# Gravitational instantons, CP asymmetry and axion physics

Archil Kobakhidze
University of Sydney

Talk is based on:

S. Arunasalam, AK, Eur. Phys. J. C 79 (2019) 1, 49; e-Print: 1808.01796 [hep-th]

Z. Chen, AK, Eur. Phys. J. C 82 (2022) 7, 596; e-Print: 2108.05549 [hep-ph]

Z. Chen, AK, C.A.J. O'Hare, Z.S.C. Picker, G. Pierobon, e-Print: 2109.12920 [hep-ph]; 2110.11014 [hep-ph].

## Are quantum gravity effects (phenomenologically) relevant for particle physics?

Perry, 79’

Deser, Duff, Isham 80’

Dvali 05’, 22’; Dvali & Funcke 16’

# Instantons and CP violation in Standard Model

$$\mathcal{L}_{\rm CPV} = \frac{\theta_{QCD}}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \frac{\theta_{QED}}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Both of these terms are total 4-derivatives, e.g.,

$$F_{\mu\nu}\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \partial_\mu\left(K^\mu\right),\; K^\mu = \epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$$

- The QED-term can be ignored – no physical effects
- The QCD-term is physical, e.g. describes vacuum-to-vacuum transition amplitudes 'mediated' by QCD instantons. This term is mandatory to preserve causality ('cluster decomposition')!

- Neutron EDM: $d_n \simeq e\theta_{QCD} m_q/m_N^2 \implies \theta_{QCD} \lesssim 10^{-10}$ .

  The strong CP problem (Crewther, di Vecchia, Veneziano, Witten, 79')

# Anomalies and instantons

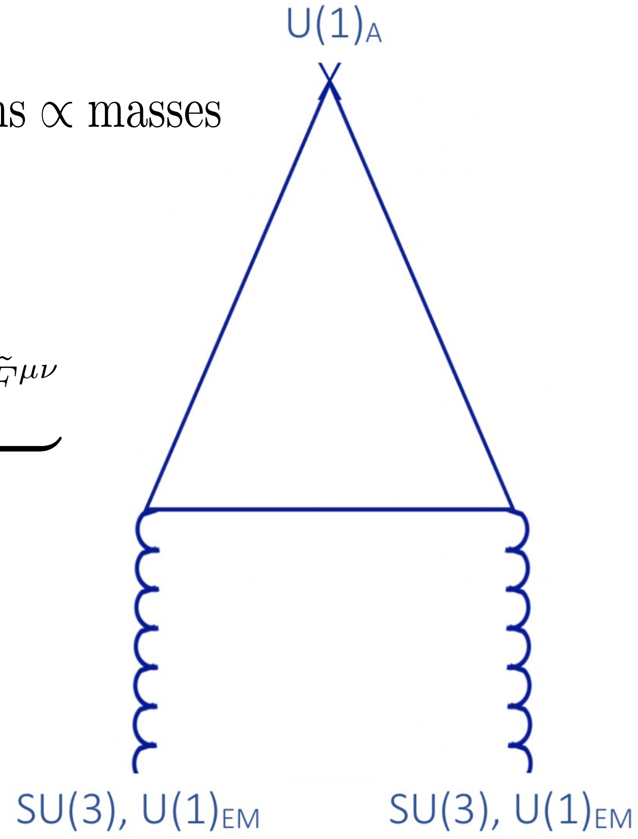
Mixed global-gauged anomalies (Adler; Bell & Jackiw, 69'):

$$\partial_\mu J^\mu_A = \frac{g^2 N_{QCD}}{16\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \frac{e^2 N_{QED}}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + \text{terms} \propto \text{masses}$$

$$\Delta Q_A \propto N_{QCD} \underbrace{\int d^4x\, \frac{g^2}{16\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}} \; + \; N_{QED} \underbrace{\int d^4x \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}}$$

Chern-Pontryagin index, $P_{SU(2)\in SU(3)} = n \in Z;\ \pi_3(S^3) = Z$

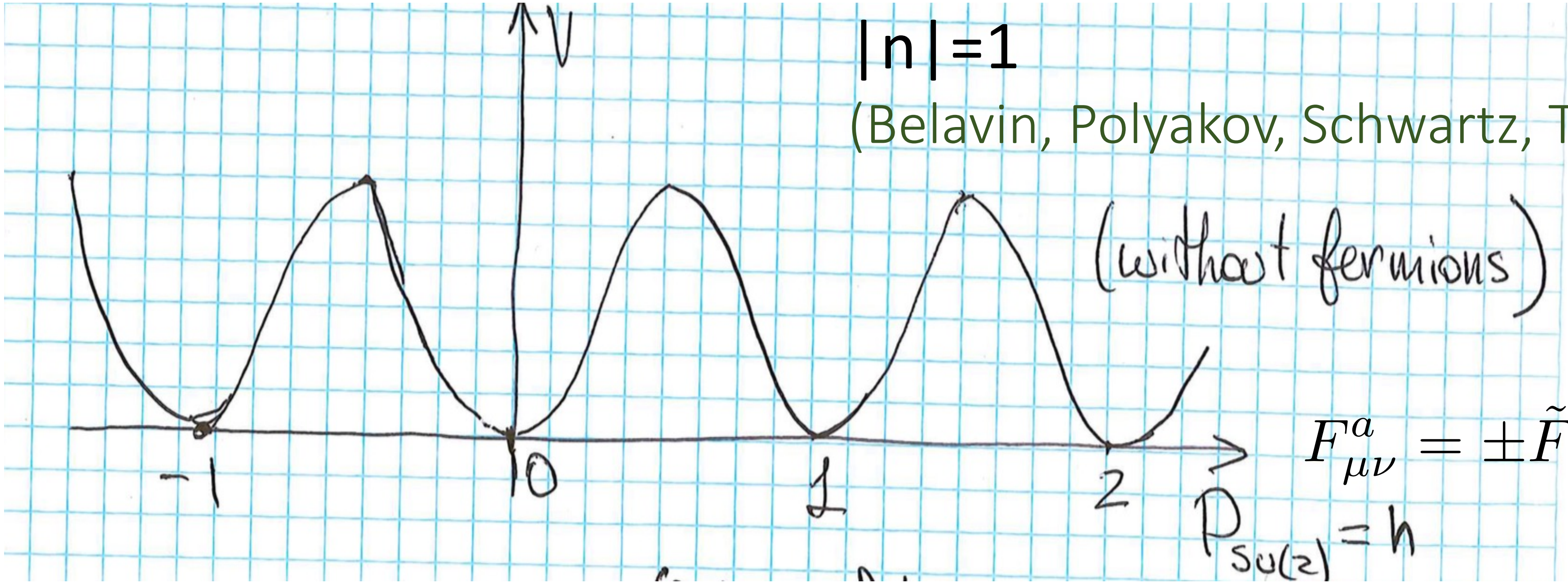
$P_{U(1)} = 0$



# Anomalies and instantons

$$\Delta Q_A \propto N_{QCD} \int_{\partial M = S^3} dx_\mu K^\mu_{QCD} \Big|_{A^a_\mu \overset{|x|\to\infty}{\longrightarrow} \frac{i}{g} U \partial_\mu U^\dagger,\ U \in SU(2)}$$

$$= N_{QCD} \nu_{1/2} = N_{QCD} \left( n_L - n_R \right)$$ (Atiyah, Singer, 63')




|n|=1

(Belavin, Polyakov, Schwartz, Tyupkin, 73')

$$F^a_{\mu\nu} = \pm \tilde{F}^a_{\mu\nu},\ S_E = \frac{8\pi^2}{g^2}$$

# Anomalies and instantons

In the presence of fermions instantons induce multi-fermion interactions (‘t Hooft vertices), e.g.,

$$\mathcal{L}_{inst} \propto \bar{u}_L u_R \bar{d}_L d_R \bar{s}_L s_R + \mathrm{h.c.}$$

$$\mathcal{A}_{inst} \propto \mathrm{e}^{-S_E} = \mathrm{e}^{-\frac{8\pi^2}{g^2}}$$

Explains $m_\eta > m_\pi$ (‘tHooft, 76’)

# Axion solution to the strong CP problem

Introduce a new light pseudo-scalar a(x) (Peccei-Quinn 77')

$$\mathcal{L}_{axion} \propto \frac{\theta_{QCD} + N_{QCD} a(x)/f_a}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

Instanton-induced potential:

$$V(a) = -2K \cos\left(N_{QCD} a(x)/f_a + \theta_{QCD}\right), \; K \approx m_\pi^2 f_\pi^2$$

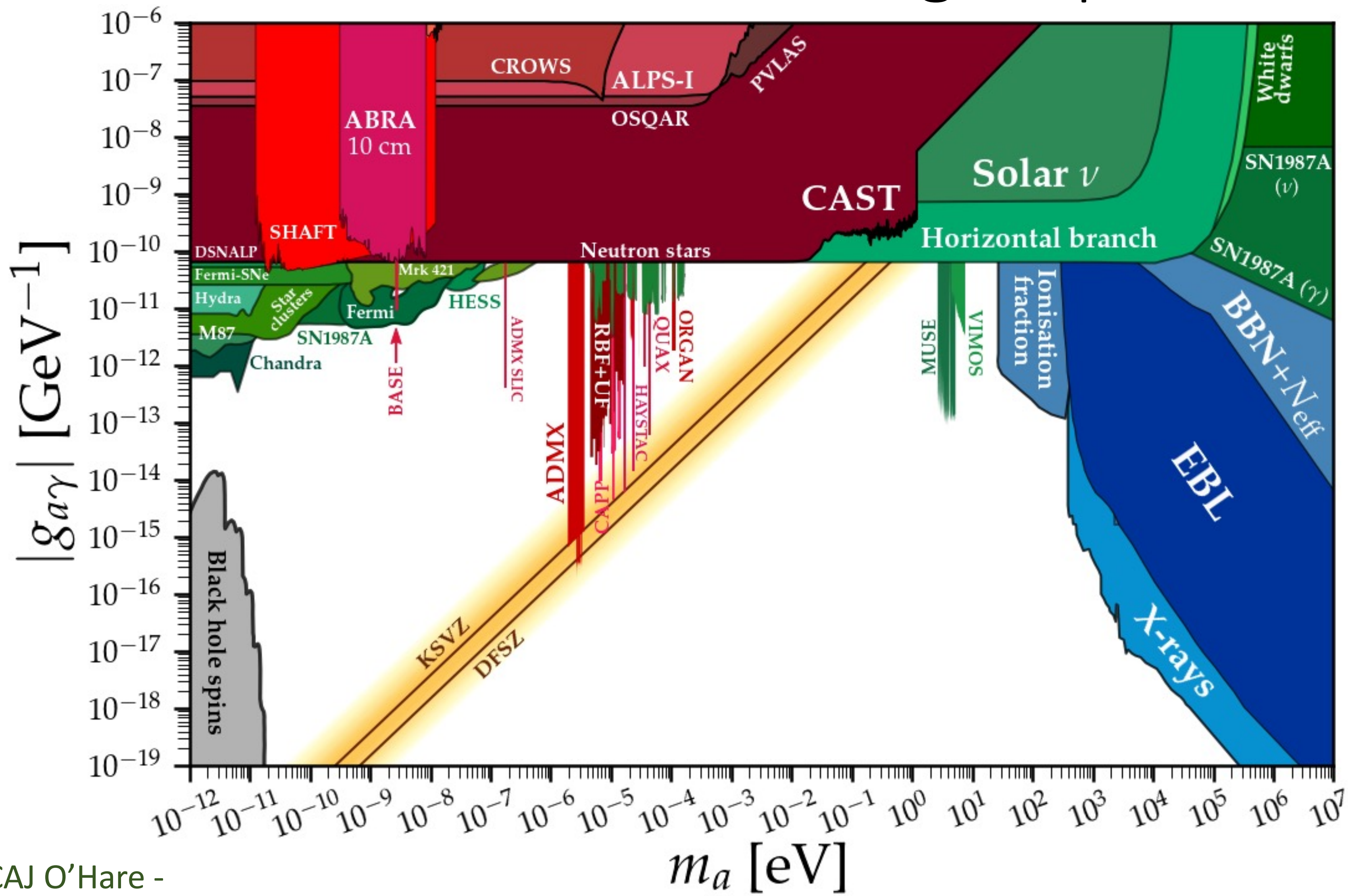
In the minimum CP-violating phase cancels out:

$$\langle a \rangle = -f_a \theta_{QCD}/N_{QCD}$$

Light, feebly coupled, essentially stable => dark matter candidate

$$m_a \approx m_\pi \frac{f_\pi}{f_a} \; (f_\pi << f_a)$$

# Axion solution to the strong CP problem

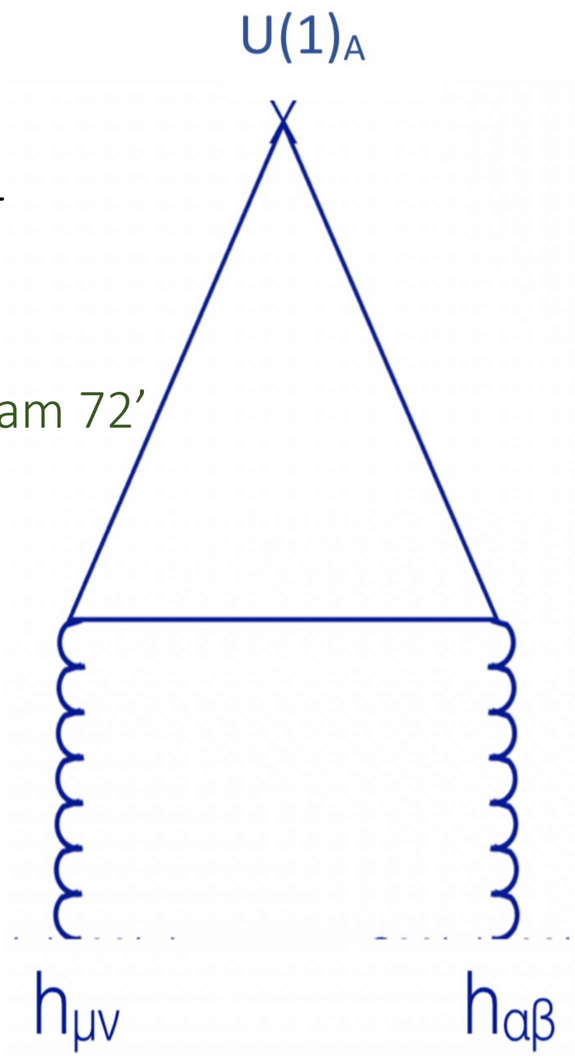



CAJ O'Hare -
https://github.com/cajohare/AxionLimits/blob/master/docs/ap.md

# Anomalies in the Standard Model + Gravity

$$\nabla_\mu J^\mu_A = \frac{N_g}{192\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}, \quad \tilde{R}^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta}{}^{\rho\sigma}$$

Delbourgo, Salam 72'




Gravitational instantons?

# Gravitational instantons: generalities

- In quantum gravity imitate instanton effects by summing up over manifolds $(M, g_{\mu\nu})$ with an arbitrary topology in the Euclidean path integral

- Problem:

$$S_{\mathrm{EG}} \lesseqgtr 0$$

# Gravitational instantons: generalities

- Positive action theorem (Schoen, Yau 79'; Witten 81')

For Ricci flat, $R_{\mu\nu} = 0$, (a.k.a, vacuum manifolds):

(a) Asymptotically Euclidean (AE) spaces, $S_{\rm EG} \geq 0$ ; $S_{\rm EG} = 0$ , if and only if the space is flat

(b) Asymptotically Locally Euclidean (ALE) spaces, $S_{\rm EG} \geq 0$ ; $S_{\rm EG} = 0$ for self (anti-self)-dual configurations!

$$S_{\rm EG}^{\rm inst} = 0!$$

# Gravitational instantons: generalities

- For $R_{\mu\nu} = 0$ AE/ALE there are no fermion zero-modes even for massless fermions

$$\nabla\!\!\!/\,\psi = 0,\ \nabla\!\!\!/ = \gamma^\mu \nabla_\mu$$
$$= \gamma^\mu \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \sigma_{ab} \right)$$

- Hence, $\bar{\nabla\!\!\!/}\nabla\!\!\!/ = -\nabla^2 + \frac{1}{8} R_{\mu\nu ab}\, \sigma^{\mu\nu} \sigma^{ab} = -\nabla^2 > 0$

$$\nabla^2 \psi = 0 \Longrightarrow \psi = 0$$

Such instantons would not induce chiral symmetry breaking

# Eguchi-Hanson instanton

(Eguchi, Hanson 78'; Gibbons, Hawking, Perry 78')

- Anti-self dual solution:

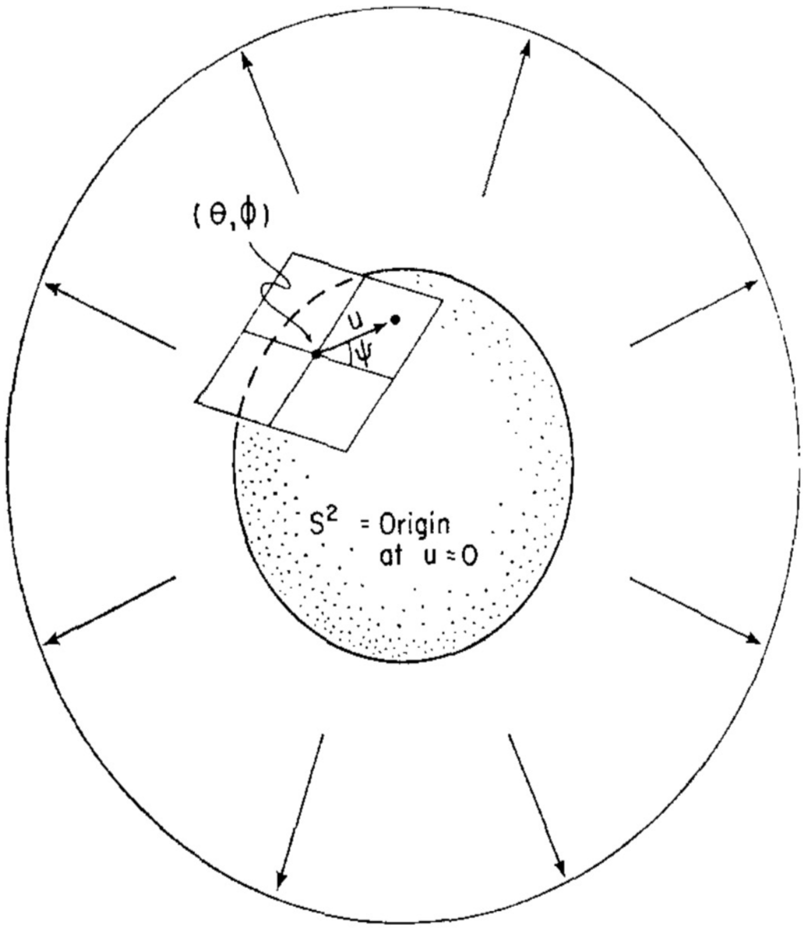
$$ds^2 = \frac{dr^2}{1-\left(\frac{a}{r}\right)^4} + r^2\left(\sigma_x^2 + \sigma_y^2 + \left[1-\left(\frac{a}{r}\right)^4\right]\sigma_z^2\right)$$

(i) a is an arbitrary length parameter, the instanton size

(ii) a -> 0, flat space limit

(iii) Coordinate (non-physical) singularity at r=a

# Eguchi-Hanson instanton




taken from Eguchi, Hanson, Annals of Physics 120, 82 (1979)

- Geodesic completeness: $0 \leq \psi \leq 2\pi$
- We 'half' the space, it got a boundary
  $$r \to \infty,\ S^3/Z_2 = RP^3$$
- Note:

$$\pi_1(S^3/Z_2) = Z_2 \ vs \ \pi_1(S^3) = 0$$

# Eguchi-Hanson instanton

$$S_{\rm EG} = \underbrace{\frac{-1}{16\pi G}\int d^4x\sqrt{g}R}_{=0,\ \text{Ricci flat}} - \frac{1}{8\pi}\int_{\partial M(r\to\infty)} K d\Sigma$$

$$= \frac{\pi}{8}\left[3r^2 - \frac{a^4}{r^2} - 3r^2\left(1 - a^4/r^4\right)^{1/2}\right]\Bigg|_{r\to\infty}$$

$$= \frac{\pi}{16}\frac{a^4}{r^2}\Bigg|_{r\to\infty} = 0$$

- Index of the Dirac operator:

$$\begin{aligned} \nu_{1/2}(\nabla\!\!\!/) &= \frac{1}{24}\left(p - q(S^3/Z_2)\right) - \frac{1}{2}\left(\eta_{1/2} - h_{1/2}\right) \\ &= \frac{1}{24}(3-0) - \frac{1}{2}(1/4 - 0) = 0 \end{aligned}$$

# $U(1)_{EM}$-charged Eguchi-Hanson instanton

- Let's consider QED. In the limit of massless electron we have chiral anomaly. In flat spacetime, $U(1)_{EM}$ vacuum is topologically trivial, hence no CP-violation occurs in the effective Lagrangian

- Abelian nature of $U(1)_{EM}$ – no explanation of electric charge quantisation

- Inclusion of gravity -> CP-violation + charge quantisation!

# New CP violation in the Standard Model

$$\mathcal{L} \propto \theta_3 G\tilde{G} + \theta_2 W\tilde{W} + \theta_1 B\tilde{B} + \theta_{EH} R\tilde{R}$$

- One of the phases is unphysical and can be removed by B+L phase redefinition of quarks and leptons => 3 phases in the strong, electroweak and gravitational sectors remain!
- Interesting phenomenological and cosmological implications

# Coloured Eguchi-Hanson instanton

- We can extend the above construction to Yang-Mills (SU(2))-Eguchi-Hanson instantons

$$A^a_\mu = \frac{1}{2}\eta^a_{AB}\omega^{AB}_\mu \; ,$$

$$\omega^{01}_\theta = \omega^{23}_\theta = \omega^{02}_\phi = \omega^{31}_\phi = \frac{1}{2}\sqrt{1-\frac{a^4}{r^4}} \; ,$$

$$\omega^{03}_\psi = \omega^{12}_\psi = \frac{1}{2}\left(1+\frac{a^4}{r^4}\right) \; ,$$

- Action $S_{\rm CEH} = \frac{1}{4g^2}\int d^4x\sqrt{g}F^{a\mu\nu}F^a_{\mu\nu} = \frac{4\pi^2}{g^2}\times 3$

# Coloured Eguchi-Hanson instanton

- Chern-Pontryagin index

$$p = \frac{1}{16\pi^2} \int d^4x \sqrt{g} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}$$
$$= \frac{2}{32\pi^2} \int d^4x \sqrt{g} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} = 3 \,.$$

- Fermion index (R – irrep of SU(2))

$$\nu^{(R)}_{1/2} = \begin{cases} \frac{1}{4} d_R (d_R^2 - 2) \,, \text{ for } d_R = 2, 4, \ldots \\ \frac{1}{4} d_R (d_R^2 - 1) \,, \text{ for } d_R = 1, 3, \ldots \end{cases}$$

- 'tHooft vertecies: $\propto \mathrm{e}^{-\frac{12\pi^2}{g^2}} \det\left(\psi_L \bar{\psi}_R\right)$ vs $\propto \mathrm{e}^{-\frac{8\pi^2}{g^2}} \det\left(\psi_L \bar{\psi}_R\right)$

# Coloured gravitational instantons and axion

- We have extra CP violation due to the colored gravitational instantons

$$S_{eff}[\Phi] = S[\Phi] + \frac{\theta_{QCD}}{32\pi^2}\int d^4x\sqrt{g}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \frac{\theta_{EH}}{48\pi^2}\int d^4x\sqrt{g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

- Effective vacuum angle $\theta_g = 3\theta_{QCD}/2 + \theta_{EH}$
- Instanton action:

$$S_{\text{inst}} = \int d^4x\sqrt{g}\left(2M_{\text{P}}^2\, R + \frac{1}{2g^2}\text{Tr}F_{\mu\nu}F^{\mu\nu} + ...\right) = \frac{4\pi^2}{g^2}\times 3$$

- Does the original axion solution to the strong CP problem hold?

# Computing the axion potential

$$\Delta\mathcal{L} = -d\left(\frac{2\pi}{\alpha_s(\Lambda)}\right)^6 \exp\left[-\frac{2\pi}{\alpha_s(\Lambda)} + iN\frac{a}{f_a} + i\theta\right]$$
$$\times\ \frac{d\rho}{\rho^5}(\Lambda\rho)^b \det\left(q_{iL}\bar{q}_{jR}\right)$$
$$-\ \bar{d}\left(\frac{2\pi}{\alpha_s(\Lambda)}\right)^8 \exp\left[-\frac{3\pi}{\alpha_s(\Lambda)} + iN_g\frac{a}{f_a} + i\theta_g\right]$$
$$\times\ \frac{d\rho}{\rho^5}(\Lambda\rho)^{3b/2} \det\left(q_{iL}\bar{q}_{jR}\right) + \text{h.c.}$$

$$V(a) = -2K\cos\left(N\frac{a}{f_a} + \theta\right) - 2\kappa K\cos\left(N_g\frac{a}{f_a} + \theta_g\right)$$

The CEH contribution is large – $\kappa \approx 0.04 - 0.6$

The one-axion solution is no-longer valid!

Chen, AK 21’

# Companion axion model

- Extend PQ symmetry $U(1)_{PQ} \times U(1)'_{PQ} \longrightarrow 1$

- Two coupled QCD axions

$$V(a,a') = -2K\cos\left(N\frac{a}{f_a} + N'\frac{a'}{f'_a} + \theta\right) - 2\kappa K\cos\left(N_g\frac{a}{f_a} + N'_g\frac{a'}{f'_a} + \theta_g\right)$$

Chen, AK, O'Hare, Picker, Pierobon 21'

# Companion axion model

## Masses and mixing

- Hierarchical case, $\epsilon = f_a/f'_a << 1$

$$m_1^2 \approx \frac{2K\left(N^2+\kappa N_g^2\right)}{f_a^2}\,,$$

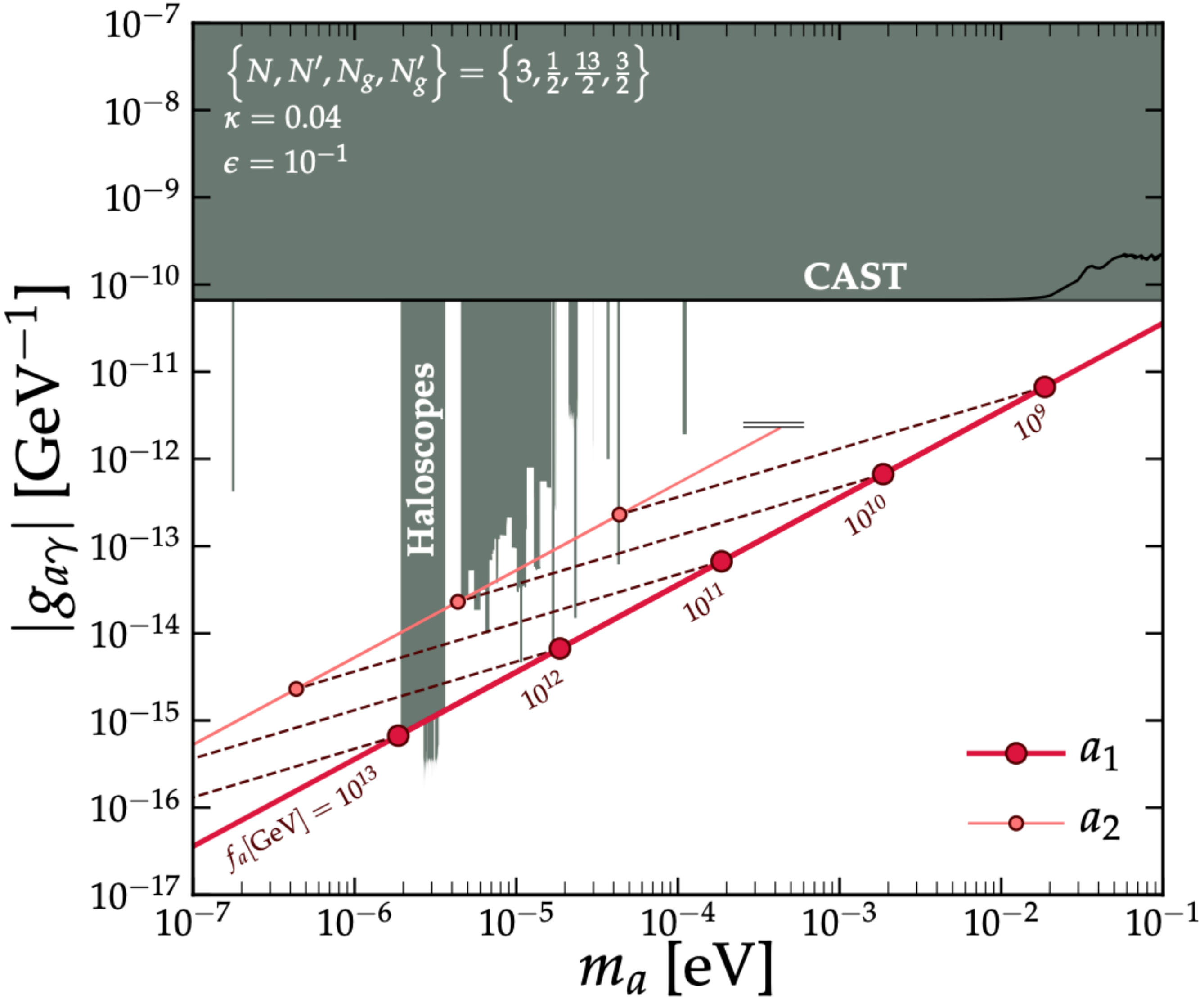
$$m_2^2 \approx \frac{\kappa\left(NN'_g - N_g N'\right)^2}{N^4+\kappa N^2 N_g^2}\epsilon^2 m_1^2 \sim \kappa\epsilon^2 m_1^2 \qquad \alpha \approx \frac{NN'+\kappa N_g N'_g}{N'^2+\kappa N_g'^2}\epsilon \sim \epsilon \ll 1\,.$$

- Large mixing, $\epsilon \sim 1$ (axion-axion oscillation effects)

$$m_1^2 \approx \frac{2K}{f_a^2}\left[N^2+N'^2+\kappa\frac{N^2N_g^2+N'^2N_g'^2}{N^2+N'^2}\right]$$

$$m_2^2 \approx \frac{2K\kappa}{f_a^2}\,\frac{N^2N_g'^2+N'^2N_g^2}{N^2+N'^2} \sim \kappa m_1^2 \qquad \tan 2\alpha \approx -\frac{2\left(NN'+\kappa N_g N'_g\right)}{\left(N^2-N'^2\right)+\kappa\left(N_g^2-N_g'^2\right)}$$

# Companion axion model [2109.12920]



# Companion axion model: bounds and projections

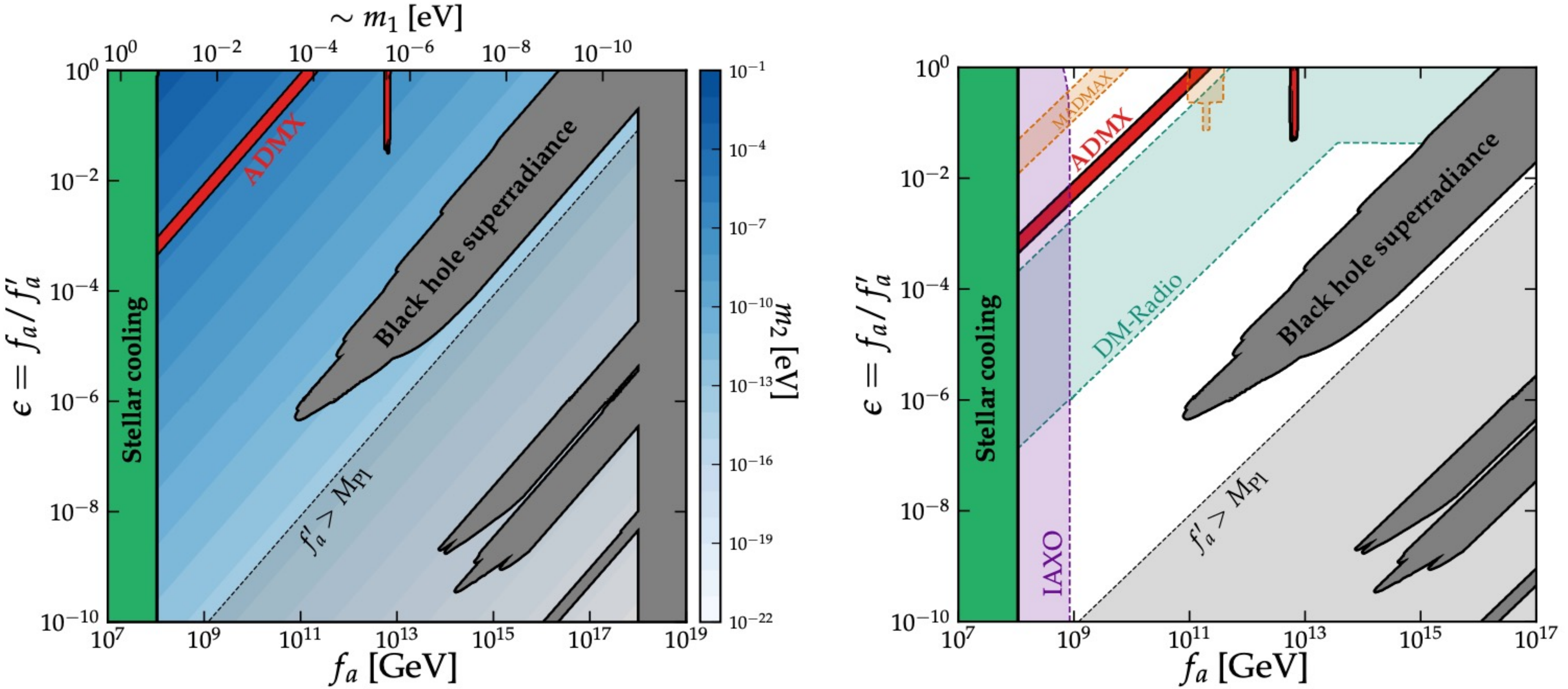
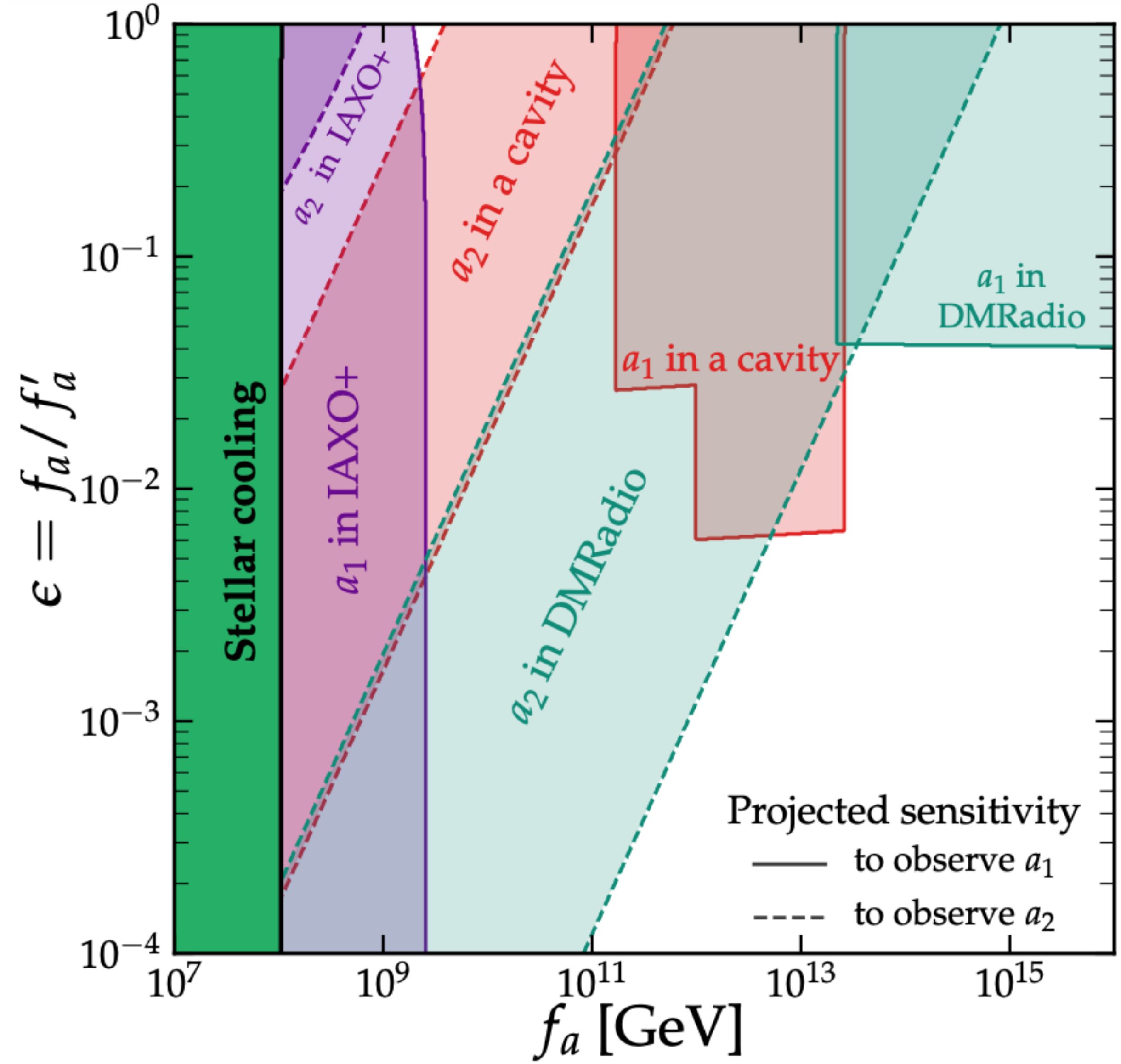



FIG. 2. **Left:** Current bounds on the companion-axion model. The colorscale corresponds to the value of the lighter axion's mass, whereas the heavier axion's mass is shown (roughly) by the upper horizontal axis. We can rule out parts of this parameter space using stellar cooling arguments, ADMX, and black hole superradiance. **Right:** As in the left-hand panel, but now showing projected constraints from future experiments: MADMAX [58], IAXO [59] and DMRadio/ABRACADABRA [60, 61].

# Companion axion model: bounds and projections



# Companion axion dark matter [2110.11014]

- Both axions are long-lived and contribute to the dark matter energy density
- Three qualitatively different scenarios for axion production:

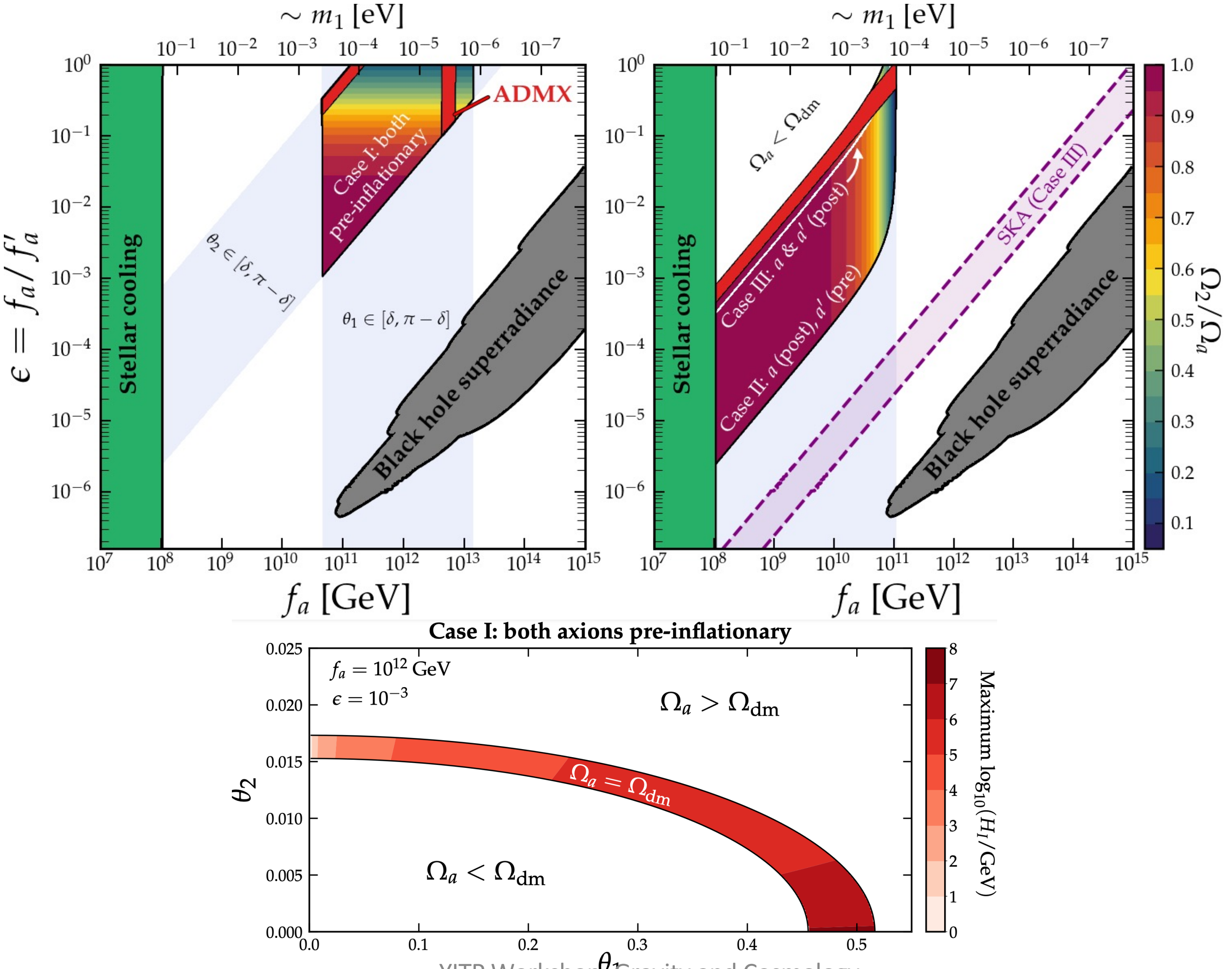
I. Both axions produced before inflation, $f_a,\ f'_a > H_{\rm inf}$

II. QCD-like axion post- and the companion axion pre-inflation, $f_a < H_{\rm inf},\ f'_a > H_{\rm inf}$

III. Both axions produced after inflation, $f_a,\ f'_a < H_{\rm inf}$

- Typically, the companion axion is a dominant dark matter component

# Companion axion dark matter [2110.11014]



# Axion strings and domain walls [2110.11014]

$$V(a) = -2K\cos\left(N\frac{a}{f_a} + \theta\right), \quad Z_N : a \to a + \frac{2\pi k f_a}{N}$$

$$U(1)_{\mathrm{PQ}} \to Z_N \to 1$$

- The standard axion potential exhibits $Z_N$ symmetry which is spontaneously broken by axion condensate => a long-lived network of domain walls attached to strings form => axion domain wall problem
- Introduce explicit PQ (and $Z_N$) breaking interactions:

$$V(a) = -2K\cos\left(N\frac{a}{f_a} + \theta\right) + \delta V(a), \quad \frac{\delta V}{V} < 10^{-9}$$

# Axion strings and domain walls [2110.11014]

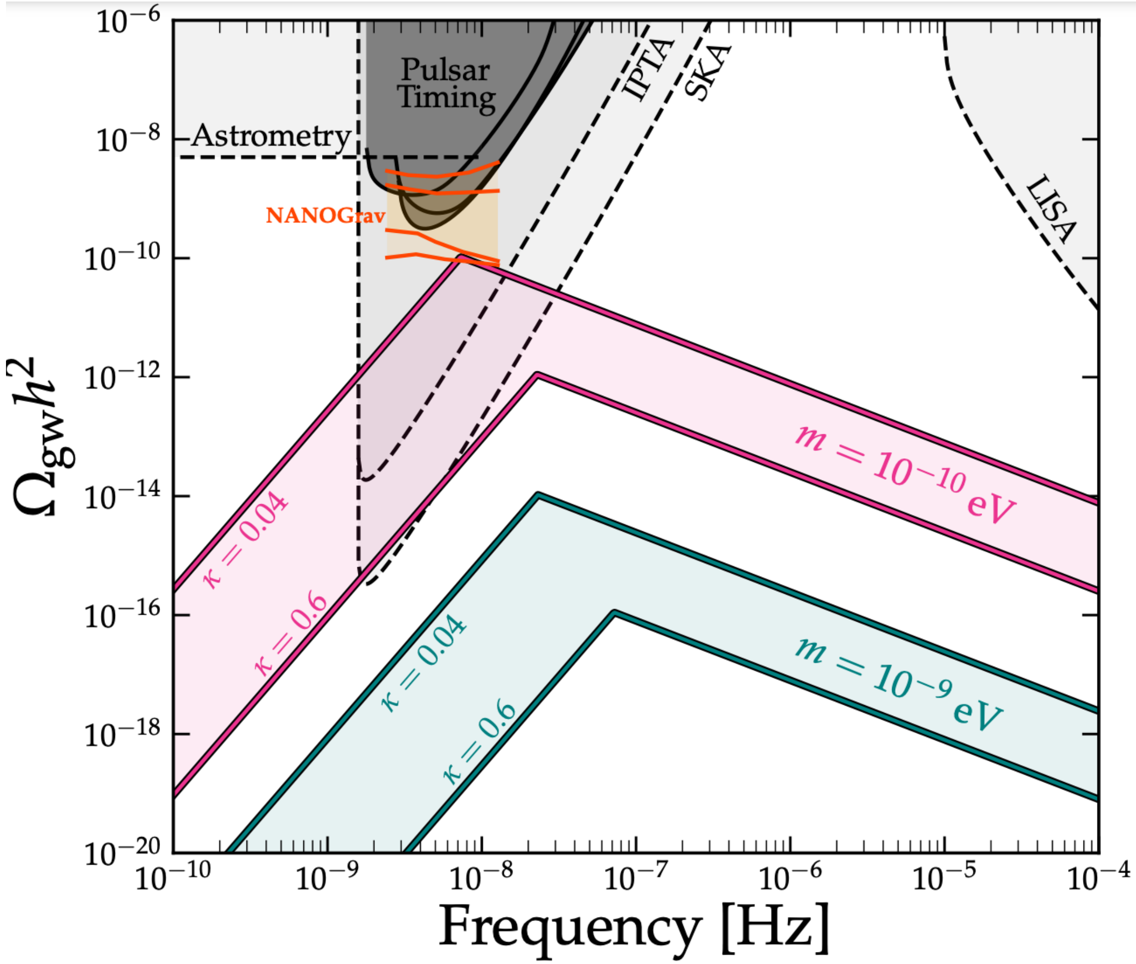
$$V(a,a') = -2K\cos\left(N\frac{a}{f_a} + N'\frac{a'}{f'_a} + \theta\right) - 2\kappa K\cos\left(N_g\frac{a}{f_a} + N'_g\frac{a'}{f'_a} + \theta_g\right)$$

$$U(1)_{PQ} \times U(1)'_{PQ} \longrightarrow 1$$

- A typical companion axion model does not exhibits the domain wall problem => domain walls does not even form except when both axions have masses in a small mass range:

$$10^{-10}\ \mathrm{eV} \lesssim m_i \lesssim 10^{-9}\ \mathrm{eV}$$

# Gravitational waves and PBH predictions [2110.11014]




Collapse of the false vacuum domain walls of the horizon-size lead to black hole formation:

$$M_{\rm PBH} \sim \frac{\sqrt{3}}{4\sqrt{2}} \frac{M_P^3}{(\pi\kappa K)^{1/2}} \sim 150\ M_\odot \left(\frac{\kappa}{0.1}\right)^{-1/2}$$

$$p_{\rm coll} \sim e^{-(T_{\rm ann}/T_{\rm coll})^2} \sim 10^{-22} - 10^{-9}$$

$$f_{\rm PBH} = \frac{\rho_{\rm PBH}}{\rho_{\rm dm}} \simeq 34.9\ p_{\rm coll} \frac{M_{\rm P}^4}{H_0^2 M_{\rm PBH}^2} \left(\frac{T_0}{T_{\rm coll}}\right)^3 \sim 10^{-13} - 1$$

# Conclusions

- Contrary to a widespread belief, (nonperturbative) quantum gravity effects may have significant phenomenological implications in particle physics
- Instanton picture: Gauge-Eguchi-Hanson instantons
    - Spin structure => electric charge quantisation;
    - Extra, unexplored sources of CP violation in the Standard Model;
    - Potential implications in cosmology & elsewhere
- Colored gravitational instantons -> additional companion axion
    - Rich phenomenology with interesting predictions for ongoing and planned axion searches
    - Cosmology – dark matter, nano-Hz gravitational wave signals; LIGO-sized PBHs

# Eguchi-Hanson instanton
(Eguchi, Hanson 78'; Gibbons, Hawking, Perry 78')

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac}\omega_\nu^{cb} - \omega_\nu^{ac}\omega_\mu^{cb};$$

$$T_{\mu\nu}^{a} = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu^{ab} e_{\nu b} - \omega_\nu^{ab} e_{\mu b} = 0 \text{ (torsion-free)};$$

$$g_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b \text{ (metric formulation)}$$

- We are looking for (anti)self-dual solutions:

$$\omega_\mu^{ab} = \pm\tilde{\omega}_\mu^{ab} = \pm\frac{1}{2}\epsilon^{abcd}\omega_\mu^{cd};$$

$$R_{\mu\nu}{}^{ab} = \pm\tilde{R}_{\mu\nu}{}^{ab} = \pm\frac{1}{2}\epsilon^{abcd} R^{\mu\nu\ cd}$$

# Eguchi-Hanson instanton

- Coordinates: (r, θ,ϕ,ψ)
- Flat space: $$ds^2 = dr^2 + r^2\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right)$$

$$\sigma_x = \frac{1}{2}\left(\sin\psi \mathrm{d}\theta - \sin\theta\cos\psi \mathrm{d}\phi\right);$$

$$\sigma_y = \frac{1}{2}\left(-\cos\psi \mathrm{d}\theta - \sin\theta\sin\psi \mathrm{d}\phi\right);$$

$$\sigma_y = \frac{1}{2}\left(-\cos\psi \mathrm{d}\theta - \sin\theta\sin\psi \mathrm{d}\phi\right);$$

$$\sigma_z = \frac{1}{2}\left(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi\right)$$

$$0 \leq \theta < \pi,\ 0 \leq \phi < 2\pi,\ 0 \leq \psi \leq 4\pi$$

# $U(1)_{EM}$-charged Eguchi-Hanson instanton

- Vacua are classified by

$$H^2(S^3/Z_2, \pi_1(U(1))) = Z_2 \ [Z_n \text{ for n-centrered instantons}]$$

- We look for (anti)self-dual solutions to Einstein-Maxwell equations

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \Rightarrow T^{\rm EM}_{\mu\nu} = 0 \quad (dF = d^*F = 0)$$

- Anti-self dual solution:

$$A_r = A_\theta = 0,\ A_\psi = \frac{qa^2}{2r^2},\ A_\phi = \frac{qa^2}{2r^2}\cos\theta$$

q – is a charge of the instanton

# U(1)$_{EM}$-charged Eguchi-Hanson instanton

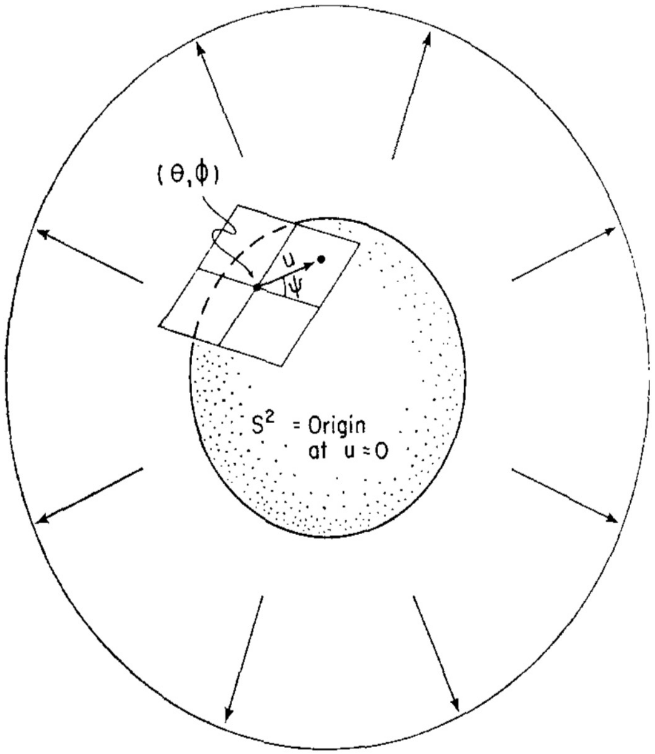
- Compute action: $$S_{EcG} = \frac{1}{4e^2}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}$$ $$= \frac{4\pi^2 q^2}{e^2} = \frac{\pi q^2}{\alpha}$$

- Note π vs 2π – the effect of 'half' space;
- $\alpha(a \to 0) \to \infty$, hence small size instantons contribute predominantly to transition amplitudes;
- No flat space analogue;
- Can charge q be arbitrarily small? (enhanced amplitudes?)

# Charge quantisation

- The existence of fermions (spin structure)

$$-e\int_0^{2\pi}\int_0^{\pi} F_{\theta\phi}\,d\theta d\phi = -e\int_0^{2\pi}\int_0^{\pi} -\frac{a^2 q\sin(\theta)}{2r^2}\,d\theta d\phi$$
$$= 2\pi e q$$
$$= 2n\pi$$




Arunasalam, AK 18'

# $U(1)_{EM}$-charged Eguchi-Hanson instanton

- Spinor structure is supported if, and only if electric charge is quantised:

$$qQ_e = n \in Z$$

(q=3 is the smallest charge, since $Q_d$=1/3).

- Vacuum-to-vacuum transitions are non-zero => CP-violating term in QED is supported:

$$\theta_{QED} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

- Index theorem => fermion zero modes

# Fermion zero-modes in the chiral limit

$$\not{D}^2\Psi_1 =$$

$$\frac{(a^4-r^4)}{r^4}\partial_{rr}\Psi_1(r,\theta) - \frac{(a^4+3r^4)}{r^5}\partial_r\Psi_1(r,\theta)$$
$$+\frac{4i\left(\sqrt{1-\frac{a^4}{r^4}}+i\cot(\theta)\right)}{r^2}\partial_\theta\Psi_1(r,\theta) - \frac{4}{r^2}\partial_{\theta\theta}\Psi_1(r,\theta)$$
$$+\frac{\left(3a^8+a^4r^4\left(q^2y^2-3\right)+r^4\left(r^4-a^4\right)\left(\csc^2(\theta)+2i\sqrt{1-\frac{a^4}{r^4}}\cot(\theta)\right)+r^8\right)}{r^6\left(r^4-a^4\right)}\Psi_1(r,\theta)$$
$$-\frac{2a^2qy\left(a^4-2r^4\right)}{r^4\left(r^4-a^4\right)}\Psi_2(r,\theta)$$

$$\not{D}^2\Psi_2 =$$

$$\frac{(a^4-r^4)}{r^4}\partial_{rr}\Psi_2(r,\theta) - \frac{(a^4+3r^4)}{r^5}\partial_r\Psi_2(r,\theta)$$
$$+\frac{-4i\left(\sqrt{1-\frac{a^4}{r^4}}-i\cot(\theta)\right)}{r^2}\partial_\theta\Psi_2(r,\theta) - \frac{4}{r^2}\partial_{\theta\theta}\Psi_2(r,\theta)$$
$$+\frac{\left(3a^8+a^4r^4\left(q^2y^2-3\right)+r^4\left(r^4-a^4\right)\left(\csc^2(\theta)-2i\sqrt{1-\frac{a^4}{r^4}}\cot(\theta)\right)+r^8\right)}{r^6\left(r^4-a^4\right)}\Psi_2(r,\theta)$$
$$-\frac{2a^2qy\left(a^4-2r^4\right)}{r^4\left(r^4-a^4\right)}\Psi_1(r,\theta)$$

$$\psi_0 \propto \frac{a^2}{r^5} \text{ (in the singular gauge)}$$

Arunasalam, AK 18'