \frac{3}{\eta_T}\left(\frac{A_+ - A_-}{A_+}\right)\delta(\eta - \eta_T)$$

- general piecewise solution

$$v_k = \frac{\alpha_k}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right)e^{ik\eta}$$

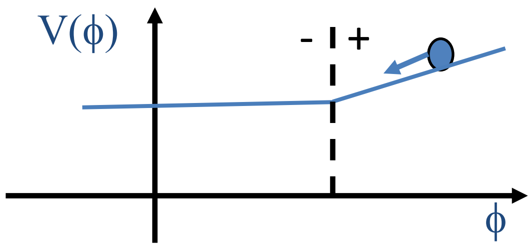
  – before transition:

$$\alpha_{k+} = 1, \;\; \beta_{k+} = 0 \quad \text{(Bunch-Davies vacuum)}$$

  – after transition:

$$\alpha_{k-} \neq 1 + \mathcal{O}(\Delta A/A_+), \;\; \beta_{k-} \neq \mathcal{O}(\Delta A/A_+) \quad \text{(excited state)}$$

piecewise linear potential:



- Starobinsky (1992):

$$V(\varphi) = \begin{cases} A_+(\varphi - \varphi_T) & \text{for} \quad \varphi > \varphi_T \\ A_-(\varphi - \varphi_T) & \text{for} \quad \varphi < \varphi_T \end{cases}$$

- homogeneous solution after transition

$$\delta\varphi_0 = \frac{v_0}{a} = C_- + D_-\eta^3$$

  – particular solution for excited state on super-Hubble scales $(k < k_T)$

$$C_- = \frac{-iH}{\sqrt{2k^3}} \left\{ 1 + \frac{2}{5}\frac{\Delta A}{A_-}\left(\frac{\boldsymbol{k}}{\boldsymbol{k_T}}\right)^2 + \mathcal{O}\left(\frac{k}{k_T}\right)^3 \right\}$$

$$D_-\eta^3 = \frac{H}{3\sqrt{2k^3}}\left(\frac{k_T}{aH}\right)^3 \mathcal{O}\left(\frac{\boldsymbol{k}}{\boldsymbol{k_T}}\right)^2$$

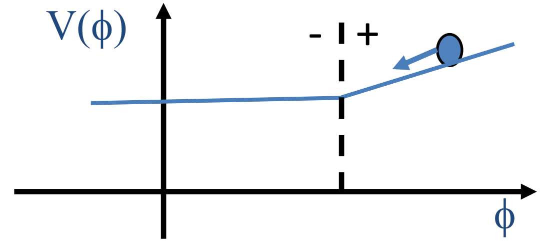
  – **gradient terms** in the adiabatic growing mode before the transition, source the non-adiabatic decaying mode at the transition

Leach, Liddle, Sasaki & Wands (2003)

  – **separate universe approach breaks down on some scales at transition**

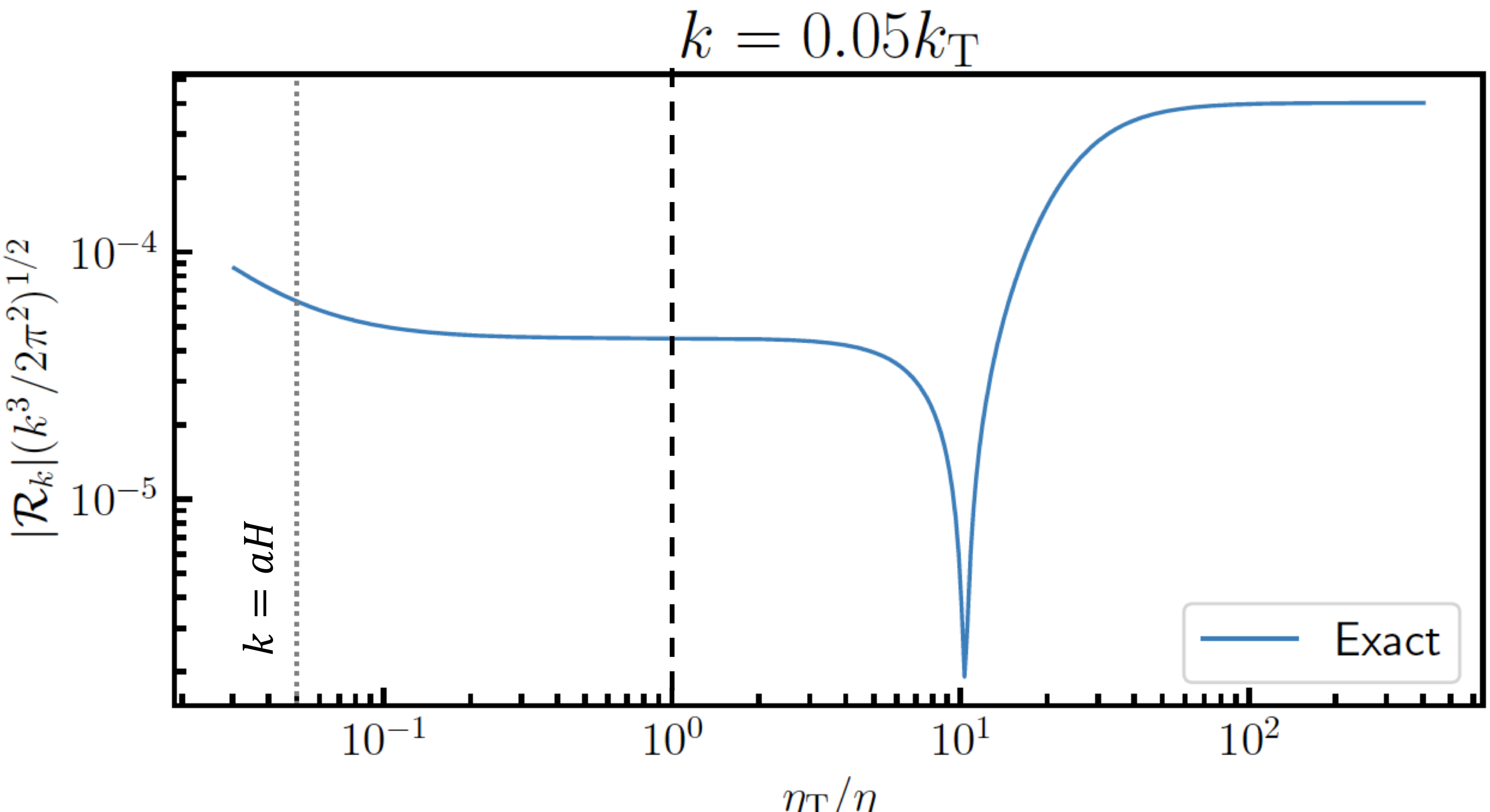
Jackson et al, arXiv:2311.03281

piecewise linear potential:



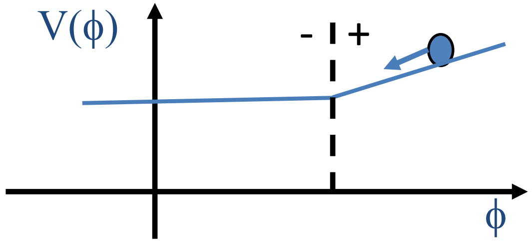
- comoving curvature before and after transition

$$R = \frac{v}{z} = \frac{H\delta\varphi}{\dot{\varphi}} \quad = \zeta$$ = constant for adiabatic perturbations on large-scales



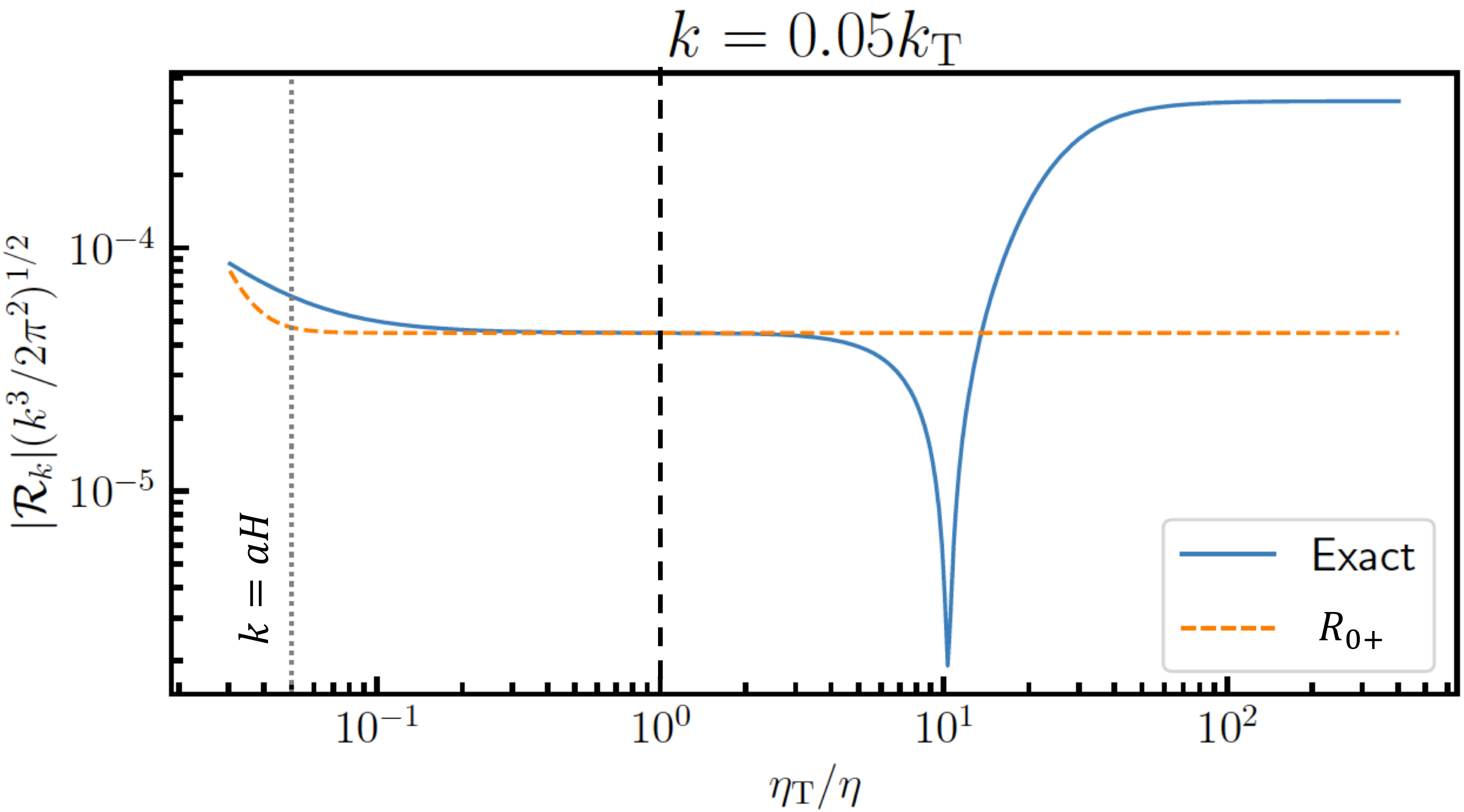
credit Joe Jackson

piecewise linear potential:



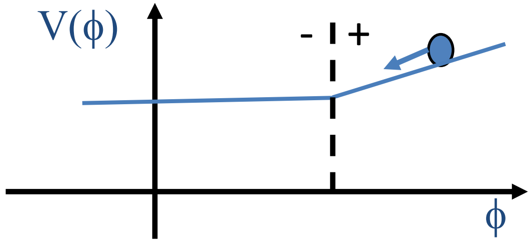
- comoving curvature before and after transition

$$R = \frac{v}{z} = \frac{H\delta\varphi}{\dot{\varphi}} \quad = \zeta$$ = constant for adiabatic perturbations on large-scales



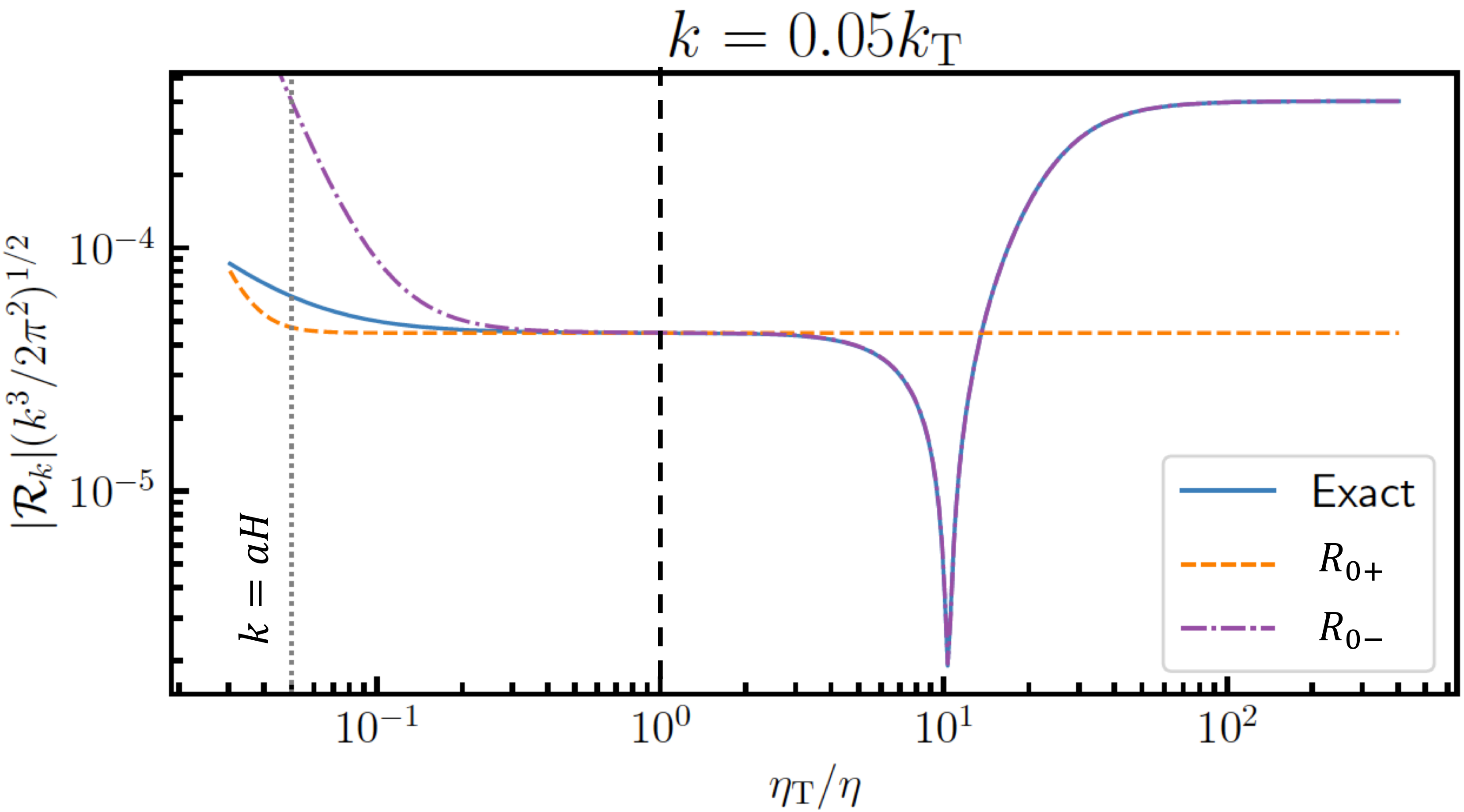
credit Joe Jackson

piecewise linear potential:



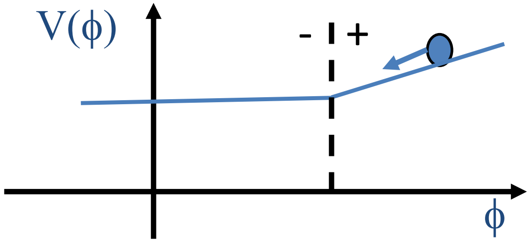
- comoving curvature before and after transition

$$R = \frac{v}{z} = \frac{H\delta\varphi}{\dot{\varphi}} = \zeta$$ = constant for adiabatic perturbations on large-scales

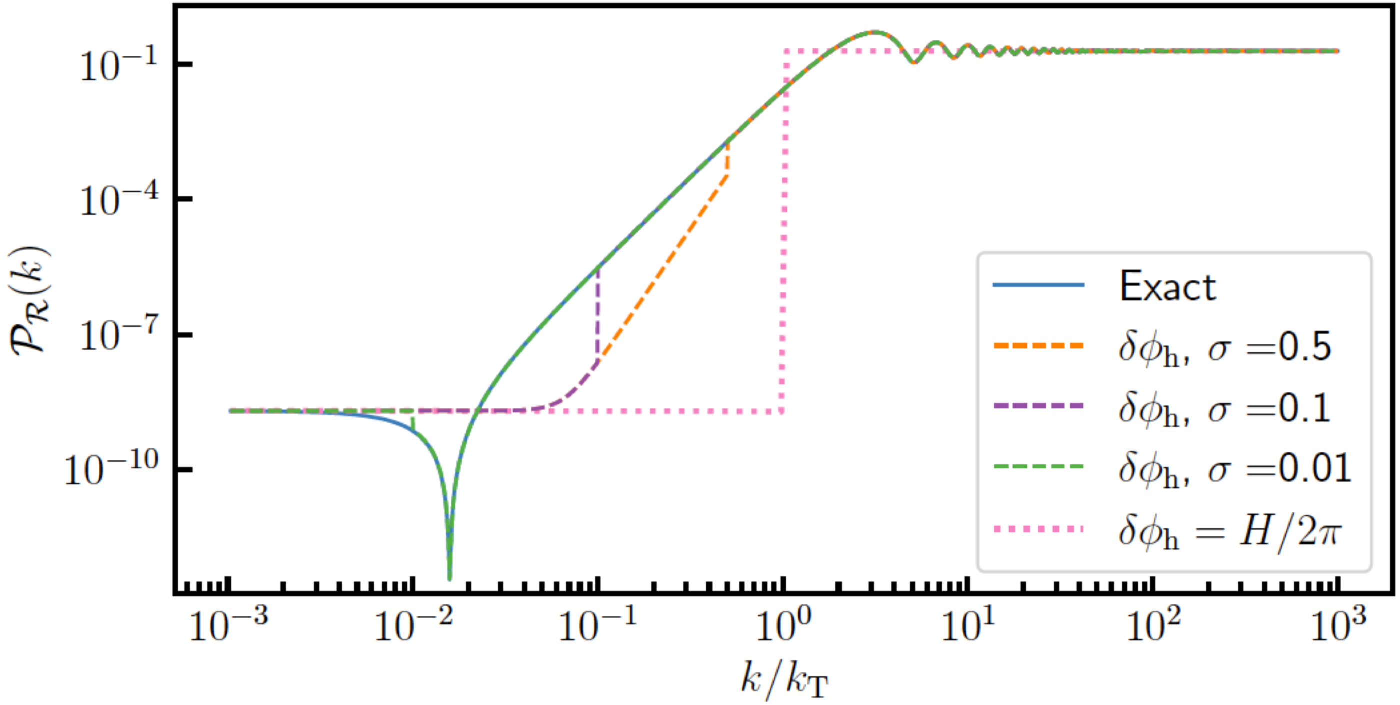


credit Joe Jackson

piecewise linear potential:



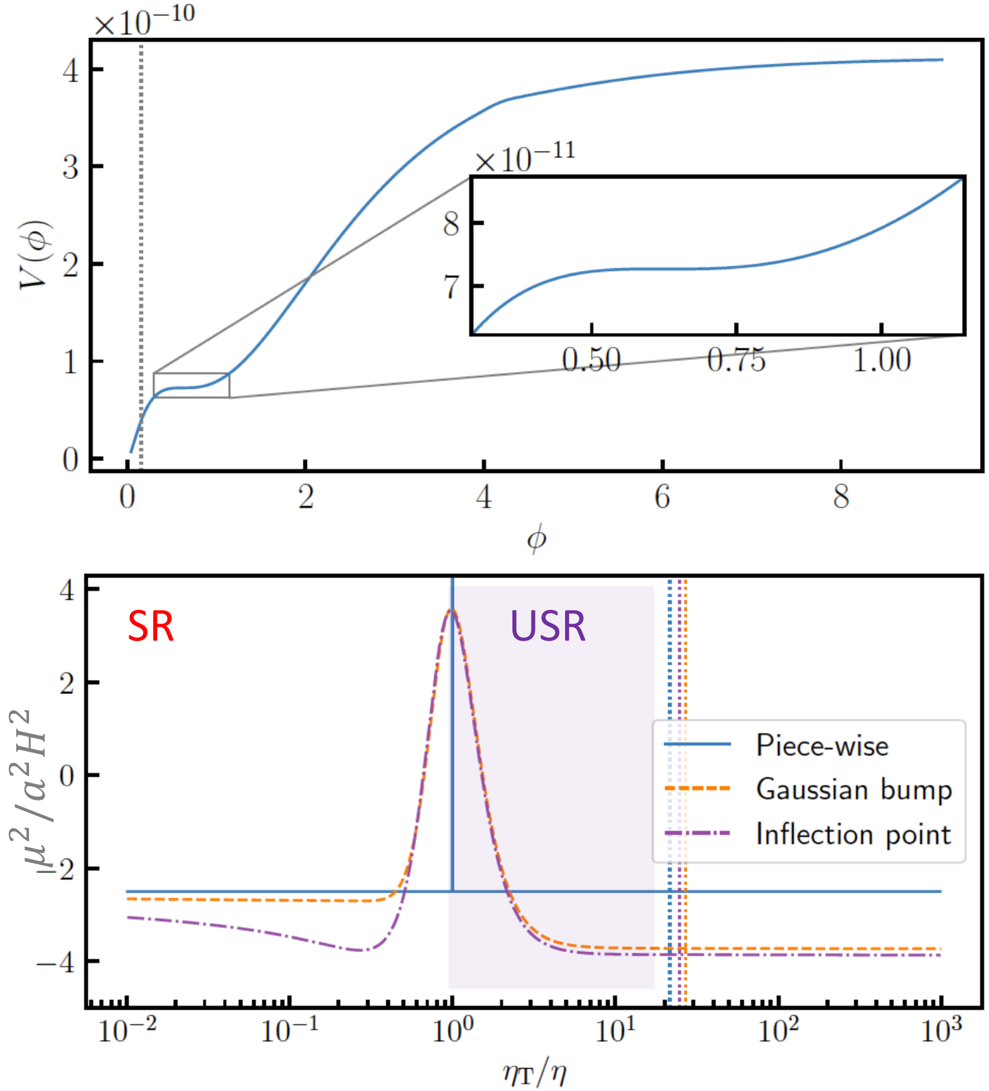
- δN using separate universe approximation for $k < \sigma a H$



Jackson et al arXiv:2311.03281

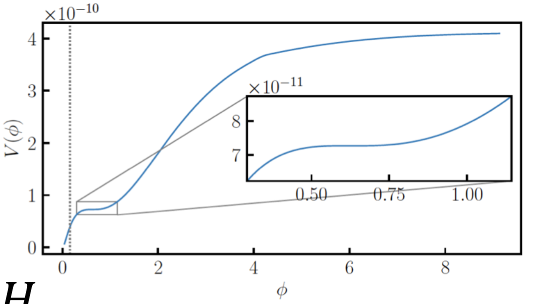
$$\delta N_k \simeq \frac{\partial N}{\partial \phi_{\rm in}}\left(\phi_*, \dot{\phi}_*\right)\delta\phi_{k*} + \frac{\partial N}{\partial \dot{\phi}_{\rm in}}\left(\phi_*, \dot{\phi}_*\right)\delta\dot{\phi}_{k*}$$

smooth inflection point: Rasanen & Tomberg (2019)

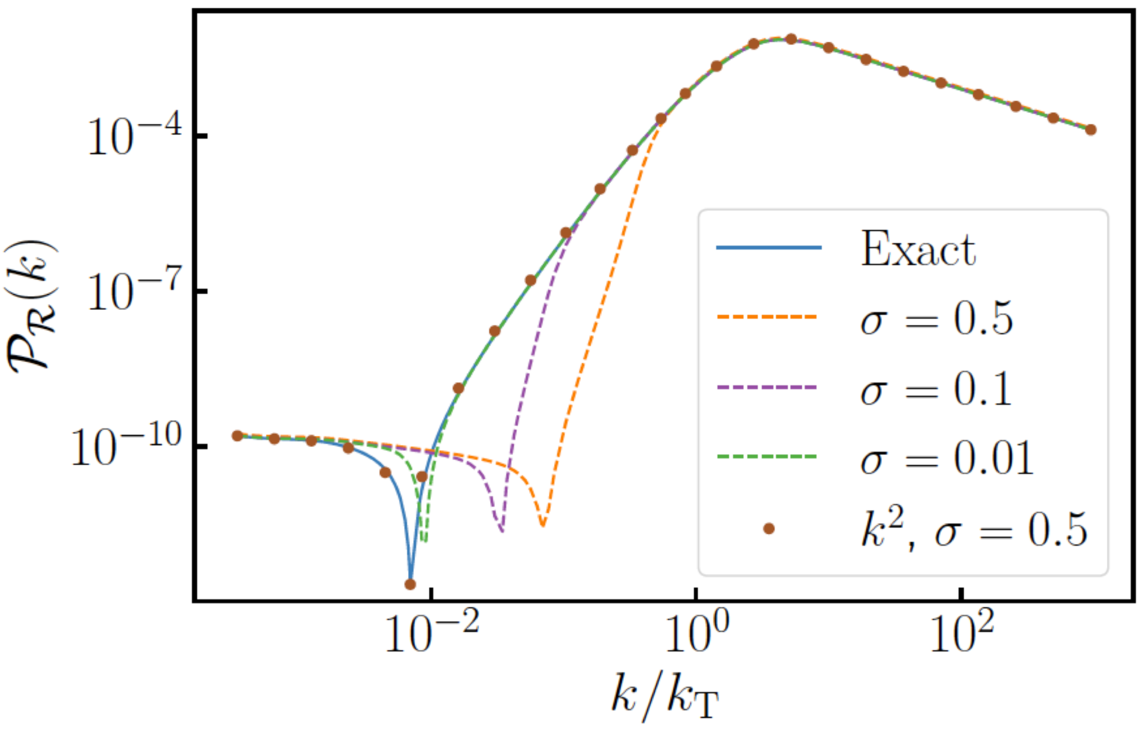


Jackson et al arXiv:2311.03281

smooth inflection point:



- δN using separate universe approximation for $k < \sigma a H$



Jackson et al arXiv:2311.03281

summary

- separate universe approach at a sudden transition
  - breaks down on finite range of super-Hubble scales
  - gradient terms generate non-adiabatic perturbation

- δN formalism to calculate primordial curvature perturbation
  - include the field's momentum beyond slow-roll limit
  - include particle production at transition (sub-H modes not in BD vacuum)
  - works before and after the transition, but not at the transition
    - need to include k^2 corrections in a gradient expansion

- stochastic inflation requires two correlated sources on noise
  - at Hubble crossing *and* at the transition (non-Markovian)
  - are quantum fluctuations effectively classical stochastic noise?
  - *too many contradictory views already on stochastic inflation beyond slow-roll!*