Full analysis of the scalar-induced gravitational waves for the curvature perturbation with local-type non-Gaussianities

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PBH formation

Mass-scale relation

$$M_{\rm pbh} = \gamma M_{\rm H}|_{\rm at\ formation} \simeq 2.3 \times 10^{18} \left(\frac{H_0}{f_*}\right)^2 M_\odot$$

Carr, Hawking (1974)

PBH abundance

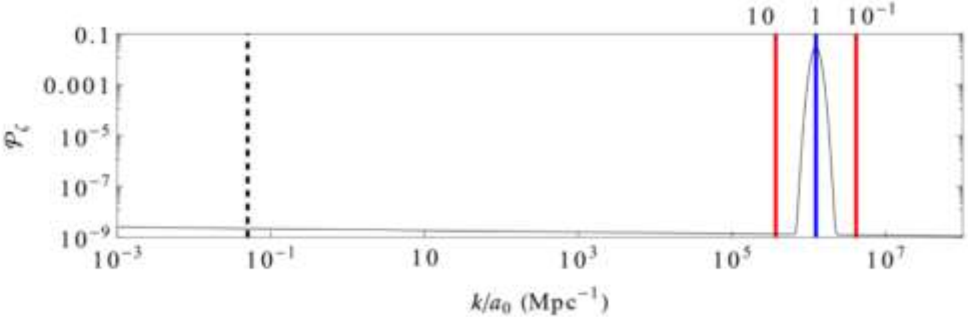
$$f_{\rm pbh} = 2.7 \times 10^8 \left(\frac{\gamma}{0.2}\right)^{1/2} \left(\frac{g_{*,\,\rm form}}{10.75}\right)^{-1/4} \left(\frac{M}{M_\odot}\right)^{-1/2} \beta$$

Nakama, Silk, Kamionkowski (2017)

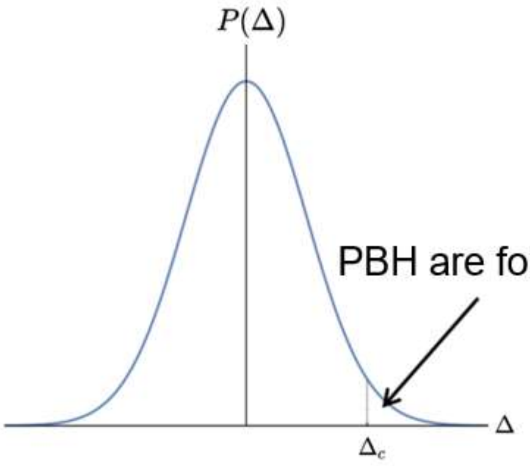
PBH mass function

$$\beta \equiv \frac{\rho_{\rm PBH}}{\rho_{\rm tot}}\bigg|_{\rm at\ formation} = \int_{\Delta_c}^{\infty} P(\Delta) d\Delta$$

Press, Schechter (1974)



Clesse,García-Bellido, & Orani, arXiv:1812.11011



PBH are formed

PBH formation

Are primordial curvature perturbations

Gaussian

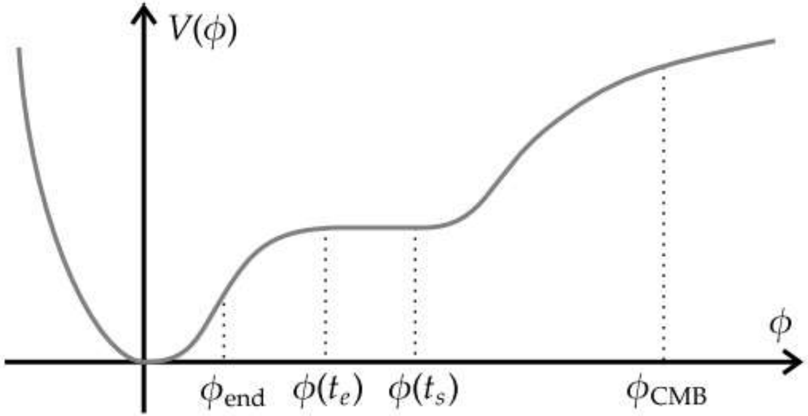
Or

Non-Gaussian?

Action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[M_{\mathrm{pl}}^2 R - (\partial_\mu\phi)^2 - 2V(\phi)\right]$$

Potential



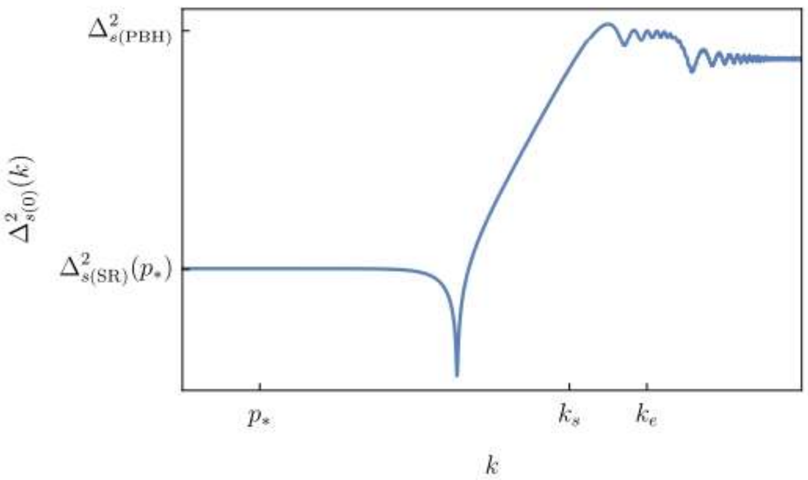
Ultra-slow roll model

Tree level

$$\langle \hat{\zeta}^I_{\mathbf{k}}(t_*)\hat{\zeta}^I_{\mathbf{k}'}(t_*)\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')|\zeta_k(t_*)|^2 \\ \equiv (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')P_\zeta(k).$$

Higher order (non-Gaussianities)

$$\langle \hat{\zeta}_{\mathbf{k_1}}\hat{\zeta}_{\mathbf{k_2}}\hat{\zeta}_{\mathbf{k_3}}\rangle = (2\pi)^3\delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B_\zeta(k_1,k_2,k_3),$$



Kristiano & Yokoyama (202)

All inflation models inevitably produce non-Gaussianities!

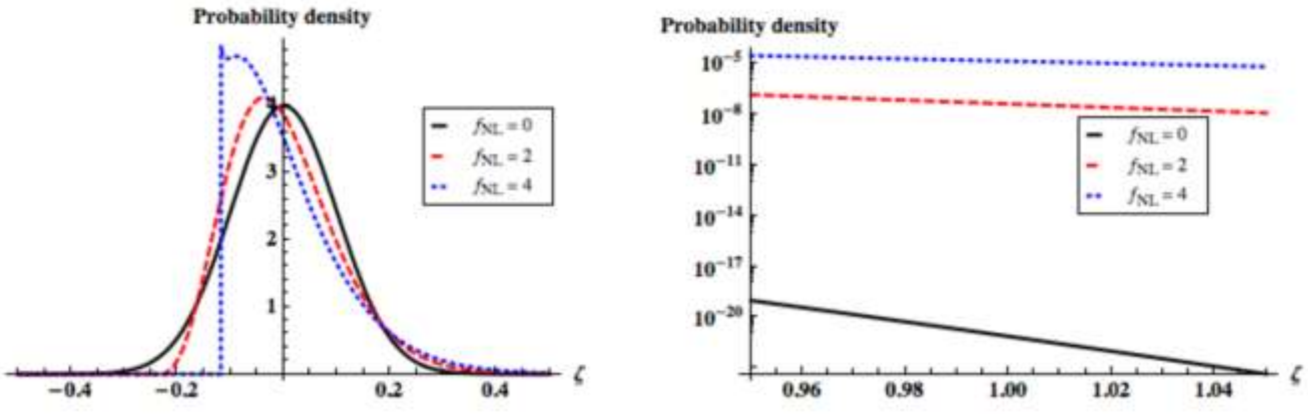
Are Non-Gaussian effects important?

Local-type non-Gaussianities

$$\boxed{\zeta}(\zeta_g) = \zeta_g + F_{\rm NL}\left(\boxed{\zeta_g^2} - \langle\zeta_g^2\rangle\right) + G_{\rm NL}\zeta_g^3,$$

Non-Gaussian

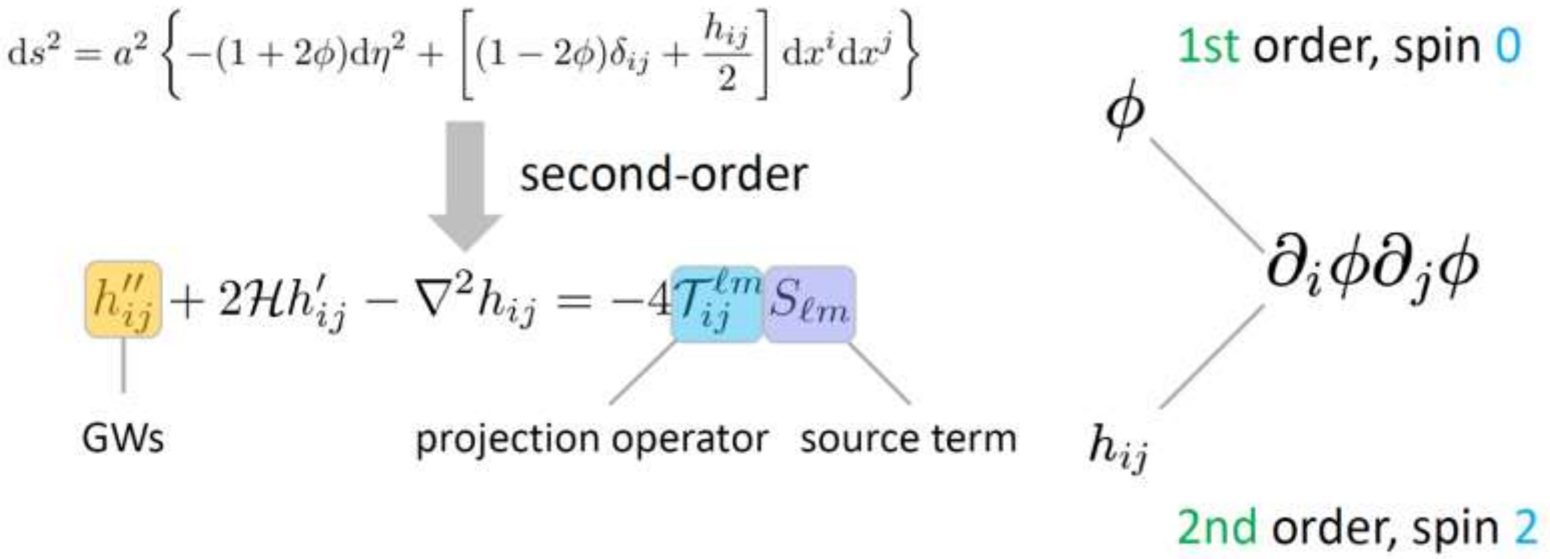
Gaussian



Young & Byrnes (2013)

PBH formation is extremely sensitive to non-Gaussianities!

Scalar-induced GWs



$$ds^2 = a^2\left\{-(1+2\phi)d\eta^2 + \left[(1-2\phi)\delta_{ij} + \frac{h_{ij}}{2}\right]dx^i dx^j\right\}$$

second-order

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = -4\mathcal{T}_{ij}^{\ell m} S_{\ell m}$$

GWs

projection operator

source term

1st order, spin 0

ϕ

$\partial_i\phi\partial_j\phi$

h_{ij}

2nd order, spin 2

$$S_{\ell m} = \phi\partial_\ell\partial_m\phi + 2\partial_\ell\phi\partial_m\phi - \partial_\ell\left(\frac{\phi'}{\mathcal{H}} + \phi\right)\partial_m\left(\frac{\phi'}{\mathcal{H}} + \phi\right)$$

Non-Gaussian SIGWs

Energy spectrum

$$\Omega_{\mathrm{GW}}(k) \equiv \frac{1}{\rho_c}\frac{\mathrm{d}\rho_{\mathrm{GW}}(k)}{\mathrm{d}\ln k}$$

For SIGWs at leading order

$$\Omega_{\mathrm{GW}}(k)=\frac{k^3}{6\pi^2}\int\frac{\mathrm{d}^3q\,\mathrm{d}^3q'}{(2\pi)^6}\cos 2(\phi-\phi')I(u,v)\,I(u',v')\,\langle\langle\zeta_{\boldsymbol{q}}\zeta_{\boldsymbol{k}-\boldsymbol{q}}\zeta_{\boldsymbol{q}'}\zeta_{\boldsymbol{k}'-\boldsymbol{q}'}\rangle\rangle$$

$$\begin{aligned}I(u,v)\,I(u',v')= & \frac{9\left(u^2+v^2-3\right)\left(u'^2+v'^2-3\right)}{1024u^3u'^3v^3v'^3}\left[4u^2-\left(u^2-v^2+1\right)^2\right]\left[4u'^2-\left(u'^2-v'^2+1\right)^2\right]\\ & \times\left\{\left[\left(u^2+v^2-3\right)\ln\left(\left|\frac{(u-v)^2-3}{(u+v)^2-3}\right|\right)+4uv\right]\left[\left(u'^2+v'^2-3\right)\ln\left(\left|\frac{(u'-v')^2-3}{(u'+v')^2-3}\right|\right)+4u'v'\right]\right.\\ & \left.+\pi^2\left(u^2+v^2-3\right)\left(u'^2+v'^2-3\right)\Theta\left(u+v-\sqrt{3}\right)\Theta\left(u'+v'-\sqrt{3}\right)\right\},\end{aligned}$$

Non-Gaussian SIGWs

The leading order is the Gaussian part, in which case we have

$$\begin{aligned}\langle\zeta_{\boldsymbol{q}}\zeta_{\boldsymbol{k}-\boldsymbol{q}}\zeta_{\boldsymbol{q}'}\zeta_{\boldsymbol{k}'-\boldsymbol{q}'}\rangle_g = \quad & \langle\zeta_g(\mathbf{q})\,\zeta_g(\mathbf{q}')\rangle\,\langle\zeta_g(\mathbf{k}-\mathbf{q})\,\zeta_g(\mathbf{k}'-\mathbf{q}')\rangle + \langle\zeta_g(\mathbf{q})\,\zeta_g(\mathbf{k}'-\mathbf{q}')\rangle\,\langle\zeta_g(\mathbf{k}-\mathbf{q})\,\zeta_g(\mathbf{q}')\rangle \\ & + \langle\zeta_g(\mathbf{q})\,\zeta_g(\mathbf{k}-\mathbf{q})\rangle\,\langle\zeta_g(\mathbf{q}')\,\zeta_g(\mathbf{k}'-\mathbf{q}')\rangle\,,\end{aligned}$$

Wick’s theorem!

$$\begin{aligned}\Omega^g_{\mathrm{GW}}(k) = \quad & \frac{k^3}{6\pi^2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}I^2(u,v)\,P_g(q)\,P_g(|\boldsymbol{k}-\boldsymbol{q}|)\times 2 \\ = \quad & \frac{1}{3}\int_0^\infty \mathrm{d}u\int_{|1-u|}^{1+u}\mathrm{d}v I^2(u,v)\frac{1}{u^2v^2}\mathcal{P}_g(uk)\,\mathcal{P}_g(vk)\,.\end{aligned}$$

This is the Gaussian part of the energy spectrum

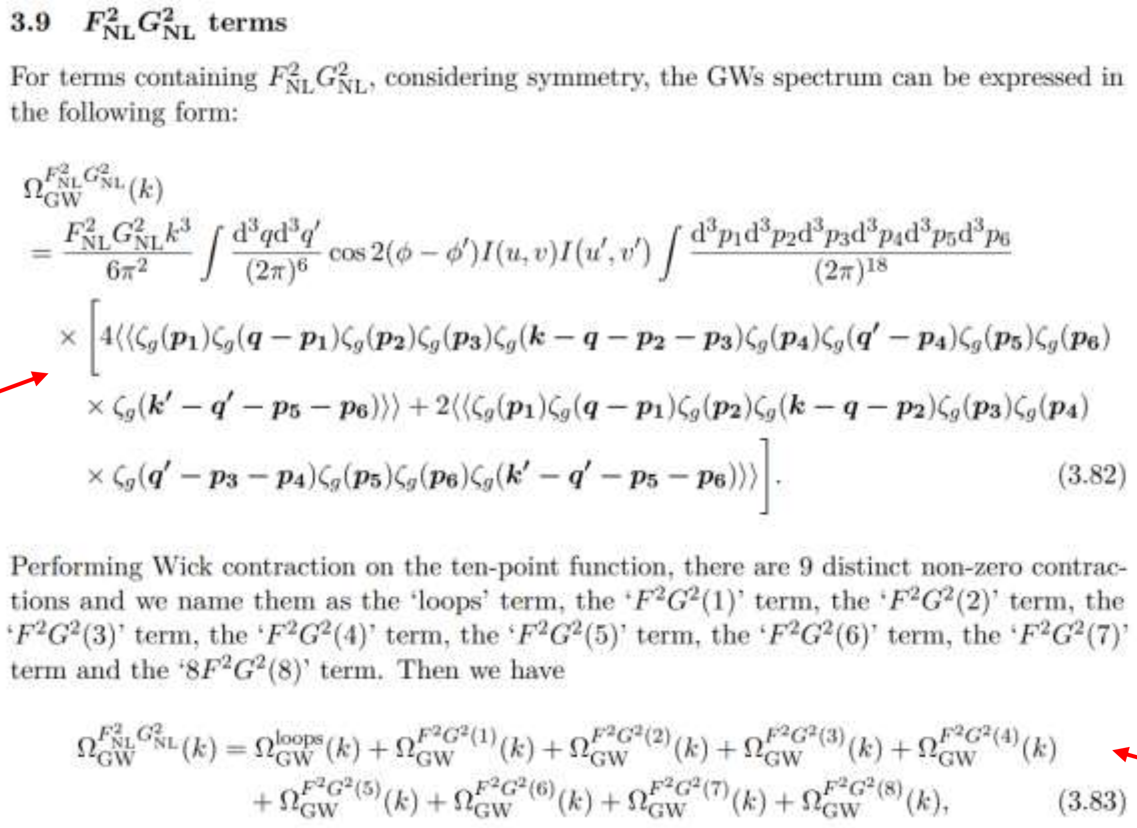
Non-Gaussian SIGWs

$\langle\zeta^4\rangle$ **+** $\zeta\left(\zeta_g\right)=\zeta_g+F_{\mathrm{NL}}\left(\zeta_g^2-\left\langle\zeta_g^2\right\rangle\right)+G_{\mathrm{NL}} \zeta_g^3,$ **+** **Wick's theorem**

=

$$\Omega_{\mathrm{GW}}=\Omega^g+\Omega^{F_{NL}^2}+\Omega^{F_{NL}^4}+\Omega^{G_{NL}}+\Omega^{G_{NL}^2}+\Omega^{G_{NL}^3}+\Omega^{G_{NL}^4}+\Omega^{F^2G}+\Omega^{F^2G^2}$$

Non-Gaussian SIGWs



3.9 $F_{\rm NL}^2 G_{\rm NL}^2$ terms

For terms containing $F_{\rm NL}^2 G_{\rm NL}^2$, considering symmetry, the GWs spectrum can be expressed in the following form:

$$\begin{aligned}
&\Omega_{\rm GW}^{F_{\rm NL}^2 G_{\rm NL}^2}(k)\\
&=\frac{F_{\rm NL}^2 G_{\rm NL}^2 k^3}{6\pi^2}\int\frac{\mathrm{d}^3q\,\mathrm{d}^3q'}{(2\pi)^6}\cos 2(\phi-\phi')I(u,v)I(u',v')\int\frac{\mathrm{d}^3p_1\mathrm{d}^3p_2\mathrm{d}^3p_3\mathrm{d}^3p_4\mathrm{d}^3p_5\mathrm{d}^3p_6}{(2\pi)^{18}}\\
&\quad\times\Big[4\langle\langle\zeta_g(\boldsymbol{p_1})\zeta_g(\boldsymbol{q}-\boldsymbol{p_1})\zeta_g(\boldsymbol{p_2})\zeta_g(\boldsymbol{p_3})\zeta_g(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{p_2}-\boldsymbol{p_3})\zeta_g(\boldsymbol{p_4})\zeta_g(\boldsymbol{q'}-\boldsymbol{p_4})\zeta_g(\boldsymbol{p_5})\zeta_g(\boldsymbol{p_6})\\
&\quad\times\zeta_g(\boldsymbol{k'}-\boldsymbol{q'}-\boldsymbol{p_5}-\boldsymbol{p_6})\rangle\rangle+2\langle\langle\zeta_g(\boldsymbol{p_1})\zeta_g(\boldsymbol{q}-\boldsymbol{p_1})\zeta_g(\boldsymbol{p_2})\zeta_g(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{p_2})\zeta_g(\boldsymbol{p_3})\zeta_g(\boldsymbol{p_4})\\
&\quad\times\zeta_g(\boldsymbol{q'}-\boldsymbol{p_3}-\boldsymbol{p_4})\zeta_g(\boldsymbol{p_5})\zeta_g(\boldsymbol{p_6})\zeta_g(\boldsymbol{k'}-\boldsymbol{q'}-\boldsymbol{p_5}-\boldsymbol{p_6})\rangle\rangle\Big].
\end{aligned}\tag{3.82}$$

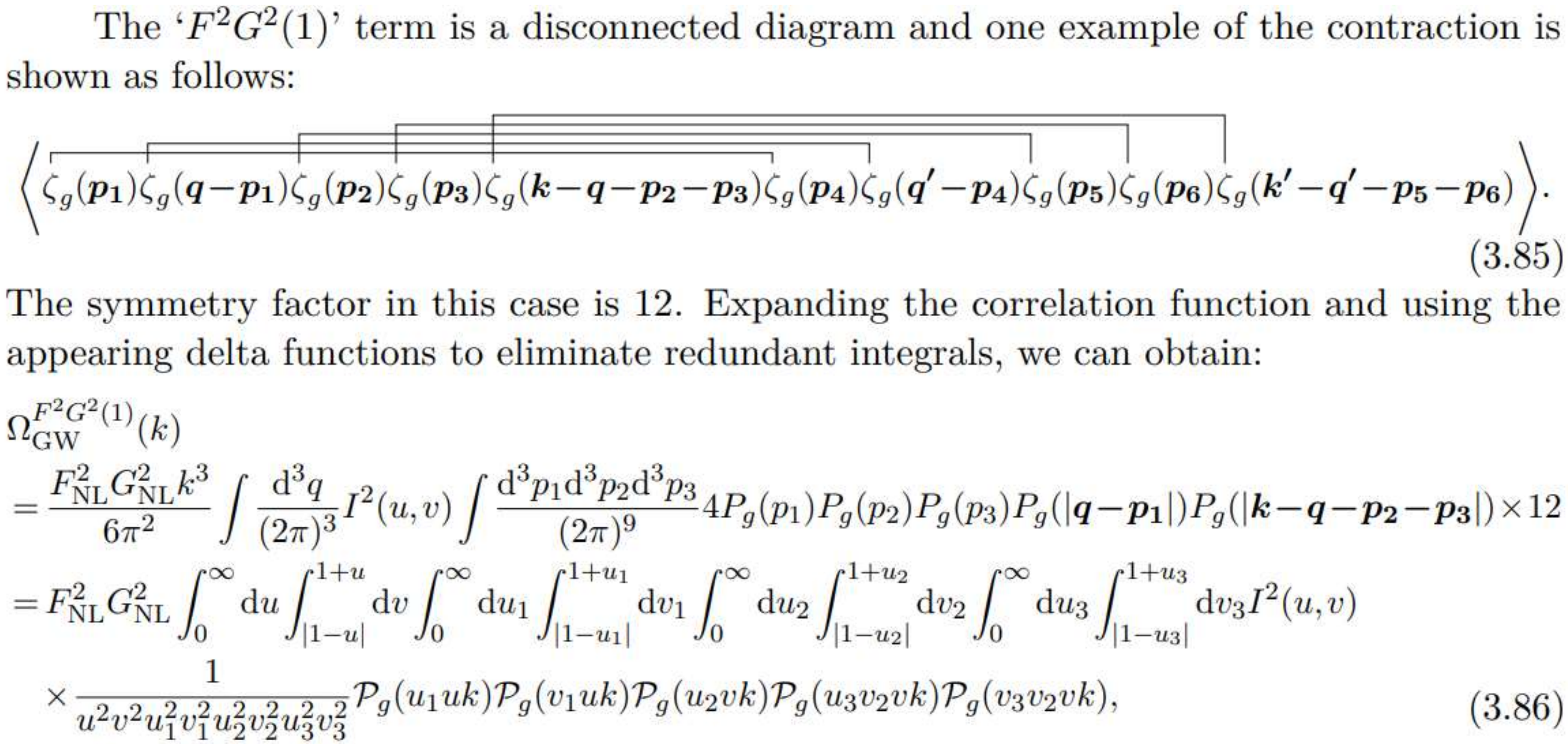
Performing Wick contraction on the ten-point function, there are 9 distinct non-zero contractions and we name them as the 'loops' term, the '$F^2G^2(1)$' term, the '$F^2G^2(2)$' term, the '$F^2G^2(3)$' term, the '$F^2G^2(4)$' term, the '$F^2G^2(5)$' term, the '$F^2G^2(6)$' term, the '$F^2G^2(7)$' term and the '$8F^2G^2(8)$' term. Then we have

$$\begin{aligned}
\Omega_{\rm GW}^{F_{\rm NL}^2 G_{\rm NL}^2}(k)=\Omega_{\rm GW}^{\rm loops}(k)+\Omega_{\rm GW}^{F^2G^2(1)}(k)+\Omega_{\rm GW}^{F^2G^2(2)}(k)+\Omega_{\rm GW}^{F^2G^2(3)}(k)+\Omega_{\rm GW}^{F^2G^2(4)}(k)\\
+\Omega_{\rm GW}^{F^2G^2(5)}(k)+\Omega_{\rm GW}^{F^2G^2(6)}(k)+\Omega_{\rm GW}^{F^2G^2(7)}(k)+\Omega_{\rm GW}^{F^2G^2(8)}(k),
\end{aligned}\tag{3.83}$$

Wick's theorem!

9 kinds of contractions

Non-Gaussian SIGWs



The '$F^2G^2(1)$' term is a disconnected diagram and one example of the contraction is shown as follows:

$$\left\langle \zeta_g(\boldsymbol{p_1})\zeta_g(\boldsymbol{q}-\boldsymbol{p_1})\zeta_g(\boldsymbol{p_2})\zeta_g(\boldsymbol{p_3})\zeta_g(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{p_2}-\boldsymbol{p_3})\zeta_g(\boldsymbol{p_4})\zeta_g(\boldsymbol{q}'-\boldsymbol{p_4})\zeta_g(\boldsymbol{p_5})\zeta_g(\boldsymbol{p_6})\zeta_g(\boldsymbol{k}'-\boldsymbol{q}'-\boldsymbol{p_5}-\boldsymbol{p_6})\right\rangle. \tag{3.85}$$

The symmetry factor in this case is 12. Expanding the correlation function and using the appearing delta functions to eliminate redundant integrals, we can obtain:

$$\begin{aligned}
&\Omega_{\mathrm{GW}}^{F^2G^2(1)}(k)\\
&=\frac{F_{\mathrm{NL}}^2G_{\mathrm{NL}}^2k^3}{6\pi^2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}I^2(u,v)\int\frac{\mathrm{d}^3p_1\mathrm{d}^3p_2\mathrm{d}^3p_3}{(2\pi)^9}4P_g(p_1)P_g(p_2)P_g(p_3)P_g(|\boldsymbol{q}-\boldsymbol{p_1}|)P_g(|\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{p_2}-\boldsymbol{p_3}|)\times 12\\
&=F_{\mathrm{NL}}^2G_{\mathrm{NL}}^2\int_0^\infty\mathrm{d}u\int_{|1-u|}^{1+u}\mathrm{d}v\int_0^\infty\mathrm{d}u_1\int_{|1-u_1|}^{1+u_1}\mathrm{d}v_1\int_0^\infty\mathrm{d}u_2\int_{|1-u_2|}^{1+u_2}\mathrm{d}v_2\int_0^\infty\mathrm{d}u_3\int_{|1-u_3|}^{1+u_3}\mathrm{d}v_3I^2(u,v)\\
&\quad\times\frac{1}{u^2v^2u_1^2v_1^2u_2^2v_2^2u_3^2v_3^2}\mathcal{P}_g(u_1uk)\mathcal{P}_g(v_1uk)\mathcal{P}_g(u_2vk)\mathcal{P}_g(u_3v_2vk)\mathcal{P}_g(v_3v_2vk),
\end{aligned}\tag{3.86}$$

Log-normal power spectrum

$$\mathcal{P}_g(k) = \frac{A}{\sqrt{2\pi\sigma_*^2}} \exp\left(-\frac{\ln^2(k/k_*)}{2\sigma_*^2}\right)$$

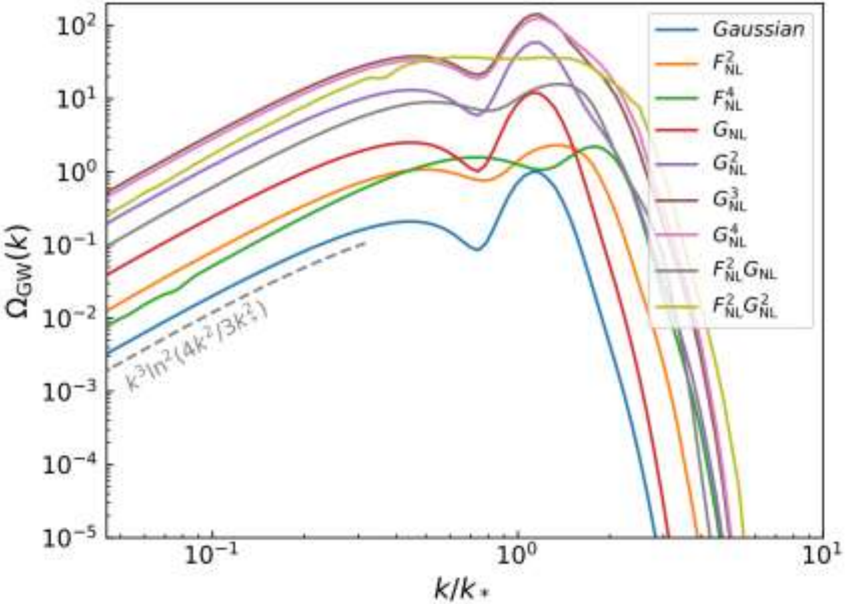
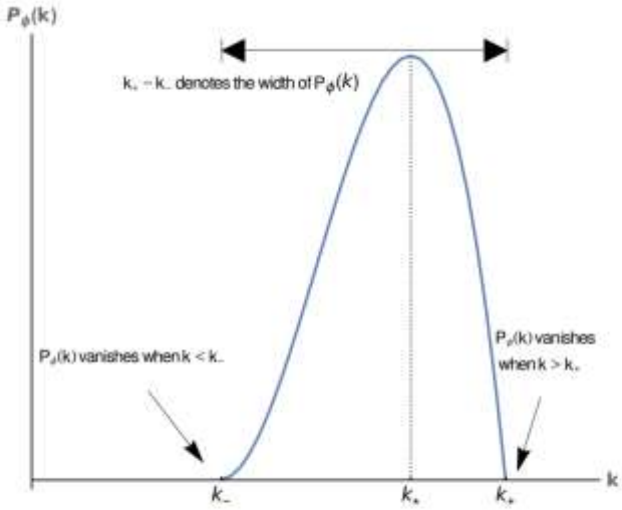


Figure 1. The unscaled (By setting $A = 1$, $F_{\rm NL} = 1$ and $G_{\rm NL} = 1$) energy spectrum of SIGW generated by a log-normal power spectrum described by eq. (4.13) with $\sigma_* = 0.2$.

Non-Gaussian SIGWs



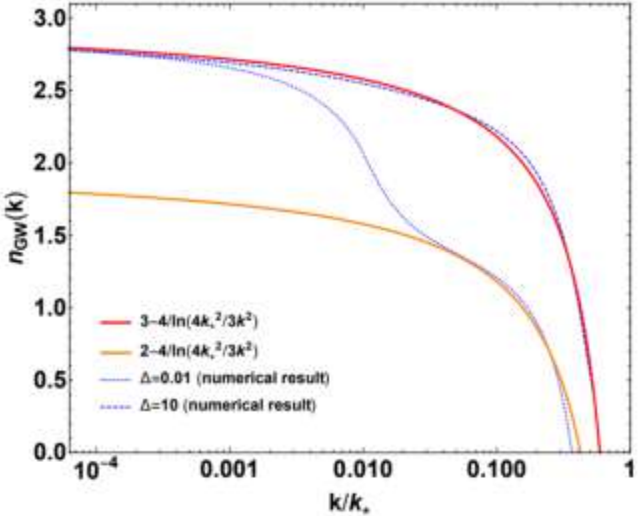
near the peak

(for narrow spectrum)

$$n_{GW}(k) = 2 - \frac{4}{\ln \frac{4k_*^2}{3k^2}}$$

far-infrared region

$$n_{GW}(k) = 3 - \frac{4}{\ln \frac{4k_*^2}{3k^2}}$$



Yuan, Chen & Huang (2020)

The slope in the infrared region has a log-dependent behavior (Gaussian case)

Assume that the power spectrum is defined in [k_-,k_+], then

$$\Omega_{\mathrm{GW}}^{tri}(k) = \frac{G_{\mathrm{NL}}^2}{4\pi^2}\int_{k_-/k}^{k_+/k}\mathrm{d}u\int_{|1-u|}^{1+u}\mathrm{d}v\int_{k_-/k}^{k_+/k}\mathrm{d}u_1\int_{|1-u_1|}^{1+u_1}\mathrm{d}v_1\int_{k_-/k}^{k_+/k}\mathrm{d}u_2\int_{|1-u_2|}^{1+u_2}\mathrm{d}v_2\int_0^{2\pi}\mathrm{d}\varphi_2\int_0^{2\pi}\mathrm{d}\varphi_3$$
$$\times I^2(u,v)\frac{uvu_1v_1u_2v_2}{(u_1u_2w_{134}v)^3}\mathcal{P}_g(u_1k)\mathcal{P}_g(u_2k)\mathcal{P}_g(w_{0134}k)\mathcal{P}_g(uk)\,.$$

Use the mean value theorem for definite integrals

$$\Omega_{\mathrm{GW}}^{tri}(k) = \frac{2G_{\mathrm{NL}}^2}{\pi^2}\left(\frac{k_+-k_-}{k}\right)^3 I^2(u^*,v^*)\frac{u^*v^*u_1^*v_1^*u_2^*v_2^*}{(u_1^*u_2^*w_{134}^*v^*)^3}\mathcal{P}_g(u_1^*k)\mathcal{P}_g(u_2^*k)\mathcal{P}_g(w_{0134}^*k)\mathcal{P}_g(u^*k)$$

In the infrared region, $u,v,u_1,v_1,u_2,v_2 >> 1$, so that

$$\Omega_{\rm GW}^{tri}(k) \propto \left(\frac{k}{k_\star}\right)^3 I^2\left(\frac{k_\star}{k}, \frac{k_\star}{k}\right) \qquad k_\star \in [k_-, k_+]$$

In the asymptotic behavior

$$I^2(u,u) \simeq \frac{9}{4}\ln^2\left(\frac{4u^2}{3}\right)$$

Finally the scaling will be

$$\Omega_{\rm GW}^{tri}(k) \propto \left(\frac{k}{k_\star}\right)^3 \ln^2\left(\frac{4k_\star^2}{3k^2}\right)$$

We need to prove that the sum of all contractions, the log-dependent slope apply…

$$\Omega_{\mathrm{GW}}(k)=\sum_i A_i\left(\frac{k}{k_{\star i}}\right)^3\ln^2\left(\frac{4k_{\star i}^2}{3k^2}\right)$$

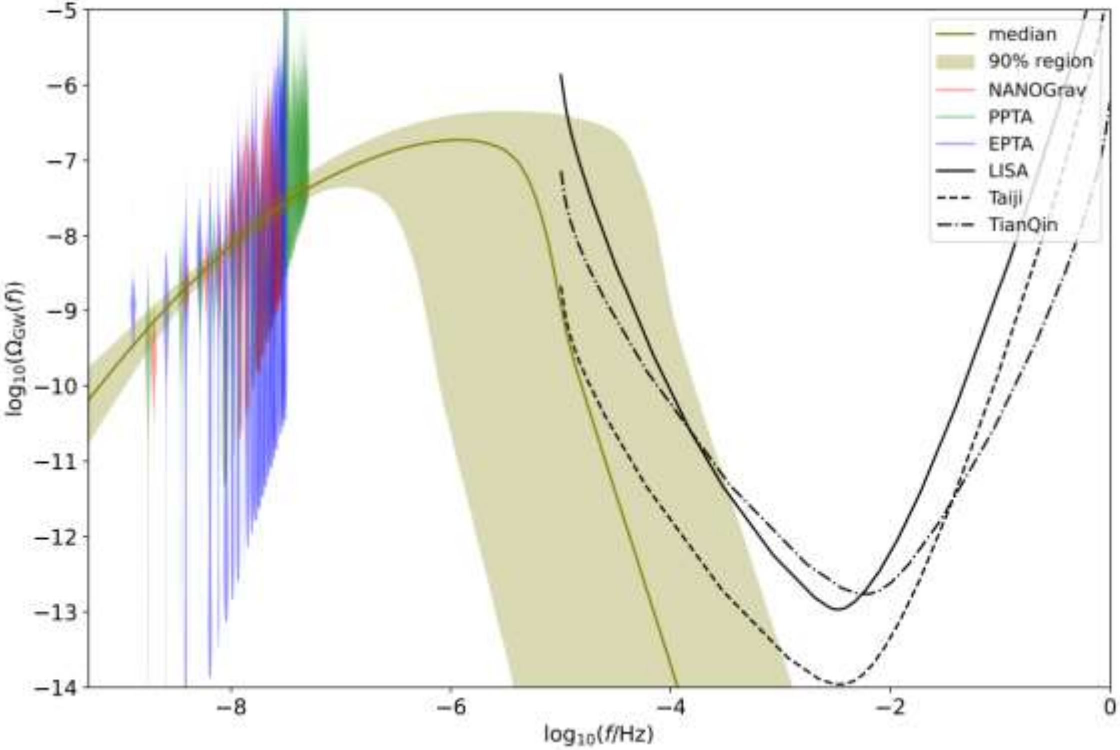
Rewrite the energy spectrum as

$$\Omega_{\mathrm{GW}}(k)=k^3\sum_i\frac{A_i}{k_{\star i}^3}\left(c_i^2+2c_i\ln\left(\frac{4k_\star^2}{3k^2}\right)+\ln^2\left(\frac{4k_\star^2}{3k^2}\right)\right),\qquad c_i\equiv\ln\left(\frac{k_{\star i}^2}{k_\star^2}\right)$$

Then we can prove that the sum of these log-dependent slopes is still log-dependent…

$$\Omega_{\mathrm{GW}}(k)\simeq k^3\ln^2\left(\frac{4k_\star^2}{3k^2}\right)\sum_i\frac{A_i}{k_{\star i}^3}.\qquad \text{if}\ \frac{k}{k_\star}\ll\frac{k_-}{k_+}$$

Non-Gaussian SIGWs



Liu, Chen & Huang (2023)

Summary

- Derive a general expression for SIGW within local-type non-Gaussianities up to third-order.

- For an enhanced curvature power spectrum, there is a log-scaling in the infrared region even if we consider non-Gaussianities.

- The log-scaling could be smoking gun in detecting SIGWs and searching for PBHs in the future.

Thank you

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