Revisiting compaction functions for PBH formation

Tomohiro Harada

Department of Physics, Rikkyo University

GC2024 @ YITP, 29/01-01/03/2024

This talk is based on
Harada, Yoo & Koga PRD108 043515 (2023) [arXiv:2304.13284].

Introduction

- Primordial black holes (PBHs) are black holes formed in the early Universe (Zeldovich & Novikov (1967), Hawking (1971)).
  - Fossils of the early Universe
  - Dark matter candidate
  - Hawking evaporation
  - High-energy physics
  - GW sources

GRAVITATIONALLY COLLAPSED OBJECTS OF VERY LOW MASS

Stephen Hawking

(Communicated by M. J. Rees)

(Received 1970 November 9)

SUMMARY

It is suggested that there may be a large number of gravitationally collapsed objects of mass 10^{-5} g upwards which were formed as a result of fluctuations in the early Universe. They could carry an electric charge of up to ± 30 electron units. Such objects would produce distinctive tracks in bubble chambers and could form atoms with orbiting electrons or protons. A mass of 10^{17} g of such objects could have accumulated at the centre of a star like the Sun. If such a star later became a neutron star there would be a steady accretion of matter by a central collapsed object which could eventually swallow up the whole star in about ten million years.

THE HYPOTHESIS OF CORES RETARDED DURING EXPANSION AND THE HOT COSMOLOGICAL MODEL

Ya. B. Zel'dovich and I. D. Novikov

Translated from Astronomicheskii Zhurnal, Vol. 43, No. 4, pp. 758-760, July-August, 1966
Original article submitted March 14, 1966

The existence of bodies with dimensions less than $R_g = 2GM/c^2$ at the early stages of expansion of the cosmological model leads to a strong accretion of radiation by these bodies. If further calculations confirm that accretion is catastrophically high, the hypothesis on cores retarded during expansion [3, 4] will conflict with observational data.

Observational constraints

- Observational constraints on the abundance of PBHs
  - Dark matter mass windows: $\sim \mathbf{10^{16} - 10^{23}}$ g for all CDM and $\sim \mathbf{10^{27} - 10^{28}}$ g and $\sim \mathbf{1 - 10^{3}} M_{\odot}$ for a large fraction of CDM

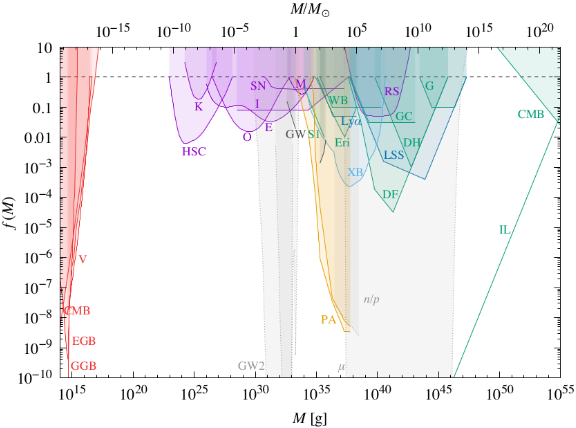


Figure: $f(M) = \Omega_{PBH}/\Omega_{CDM}$ (Carr et al. (2021))

Positive evidence for PBHs ?

- Many BBHs of $\sim 30M_\odot$ discovered by GW observation
  - Those BHs may be of cosmological origin. (Sasaki et al. (2016), Bird et al. (2016), Clesse & Garcia-Bellido (2017)).
  - Search for PBH population in LIGO-Virgo BBHs (Franciolini et al. (2022))
- Evidence for nHz GWs by NANOGrav (Agazie et al. (2023), ...) and other PTAs may be consistent with the secondary GWs of scalar perturbations that may have produced PBHs of solar mass or subsolar masses (Kohri & Terada (2021), Inomata et al. (2023), ...).
- Positive observational evidence for PBHs including subsolar triggers in O2 and O3 of LIGO/Virgo (Carr et al. (2023)).

Outline

Inflationary cosmology

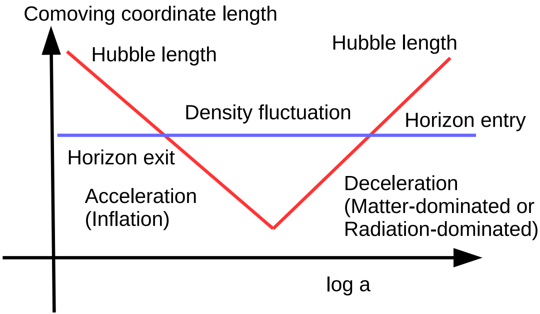


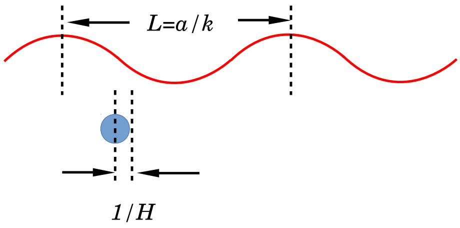
Figure: The time evolution of the Hubble length and the fluctuation scale

- The fluctuations get stretched to super-horizon in an inflationary era.
- After the inflation, the fluctuations are described by long-wavelength solutions.
- Once they enter the Hubble horizon, the long-wavelength scheme breaks down.

Long-wavelength limit

- A smoothing length is $L = a/k$, below which it is described by the FLRW, while the Hubble length is $\ell_H := H^{-1}$, where $H = \dot{a}/a$.
- Expansion parameter $\epsilon \ll 1$

$$\epsilon := \frac{\ell_H}{L} = \frac{k}{aH} \quad \text{with} \quad \frac{\partial_i \ln \Psi}{aH} = O(\epsilon)$$



- In the decelerated expansion, the limit $\epsilon \to 0$ realises as $t \to 0$.
- The term 'long-wavelength' is a bit misleading because we don't take the limit $k \to 0$ but $aH \to \infty$.

Cosmological conformal 3+1 decomposition

- Metric

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2(t)\tilde{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where $\tilde{\gamma} = \eta$ with η_{ij} being the flat 3D metric.

- $\zeta := 2\ln\psi$ is called *curvature perturbation* in cosmology:
- Flat FLRW: $\alpha = 1$, $\beta^i = 0$, $\psi = 1$ and $\tilde{\gamma}_{ij} = \eta_{ij}$

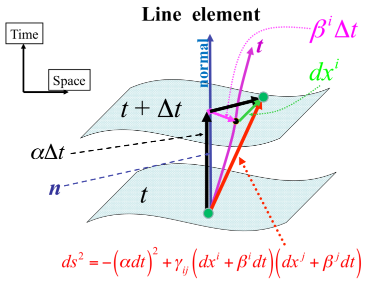


Figure: Slicing and threading with $\gamma_{ij} = \psi^4 a^2(t)\tilde{\gamma}_{ij}$

Long-wavelength solutions

- Additional assumption
  - $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$
  - $p = (\Gamma - 1)\rho$, where $\Gamma = 1 + w$
- We write $\psi = \Psi(\mathrm{x})(1 + \xi)$, $\alpha = 1 + \chi$ and $\tilde{\gamma}_{ij} = \eta_{ij} + h_{ij}$.
- ϵ expansion. The Einstein eqs imply the following:
  - Metric ψ, α, $\tilde{\gamma}_{ij}$: ψ can be nonlinearly large.

$$\Psi(\mathrm{x}) = O(\epsilon^0), \quad \xi = O(\epsilon^2), \quad \beta^i = O(\epsilon),$$
$$\chi = O(\epsilon^2), \quad h_{ij} = O(\epsilon^2)$$

  - Matter ρ, p, u^μ

$$\delta := \frac{\rho - \rho_b}{\rho_b} = O(\epsilon^2), \quad v^i := \frac{u^i}{u^t} = O(\epsilon)$$

  - Extrinsic curvature $K_{ij} = A_{ij} + \gamma_{ij}K/3$

$$K = -3H(1 + \kappa), \ \kappa = O(\epsilon^2), \ \tilde{A}_{ij} = \psi^{-4}a^{-2}A_{ij} = O(\epsilon^2)$$

Tomita (1972), Shibata & Sasaki (1999), Lyth, Malik & Sasaki (2005), Polnarev & Musco (2007)

Gauge issues

- We require the Einstein eqs and the EOM order by order in power of ϵ and solve them to obtain LWL solutions.

  Shibata & Sasaki (1999), Harada, Yoo, Nakama & Koga (2015)
- Gauge issues
  - Slicing: lapse function α
    - Constant-Mean-Curvature slice: $\kappa = 0$
    - Comoving slice: $u_i = 0$
    - ...
  - Threading: shift vector β^i
    - Conformally Flat coordinates: $h_{ij} = 0$
    - Normal coordinates: $\beta^i = 0$
    - ...
  - Caveat: The conformal Newtonian gauge is inconsistent with the ansatz of the LWL solns.

LWL solns: next-to-leading order

- $\Psi(\mathrm{x})$ generate the LWL solutions. The explicit expressions to $O(\epsilon^2)$ are obtained in different gauges (Harada, Yoo, Nakama, Koga (2015)).
- CMC slice

$$\delta_{\mathrm{CMC}} \approx f\left(\frac{1}{aH}\right)^2, \quad u_{\mathrm{CMC}j} \approx \frac{2}{3\Gamma(3\Gamma+2)H}\delta_{\mathrm{CMC},j},$$

where

$$f = f(\mathrm{x}) := -\frac{4}{3}\frac{\bar{\Delta}\Psi}{\Psi^5}$$

with $\bar{\Delta}$ being the flat Laplacian.

- Comoving slice

$$\delta_{\mathrm{com}} \approx \frac{3\Gamma}{3\Gamma+2} f\left(\frac{1}{aH}\right)^2, \quad u_{\mathrm{com}j} = 0.$$

- The above do not depend on the threading condition.

Quasi-local mass in spherical symmetry

- We focus on spherically symmetric spacetimes.

$$ds^2 = g_{AB}(x^C)dx^A dx^B + R^2(x^C)d\Omega^2,$$

where R is the areal radius and A, B and C run over 0 and 1.

- Misner-Sharp mass as total energy enclosed within a sphere of x^C

$$M := \frac{1}{2}R(1 - D_A R D^A R)$$

with D_A being the covariant derivative compatible with g_{AB}.

- The Misner-Sharp mass has an integral form (or Kodama mass):

$$M := -\int_\Sigma S^\mu d\Sigma_\mu,$$

where Σ is a 3-ball bounded by the 2-sphere of x^A.

  - Kodama current: $S^\mu := -T^\mu{}_\nu K^\nu$
  - Kodama vector: $K^\mu := -\epsilon^{AB}\partial_B R\left(\partial/\partial x^A\right)^\mu$.

Misner & Sharp (1964), Kodama (1980), Hayward (1996), Yoo (2022)

Outline

Shibata-Sasaki compaction function

- Shibata and Sasaki (1999) used the conformally flat coordinates

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2[(dr + \beta r dt)^2 + r^2 d\Omega^2],$$

with the CMC slice in spherical symmetry.

- Gave the expressions of the mass excess and the compaction function

$$\delta M_{\rm SS} := 4\pi a^3 \rho_0 \int_0^r x^2 dx \delta_{\rm CMC} \cdot \psi^6 \left(1 + \frac{2x}{\psi}\frac{\partial \psi}{\partial x}\right)$$

$$\mathcal{C}_{\rm SS} := \frac{\delta M_{\rm SS}(t,r)}{R(t,r)} = \frac{\delta M_{\rm SS}(t,r)}{r\psi^2(t,r)a}.$$

- $\mathcal{C}_{\rm SS}$ becomes time-independent in the limit $\epsilon \to 0$ or $t \to 0$, so that

$$\mathcal{C}_{\rm SS}(t,r) \approx \mathcal{C}_{\rm SS}(r) \approx \frac{\int d\Sigma \delta\rho}{R} \approx \frac{1}{2}\bar{\delta}_{{\rm CMC},H}(r),$$

where $\bar{\delta}_{{\rm CMC},H}$ is the density perturbation averaged over the horizon patch at the horizon entry $a\Psi^2 r = H^{-1}$ to the next-leading order of $O(\epsilon)$.

$\mathcal{C}_{\rm SS}$ in terms of Ψ

- $\delta_{\rm CMC}(t,r)$ and $\mathcal{C}_{\rm SS}(r)$

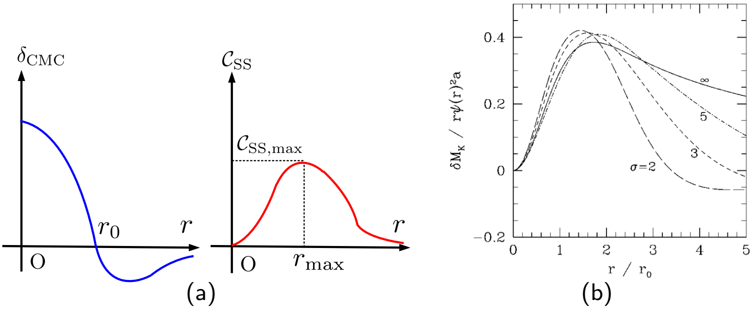


Figure: (a) $\delta_{\rm CMC}(t,r)$, $\mathcal{C}_{\rm SS}(r)$, (b) $\mathcal{C}_{\rm SS}$ for the critical cases

- Empirically, the maximum of $\mathcal{C}_{\rm SS}(r)$ (or its volume average) gives a good indicator for PBH formation. The threshold value is $\simeq 0.4$ for radiation $\Gamma = 4/3$.

Shibata & Sasaki (1999), Escriva et al. (2019)

$\mathcal{C}_{\rm SS}$ in terms of Ψ

- LWL soln in the CMC slice in spherical symmetry

$$\delta_{\rm CMC} \approx f\left(\frac{1}{aH}\right)^2, \; u_{{\rm CMC}r} \approx \frac{2}{3\Gamma(3\Gamma+2)H}\delta_{{\rm CMC},r},$$
$$\Psi = \Psi(r), \;\; f = f(r) = -\frac{4}{3}\frac{1}{r^2\Psi^5}\left(r^2\Psi'\right)'$$

- $\mathcal{C}_{\rm SS}$ in the LWL soln can be rewritten as

$$\mathcal{C}_{\rm SS} \approx \frac{1}{2}\left[1-\left(1+2\frac{d\ln\Psi}{d\ln r}\right)^2\right]$$

This does not contain Ψ''.

Harada, Yoo, Nakama & Koga (2015)

Outline

Mass and mass excess

- Mass in the spatially conformally flat coordinates

$$\begin{aligned} M &= 4\pi \int_0^r x^2 dx a^3 \alpha \psi^6 T^t{}_\mu K^\mu \\ &= 4\pi a^3 \int_0^r dx (\psi^2 x)^2 \{-[(\rho+p)u^t u_t + p](\psi^2 x)' \\ &\quad + (\rho+p) u^t u_r \frac{x}{a} (\psi^2 a)_{,t} \}. \end{aligned}$$

- Mass excess
  - The mass excess from the flat FLRW spacetime is naturally defined as

$$\delta M(t,r) = M(t,r) - M_{\mathrm{FF}}(t, \psi^2(t,r) r),$$

i.e., the difference between masses enclosed by two spheres of the same areal radius.

Mass excess in the CMC slice

- Mass excess

$$\begin{aligned}\delta M_{\rm CMC} &\approx 4\pi a^3\rho_b\int_0^r dx(\Psi^2 x)^2\left[\delta_{\rm CMC}(\Psi^2 x)' \right. \\ &\left. +\frac{2}{3(3\Gamma+2)}\delta'_{\rm CMC}(\Psi^2 x)\right] \\ &= 4\pi a^3\rho_b\left[\frac{3\Gamma}{3\Gamma+2}\int_0^r dx(\Psi^2 x)^2(\Psi^2 x)'\delta_{\rm CMC}\right. \\ &\left.+\frac{2}{3(3\Gamma+2)}\delta_{\rm CMC}(t,r)(\Psi^2(r)r)^3\right],\end{aligned}$$

where integration by parts is implemented. This reduces to

$$\mathcal{C}_{\rm CMC}(r)\approx\frac{3\Gamma}{3\Gamma+2}\mathcal{C}_{\rm SS}(r)+\frac{1}{3\Gamma+2}f(r)(\Psi^2(r)r)^2,$$

where $\mathcal{C}_{\rm CMC}:=\dfrac{\delta M_{\rm CMC}}{R}$ is the ‘legitimate’ compaction function.

$\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm CMC}$

- $\delta M_{\rm CMC}$ is different from $\delta M_{\rm SS}$ due to the nonvanishing $u_{{\rm CMC}\ r}$! There is no direct relation between $\mathcal{C}_{\rm CMC}$ and $\mathcal{C}_{\rm SS}$.
- $\mathcal{C}_{\rm SS}$ does not contain Ψ''. This is why $\mathcal{C}_{\rm SS}$ is empirically robust.
- If $\delta(t,r)$ has a spiky density shell, $\mathcal{C}_{\rm CMC}(r)$ has a large maximum of $O(\Delta^{-1/2})$ at r_1, whereas both $\mathcal{C}_{\rm SS}(r)$ and $\Psi(r)$ are kept small.

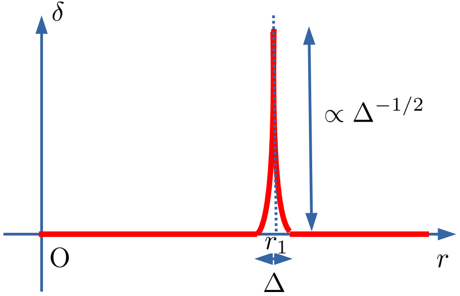


Figure: Density spike

$\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm com}$

- In the comoving slice, we have

$$\delta_{\rm com} \approx \frac{3\Gamma}{3\Gamma+2} f \left(\frac{1}{aH}\right)^2, \quad u_{{\rm com}j} = 0,$$

so that

$$\delta M_{\rm com}(t,r) \approx \frac{3\Gamma}{3\Gamma+2} \delta M_{\rm SS}(t,r)$$

- The compaction function in the comoving slice is thus directly related to $\mathcal{C}_{\rm SS}$ as

$$\mathcal{C}_{\rm com}(r) := \frac{\delta M_{\rm com}}{R} \approx \frac{3\Gamma}{3\Gamma+2} \mathcal{C}_{\rm SS}(r).$$

- The threshold value for the maximum of $\mathcal{C}_{\rm com}(r)$ is therefore $\simeq 0.27$ for $\Gamma = 4/3$.

Summary

- Despite the initial intention, $\mathcal{C}_{\rm SS}$ is not directly related to $\delta M/R$ in the CMC slice but happens to that in the comoving slice up to a constant factor depending on w.

- $\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm com}$ are unique for not containing Ψ''. This is why they are very robust to give a threshold for PBH formation.

Outline

4 Backup
- LWL solns in the Shibata-Sasaki gauge condition
- Calculations
- Alternative choice of the background mass

Outline

4 Backup

- LWL solns in the Shibata-Sasaki gauge condition
- Calculations
- Alternative choice of the background mass

Reconstruction to the next-to-leading order

- The Shibata-Sasaki gauge choice (the CMC slice and the Conformally Flat coordinates) gives the following LWL solutions:

$$\begin{aligned}
\delta &\approx f\left(\frac{1}{aH}\right)^2, \quad u_j \approx \frac{2}{3\Gamma(3\Gamma+2)a^3H^2} f'(r)\delta^r_j, \\
\chi &\approx -\frac{3\Gamma-2}{3\Gamma} f \frac{1}{(aH)^2}, \\
\beta &\approx \left\{-\frac{6}{3\Gamma+2}\int_\infty^r d\tilde{r}\frac{1}{\tilde{r}^3}\left[\frac{1}{\Psi^4(\tilde{r})}\mathcal{C}_{\rm SS}(\tilde{r}) - \frac{1}{2}\tilde{r}^2 f(\tilde{r})\right]\right. \\
&\quad \left. +\tilde{\beta}_\infty\right\}\frac{1}{a^2H} =: \tilde{\beta}(r)\frac{1}{a^2H}, \\
\xi &\approx \frac{1}{2(3\Gamma-2)}\left\{-\frac{2}{3\Gamma+2}\frac{\mathcal{C}_{\rm SS}}{r^2\Psi^4} - \frac{9\Gamma^2-3\Gamma-4}{3\Gamma(3\Gamma+2)} f\right. \\
&\quad \left. +\left(1+\frac{2r\Psi'}{\Psi}\right)\tilde{\beta}(r)\right\}\frac{1}{a^2H^2} =: \tilde{\xi}(r)\frac{1}{a^2H^2},
\end{aligned}$$

where $\tilde{\beta}_\infty$ is a constant of integration.

Mass in terms of the boundary

- The compaction function is originally given in terms of the spatial integral but can also be written in terms of the metric functions at the boundary.
- Shibata and Sasaki used the conformally flat coordinates

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2[(dr + \beta r dt)^2 + r^2 d\Omega^2],$$

  with the CMC slice in spherical symmetry.
- We use the definition of the Misner-Sharp mass

$$\frac{2M}{R} = 1 - D_A R D^A R,$$

  where $R = \psi^2(t,r)a(t)r$. This is an expression in terms of the boundary.

Consistency check

- This expression implies

$$\frac{2\delta M_{\rm CMC}}{R} \approx 1-\left(1+\frac{2r\Psi'}{\Psi}\right)^2 + 2\left[2\dot{\xi}-H\chi-\beta\left(1+\frac{2r\Psi'}{\Psi}\right)\right](\Psi^2 r)^2(a^2 H),$$

where we have used $2M_{\rm FF}/R = H^2R^2$.

- Now that we have the full set of the LWL solution, let us check consistency about the compaction function.
- Using the obtained solution for χ, β and ξ, we can show

$$\mathcal{C}_{\rm CMC}(r) \approx \frac{3\Gamma}{3\Gamma+2}\mathcal{C}_{\rm SS}(r) + \frac{1}{3\Gamma+2}f(r)(\Psi^2(r)r)^2.$$

This coincides with the expression obtained using the spatial integral.

Outline

Calculation 1

- For the LWL soln, this reduces to

$$\frac{2M}{R} \approx H^2R^2 + 1 - \left(1 + \frac{2r\Psi'}{\Psi}\right)^2 + 2\left[2\dot{\xi} - H\chi - \beta\left(1 + \frac{2r\Psi'}{\Psi}\right)\right](\Psi^2 r)^2(a^2 H),$$

where we have used $\alpha = 1 + \chi$ and $\psi = \Psi(r)(1 + \xi)$.

- $2\delta M/R$ is then written as

$$\frac{2\delta M}{R} \approx 1 - \left(1 + \frac{2r\Psi'}{\Psi}\right)^2 + 2\left[2\dot{\xi} - H\chi - \beta\left(1 + \frac{2r\Psi'}{\Psi}\right)\right](\Psi^2 r)^2(a^2 H)$$

because

$$\frac{2M_{\mathrm{FF}}}{R} = H^2R^2$$

for the corresponding FLRW solns.

Calculation 2

- The evolution equations for ψ and h_{ij} in Einstein equations imply

$$2\dot{\xi} - H\chi - \beta\left(1 + \frac{2r\Psi'}{\Psi}\right) - \frac{1}{3}r\beta' \approx 0, \tag{1}$$

$$r\beta' \approx -\frac{3}{r^2}\tilde{A}_{22}, \tag{2}$$

where

$$\tilde{A}_{ij} := \psi^{-4}a^{-2}\left(K_{ij} - \frac{\gamma_{ij}}{3}K\right),$$

and we have imposed $\kappa = 0$ and $h_{ij} = 0$.

Calculation 3

- Using (1), we can eliminate $\dot{\xi}$, χ and β, while $\tilde{A}_{ij}$ is given in the LWL soln by

$$\tilde{A}_{22} \approx \left\{ \frac{1}{2\Psi^4} \left[1 - \left(1 + \frac{2r\Psi'}{\Psi} \right)^2 \right] + \frac{2}{3} r^2 \frac{\bar{\Delta}\Psi}{\Psi^5} \right\} \frac{2}{3\Gamma + 2} \frac{1}{a^2 H}.$$

- Therefore, eliminating β' using (2), we obtain

$$\frac{\delta M}{R} \approx \frac{3\Gamma}{2(3\Gamma+2)} \left[1 - \left(1 + \frac{2r\Psi'}{\Psi} \right)^2 \right] + \frac{1}{3\Gamma+2} f(\Psi^2 r)^2,$$

or

$$\mathcal{C}(r) \approx \frac{3\Gamma}{3\Gamma+2} \mathcal{C}_{\rm SS}(r) + \frac{1}{3\Gamma+2} f(r)(\Psi^2 r)^2,$$

where we have used

$$f = -\frac{4}{3} \frac{1}{r^2 \Psi^5} \left(r^2 \Psi' \right)'.$$

Outline

4. Backup
   - LWL solns in the Shibata-Sasaki gauge condition
   - Calculations
   - Alternative choice of the background mass

Alternative choice of the background mass

- We can alternatively choose $M_{\mathrm{FF}}(t, \Psi^2(r)r)$ rather than $M_{\mathrm{FF}}(t, \psi^2(t, r)r)$ for the background mass.
- This implies an alternative mass excess and a compaction function:

$$\begin{aligned} \Delta M(t, r) &:= M(t, r) - M_{\mathrm{FF}}(t, \Psi^2(r)r), \\ \mathscr{C}_{\mathrm{CMC}}(t, r) &:= \frac{\Delta M(t, r)}{R(t, r)} \approx \mathscr{C}_{\mathrm{CMC}}(r). \end{aligned}$$

- This definition implies

$$\mathscr{C}_{\mathrm{CMC}} - \mathcal{C}_{\mathrm{SS}} = \left(3H\xi + \frac{1}{3}r\beta'\right)(\Psi^2 r)^2 a^2 H$$

in the Shibata-Sasaki gauge conditions.

- In general, $\mathcal{C}_{\mathrm{SS}}$ does not coincide with $\mathscr{C}_{\mathrm{CMC}}$ either.