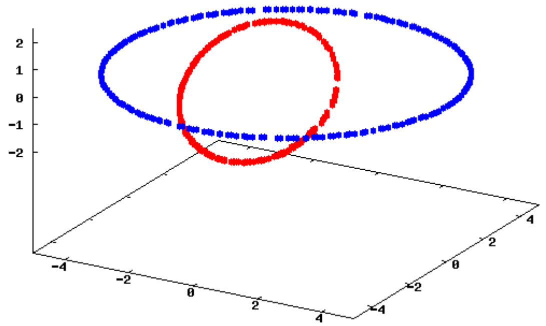
Chaotic vZLK oscillations of a binary system around SMBH and gravitational waves



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Collaboration with

P. Gupta,

H. Okawa,

H. Suzuki,

Introduction: Hierarchical Triple System (Newtonian and 1PN)

- Indirect Observation of GWs
- Direct Observation of GWs

- H. Suzuki, P. Gupta, H. Okawa, K. M.
  Mon.Not.Roy.Astron.Soc. 486 (2019) L52
- P. Gupta, H. Suzuki, H. Okawa, K.M.
  Phys.Rev.D 101 (2020) 10, 104053
- H. Suzuki, P. Gupta, H. Okawa, K.M.
  Mon.Not.Roy.Astron.Soc.500(2020)2,1645

Binary System near SMBH

- Local Inertial Frame and Binary Motion
- vZLK Oscillations near ISCO

- K.M., P. Gupta, H. Okawa
  Phys.Rev.D 107 (2023) 12, 124039
- K.M., P. Gupta, H. Okawa
  Phys.Rev.D 108 (2023) 12, 123041

Summary

Hierarchical Triplet System

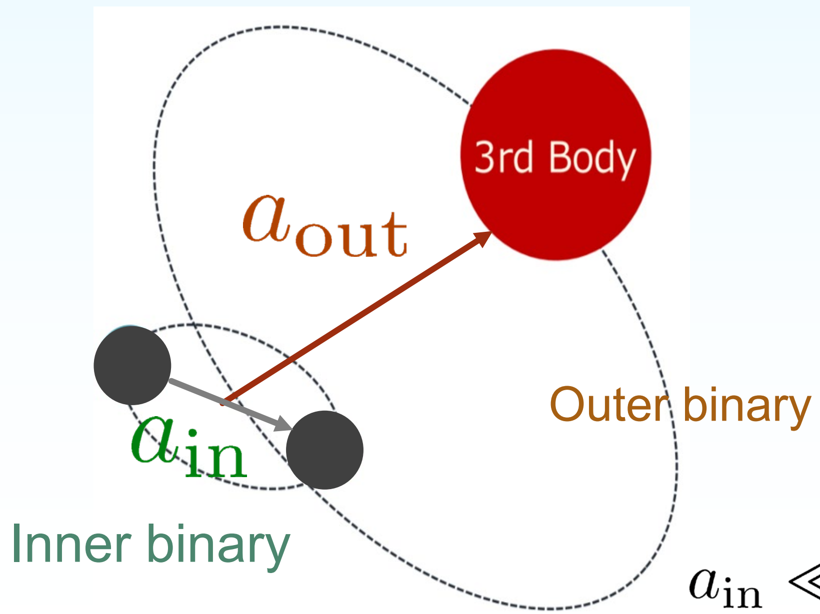
- **Three body system** nonintegrable

$|\mathbf{r}_1 - \mathbf{r}_2| \sim |\mathbf{r}_3 - \mathbf{r}_1| \sim |\mathbf{r}_3 - \mathbf{r}_2|$

$|\mathbf{r}_3 - \mathbf{r}_1|, |\mathbf{r}_3 - \mathbf{r}_2| \gg |\mathbf{r}_1 - \mathbf{r}_2|$

distance to the third body

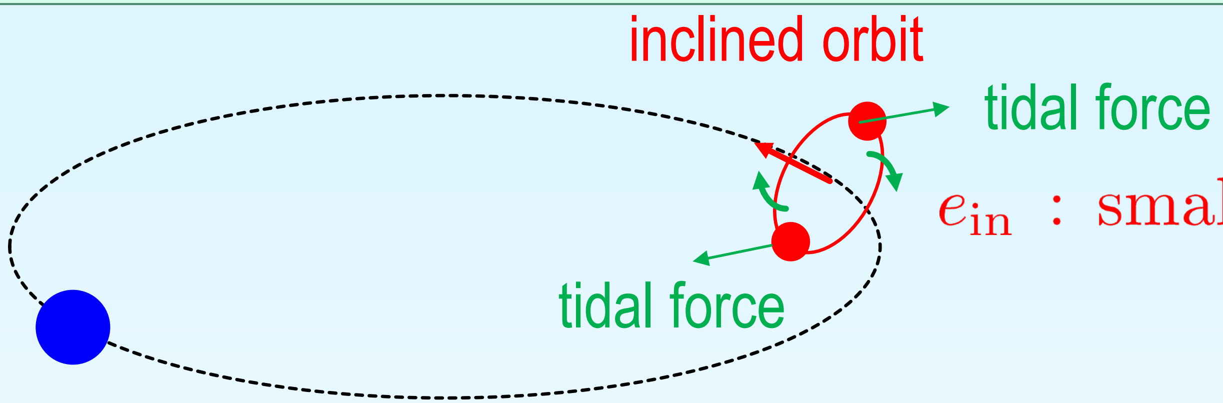
Chaotic system | **Hierarchical triple system** | **Unbounded**

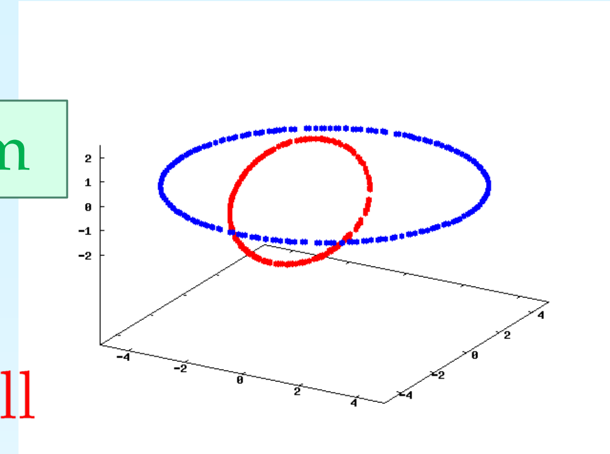


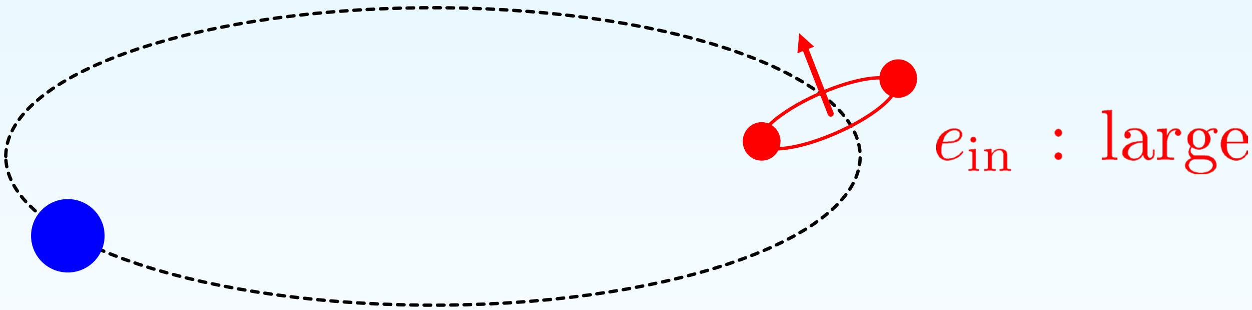
$a_{\rm in} \ll a_{\rm out}$

In a hierarchical triple system

von Zeipel-Lidov-Kozai (vZLK) mechanism







large emission of GWs

z-component of angular momentum : conserved
(Newtonian,quadrupole approx)

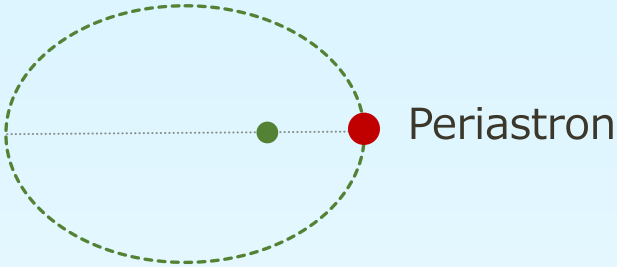
$$\sqrt{1-e_{\rm in}^2}\,\cos I = {\rm const.}$$

Secular exchange in orbital eccentricity $e_{\rm in}$ and inclination I

vZLK timescale $$T_{\rm vZLK} \sim \frac{P_{\rm out}^2}{P_{\rm in}} \gg P_{\rm out}\,,P_{\rm in}$$

Indirect Observation of Gravitational Waves from Hierarchical Triple System

In a binary pulsar   e.g. Hulse-Taylor binary



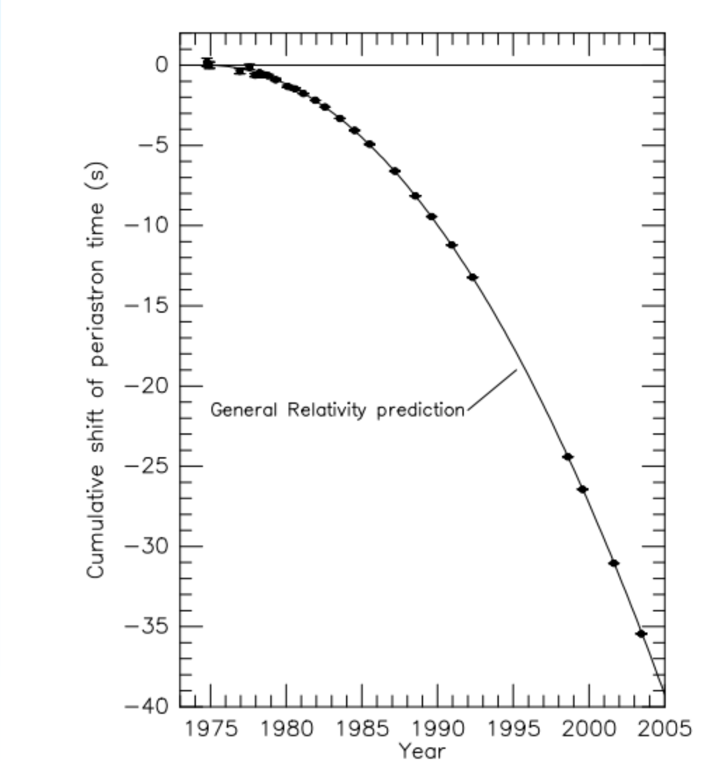
Cumulative shift of periatron time

periastron time shift: $\Delta_p \equiv T_N - P_0 N$

T_N : the *N*-th periastron time

P_0 : initial period of periastron passing

$$\Delta_p \approx \frac{\dot{P}}{2P} t^2 \quad \text{if } \dot{P} \text{ is constant}$$

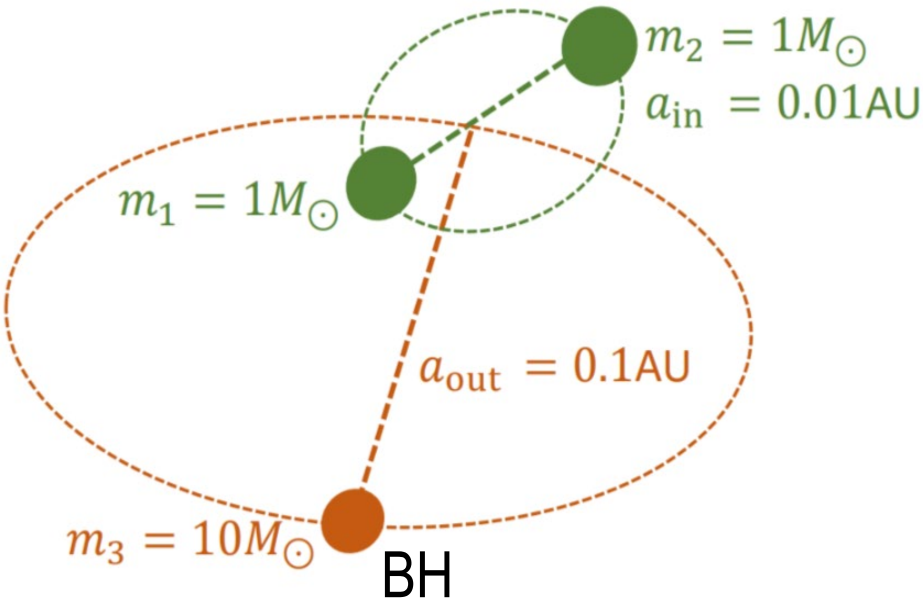


Evidence of GW emission

Indirect Observation of Gravitational Waves

H. Suzuki, P. Gupta, H. Okawa, KM,(2019)

a binary pulsar



$$e_{\rm in,0} = e_{\rm out,0} = 0$$

$$I_0 = 60^\circ$$

vZLKTimescale

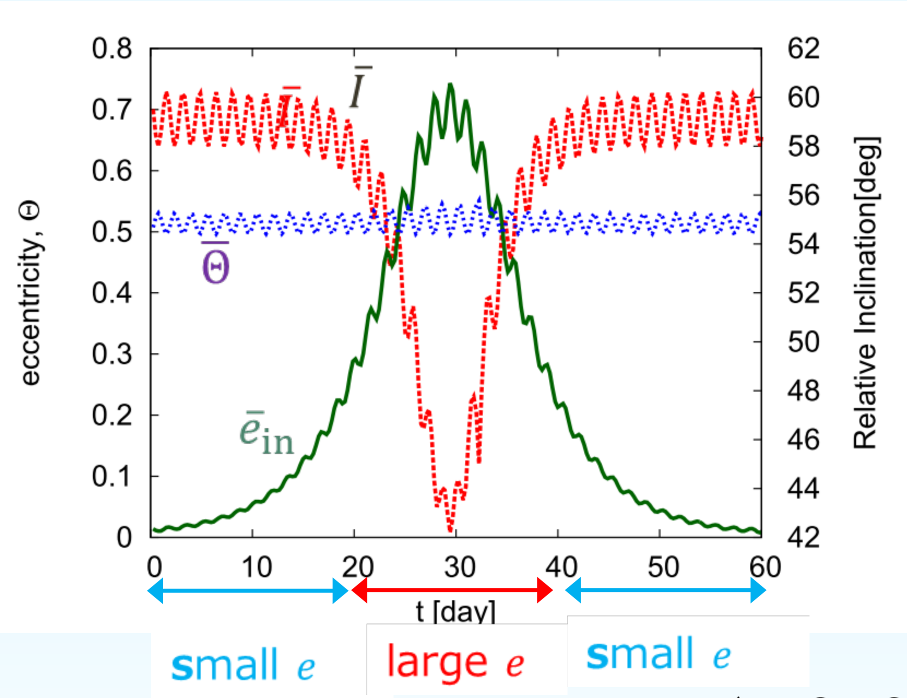
$$t_{KL} \simeq \frac{16}{15}\frac{a_{\rm out}^3}{a_{\rm in}^{3/2}}\sqrt{\frac{m_1}{Gm_3^2}}(1-e_{\rm out}^2)^{\frac{3}{2}}$$

Time Scales for our Model

$$P_{in} = 0.258\ days$$
$$P_{out} = 3.334\ days$$
$$\tau_{KL} \sim 66\ days$$
$$\tau_{merger} \sim 10^9 years$$

$$P_{in} \ll P_{out} \ll \tau_{KL} \ll \tau_{merger}$$

the eccentricity will oscillate for long time period



Oscillation between $\bar{e}_{\rm in}$ **and** $\bar{I}$

$\bar{\Theta}$ **is conserved**

← vZLK oscillations

Peter-Mathews (1963)

Power of GWs

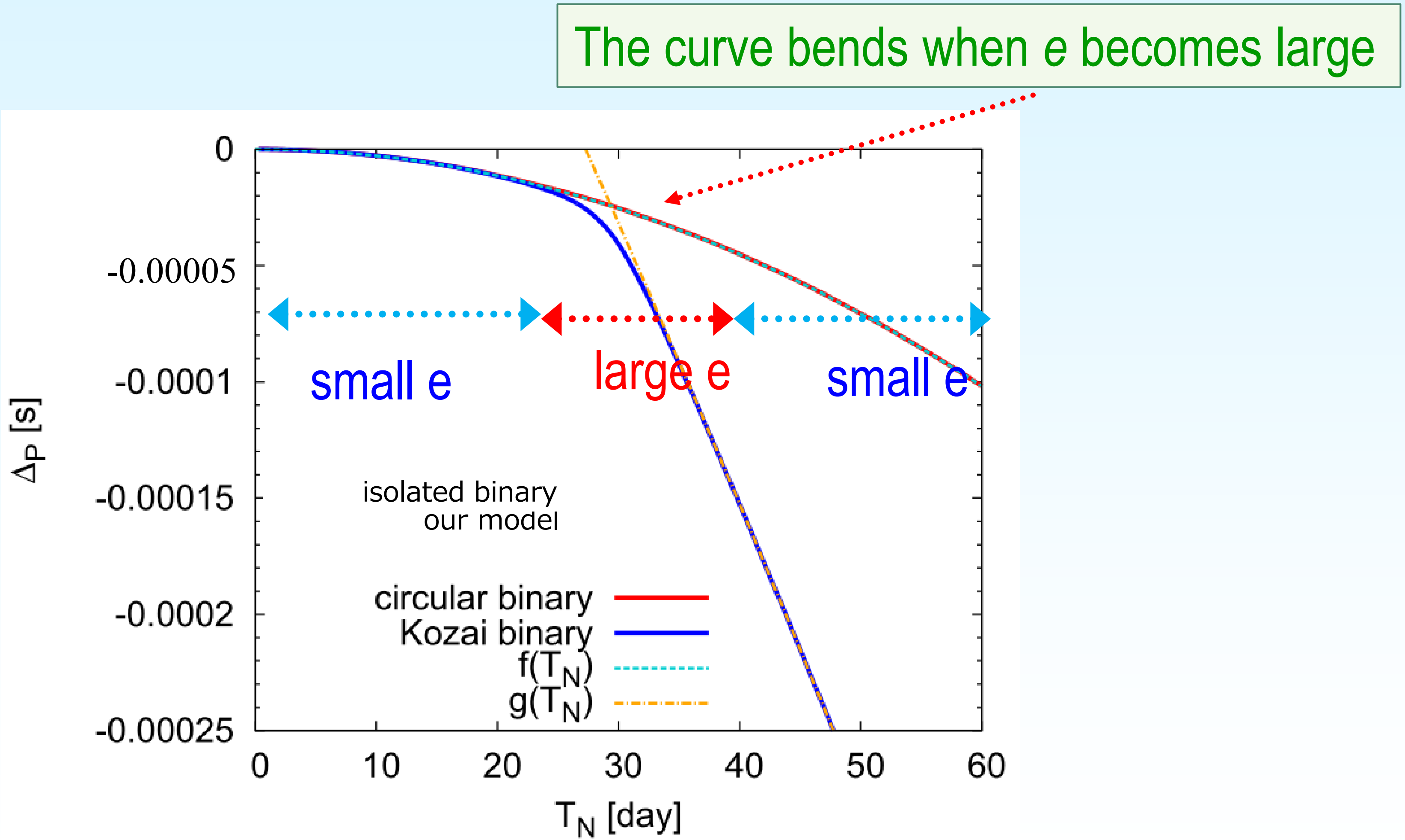
$$\left\langle \frac{dE}{dt} \right\rangle = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 (m_1+m_2)}{a^5(1-\bar{e}^2)^{7/2}}\left(1+\frac{73}{24}\bar{e}^2+\frac{37}{96}\bar{e}^4\right)$$

$\bar{e}(t)$ is the eccentricity averaged over inner binary period

Cumulative shift of periatron time

$$\Delta_p = -\frac{192\pi}{5P_0}\left(\frac{G}{c^3}\frac{2\pi}{P_0}\right)^{5/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}} \times \int_0^{T_N} dt \int_0^{t} dt' \frac{\left(1+\frac{73}{24}\bar{e}^2(t')+\frac{37}{96}\bar{e}^4(t')\right)}{(1-\bar{e}^2(t'))^{7/2}}$$

Evolution of Δ_p



Direct Observation of Gravitational Waves from Hierarchical Triple System

➤ inspiral phase

frequency

circular orbit $$f_0 = \frac{(Gm)^{1/2}}{\pi a^{3/2}}$$

For observable band $f > 10^{-4}\mathrm{Hz}$ $$\frac{(m/M_\odot)}{(a/R_\odot)^3} > 0.25$$

Unless very close binary, difficult to be observed

However, for highly eccentric orbit,
we may observe higher harmonics

peak frequency of emitted GW power

Peter-Mathews (1963)
Wen (2003)

$$f_p = n_m f_0 \qquad n_m = \frac{(1+e)^{1.1954}}{(1-e^2)^{3/2}} \quad (10^{-6} < 1 - e^2 < 1)$$

If the eccentricity is close to 1,
the peak frequency may fit to the observable band ⇐ vZLK mechanism

in hierarchical triple system

We analyze the following models by solving 1PN equations of motion

P. Gupta, H. Suzuki, H. Okawa, KM (2020)

TABLE III. The parameters of our models. We use the formula (2.6) and (2.7) for $P_{\rm in}$ and $P_{\rm out}$, while $t_{\rm KL}$ and $e_{\rm max}$ are evaluated by numerical calculation except for Models IV6, VA6, VIA6, for which we use the formula (2.8). The number N defined by (5.3) denotes how many cycles the inner orbit evolves during highly eccentric stage.

Model	m_1 $[M_\odot]$	m_2 $[M_\odot]$	m_3 $[M_\odot]$	$a_{\rm in}$ [AU]	$a_{\rm out}$ [AU]	$P_{\rm in}$ [days]	$P_{\rm out}$ [days]	$t_{\rm KL}$ [days]	$e_{\rm in,max}$	N
IA1	10	10	10	0.01	0.1	0.082	2.10	~176	~0.98	20
IA1$_3$	30	30	30	0.01	0.1	0.047	1.22	~123	~0.96	20
IB3	10	10	10^3	0.01	0.5	0.082	4.04	~255	~0.99	25
IB6	10	10	10^6	0.01	5	0.082	4.08	~96	~0.99	25
IIA3	10	10^3	10^3	0.12	1	0.478	8.15	~158	~0.99	5.8
IIB6	10	10^3	10^6	0.12	10	0.478	11.5	~185	~0.99	5.8
IIIA3	10^3	10^3	10^3	0.15	1	0.474	6.67	~226	~0.97	5.9
IIIB6	10^3	10^3	10^6	0.15	10	0.474	11.5	~142	~0.99	5.9
IVA6	10	10^6	10^6	15	100	21.2	258	17 [yrs]	-	-
VA6	10^3	10^6	10^6	15	100	21.2	258	17 [yrs]	-	-
VIA6	10^6	10^6	10^6	10^2	10^3	258	6669	1400 [yrs]	-	-

- stellar mass black hole $10 M_\odot$ or $30 M_\odot$
- intermediate-mass black hole $10^3 M_\odot$
- supermassive black hole $10^6 M_\odot$

Typical example

IA1

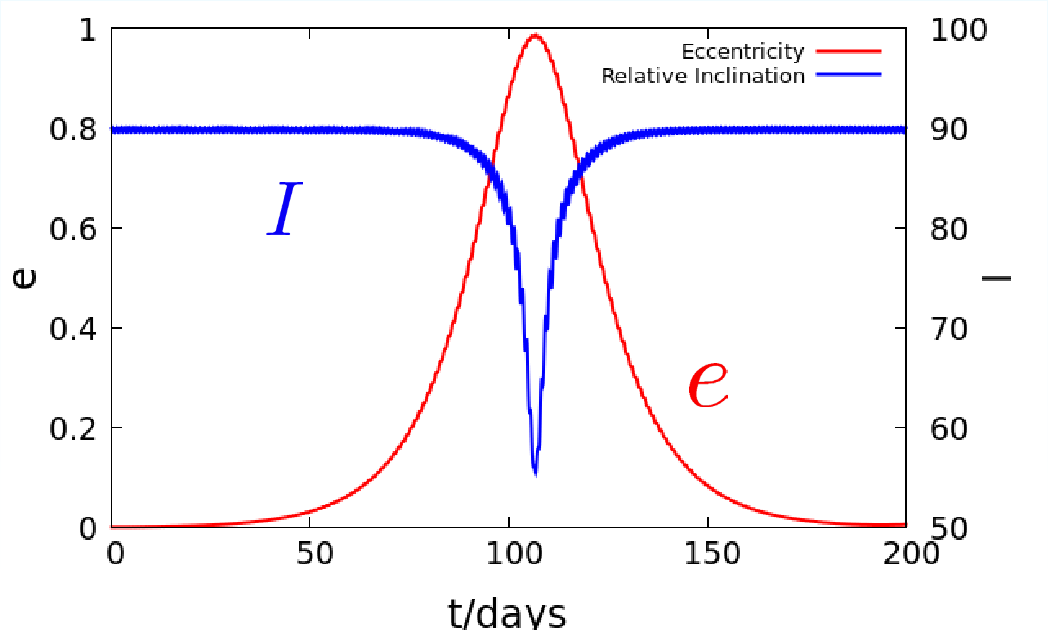
$$m_1 = m_2 = m_3 = 10M_\odot$$

$$a_{\rm in} = 0.01\text{AU} \qquad a_{\rm out} = 0.1\ \text{AU}$$

$$P_{\rm in} = 0.8\text{days} \qquad P_{\rm out} = 2.1\text{days} \qquad t_{\rm vZLK} = 163.2\text{days}$$

$$e_{\rm in,0} = 0 \qquad e_{\rm out,0} = 0 \qquad I_0 = 90^\circ$$

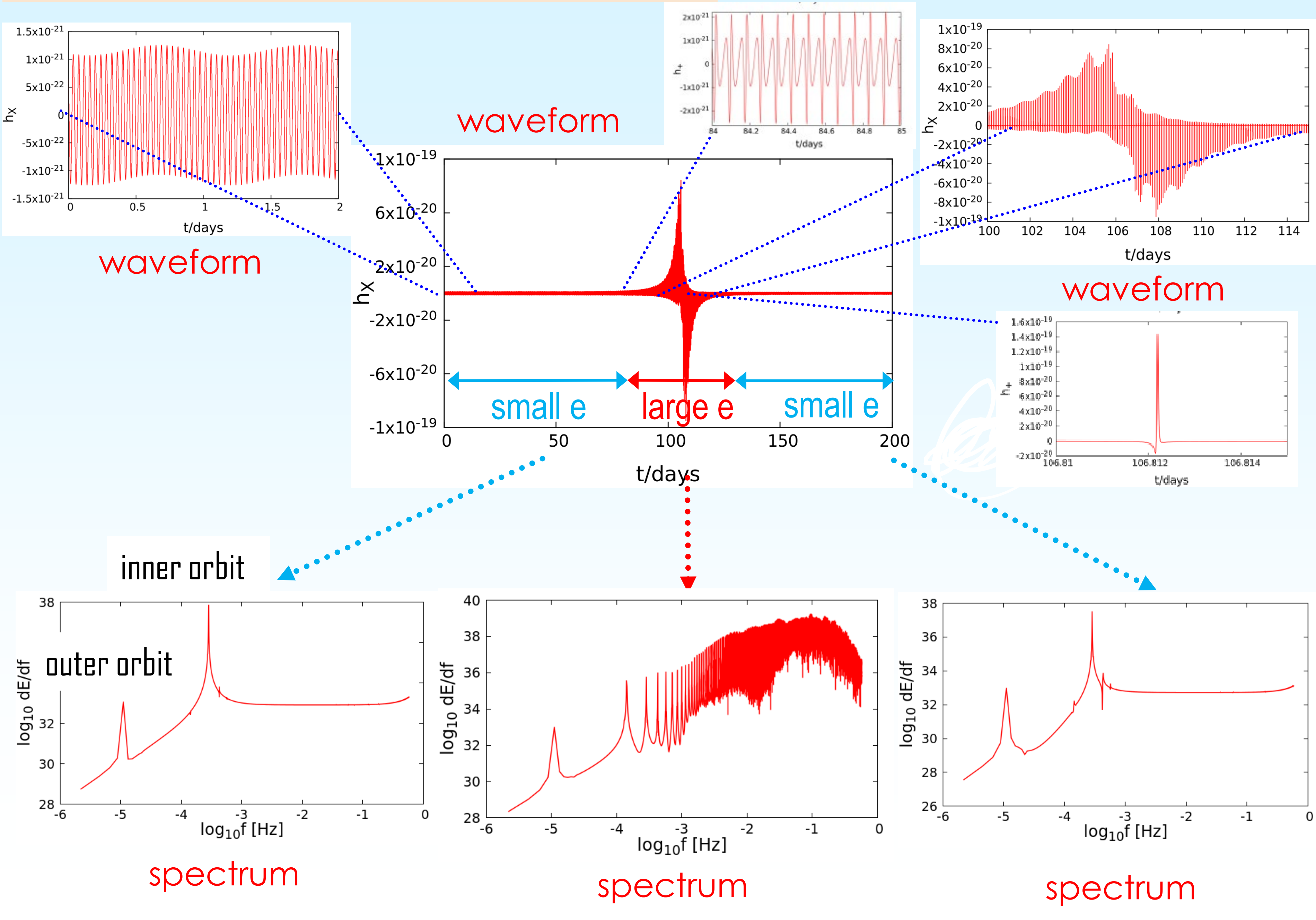
eccentricity and inclination



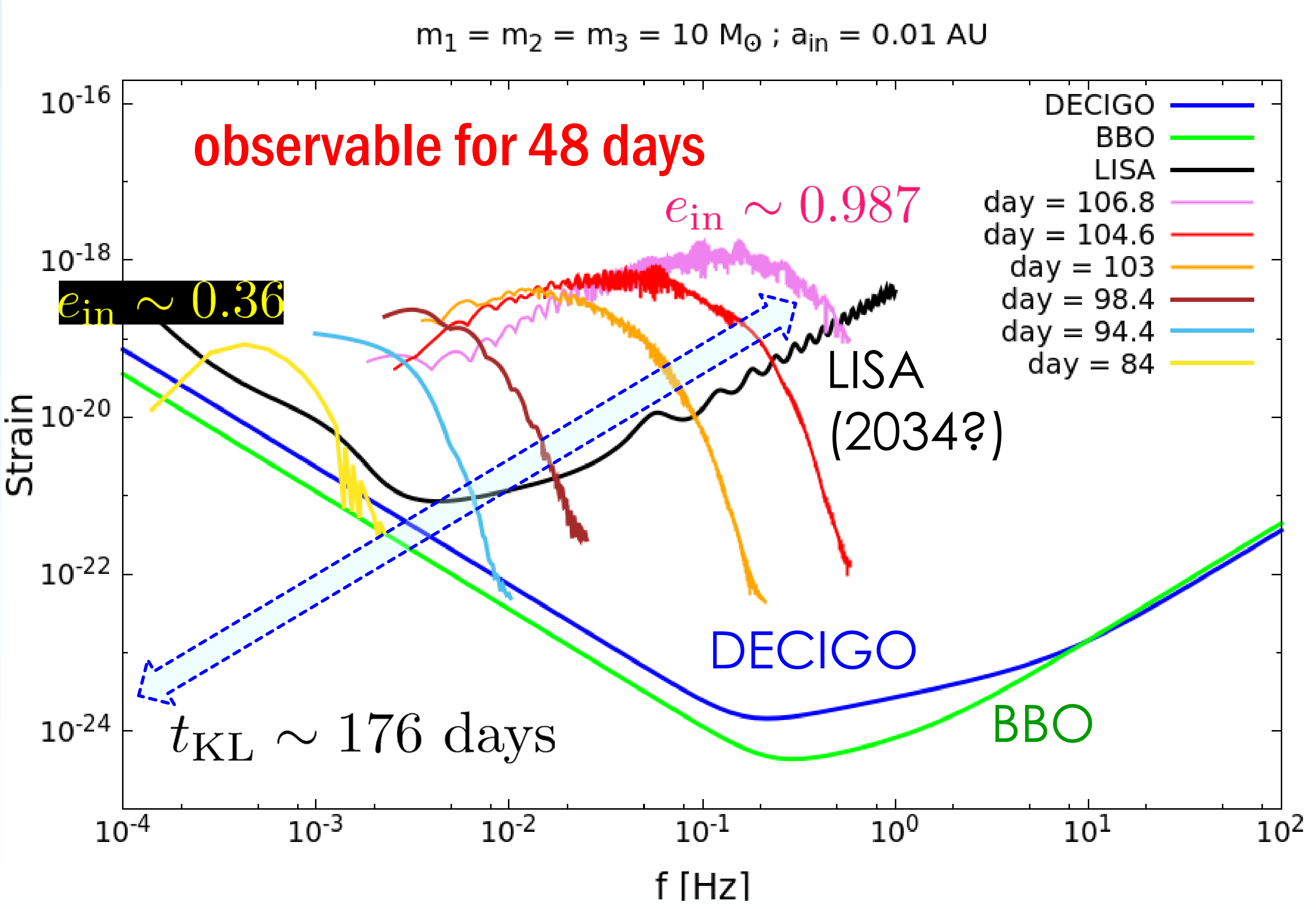
$e_{\rm max} \sim 0.98$

large h_c

Wave Form and Energy Spectra



Evolution curve of vZLK binary and sensitivity of space GW observatories



IA1

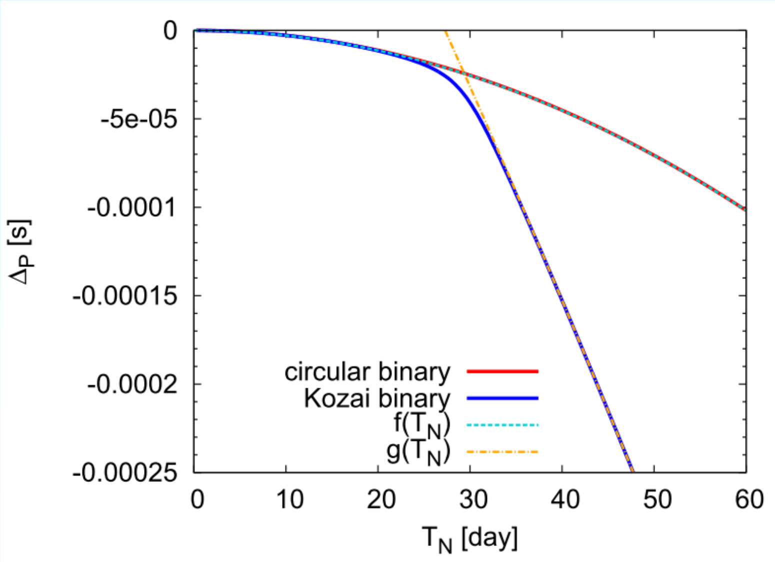
$$m_1 = m_2 = m_3 = 10M_\odot \qquad d = 10 \text{ kpc}$$
$$a_{\rm in} = 0.01\text{AU} \qquad a_{\rm out} = 0.1 \text{ AU}$$

Short Summary (Hierachical Triple System)

- A binary system with tertiary companion is interesting.

➢ Indirect observation

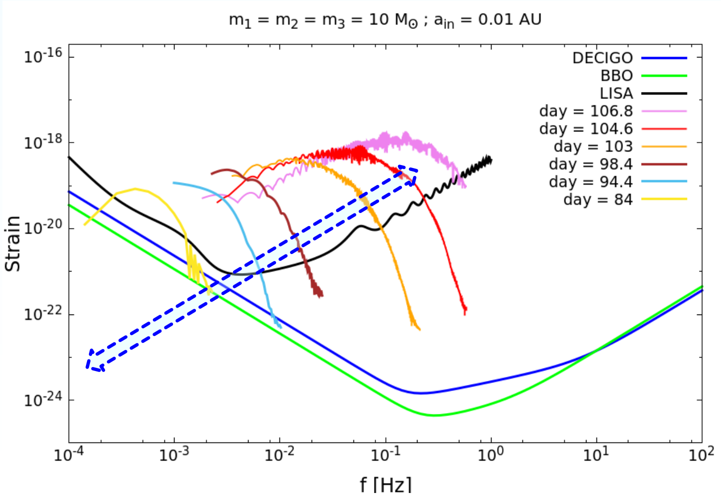
Bend of cumulative curve of periastron shift



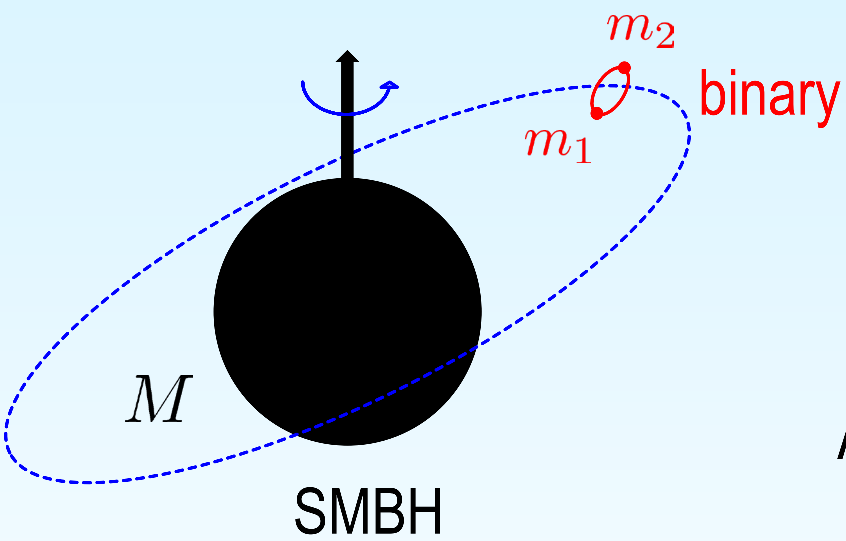
➢ Direct observation

Inspiral phase of vZLK binary

Observable period is periodic (vZLK period)



Binary System near SMBH：



a binary orbiting around SMBH

$$m_1, m_2 \ll M$$

A binary can be treated as perturbation ?

Black hole perturbation cannot be applied because of self-gravity of a binary

Instead, we consider a local inertial frame and set a "Newtonian" binary there

Local Inertial Frame and Binary Motion：

Background spacetime (SMBH) $d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu$

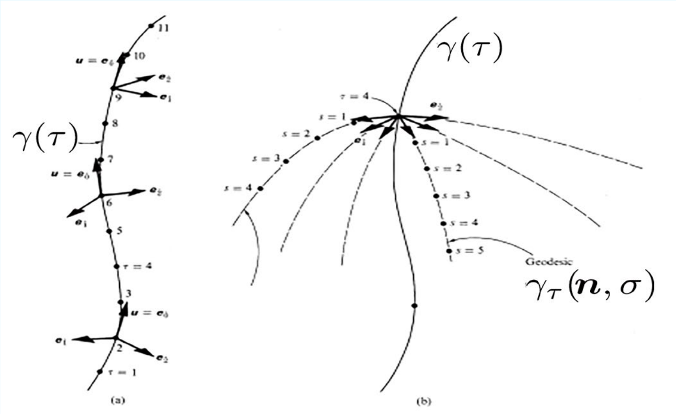
Observer's world line (γ) $z^\mu(\tau)$ $u^\mu(\tau)$ 4 velocity, a^μ acceleration, ω_μ rotation

Construct a local coordinate system

$$(c\tau, x^{\hat{a}})$$

$x^{\hat{a}}$ is measured from γ along $\Sigma(\tau)$

$\Sigma(\tau)$ is perpendicular to γ



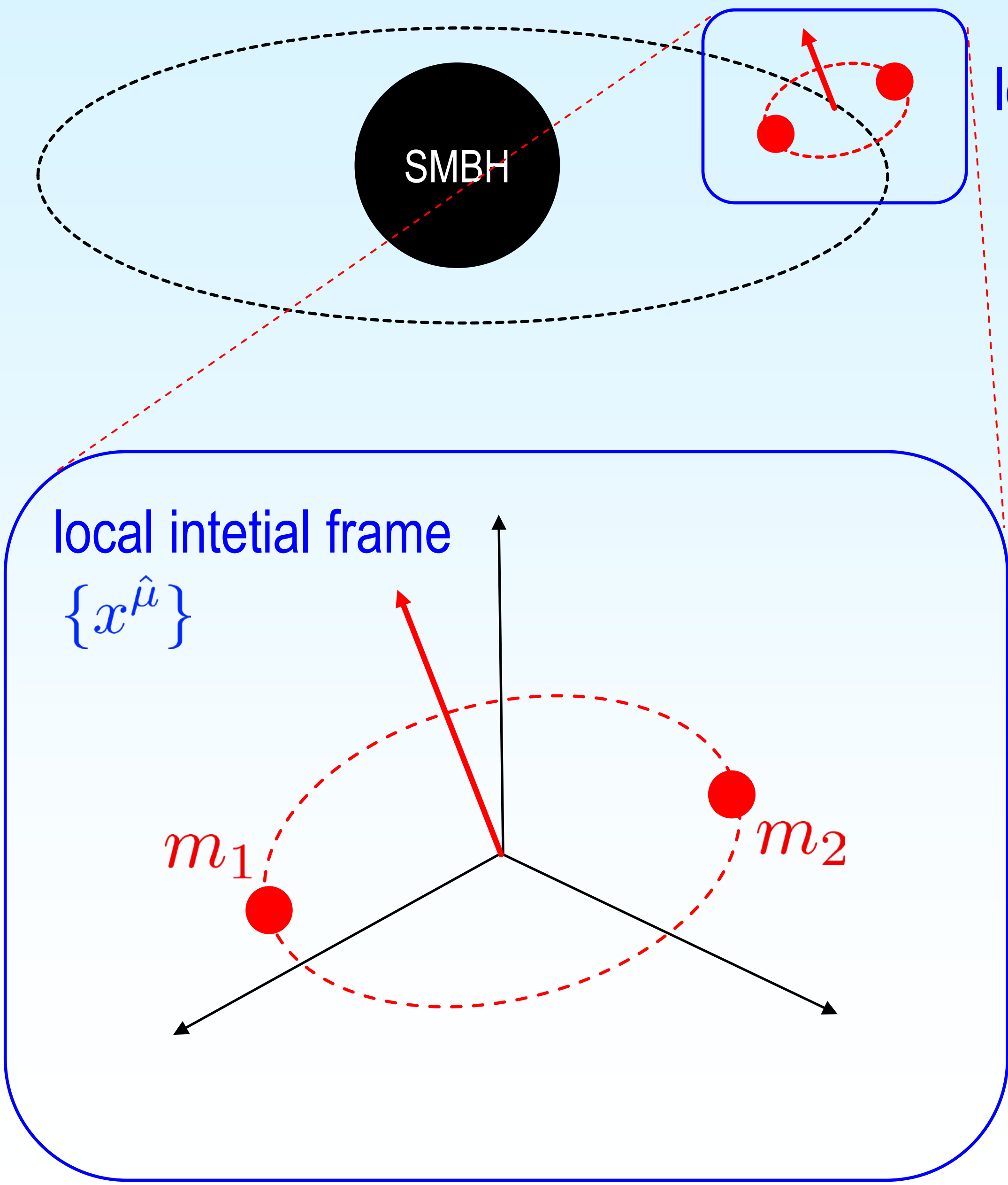
metric form of this reference frame up to the second order of $x^{\hat{a}}$

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} + \varepsilon_{\hat{\mu}\hat{\nu}} + O(|x^{\hat{k}}|^3),$$

F.K. Manasse, C.W. Misner ('63), MTW('73)
A. Gorbatsievich, A. Bobrik (2010)

$$\varepsilon_{\hat{0}\hat{0}} = -\frac{1}{c^2}\left[2a_{\hat{k}}x^{\hat{k}} + \left(c^2\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{0}\hat{\ell}} - \omega_{\hat{j}\hat{k}}\omega^{\hat{j}}{}_{\hat{\ell}}\right)x^{\hat{k}}x^{\hat{\ell}} + \frac{\left(a_{\hat{k}}x^{\hat{k}}\right)^2}{c^2}\right],$$

$$\varepsilon_{\hat{0}\hat{j}} = -\frac{1}{c^2}\left[c\,\omega_{\hat{j}\hat{k}}x^{\hat{k}} + \frac{2}{3}c^2\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}}x^{\hat{k}}x^{\hat{\ell}}\right], \qquad \varepsilon_{\hat{i}\hat{j}} = -\frac{1}{c^2}\left[\frac{1}{3}c^2\bar{\mathcal{R}}_{\hat{i}\hat{k}\hat{j}\hat{\ell}}x^{\hat{k}}x^{\hat{\ell}}\right]$$



local intetial frame

$$g_{\hat{\mu}\hat{\nu}} \approx \eta_{\hat{\mu}\hat{\nu}} + \varepsilon_{\hat{\mu}\hat{\nu}}$$

$$\varepsilon_{\hat{\mu}\hat{\nu}} = \varepsilon_{\hat{\mu}\hat{\nu}}(\bar{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}, x^{\hat{a}})$$

Set a self-gravitating binary

⇨ EOM of a binary

Self-gravitating Newtonian binary with a scale of $\ell_{\rm binary}$

$$\ell_{\rm binary} \ll \min\left[\frac{1}{|a^{\hat{j}}|}, \frac{1}{|\omega^{\hat{j}}|}, \ell_{\bar{\mathcal{R}}}\right],$$

$$\ell_{\bar{\mathcal{R}}} \equiv \min\left[|\bar{\mathcal{R}}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}|^{-\frac{1}{2}}, |\bar{\mathcal{R}}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma};\hat{\alpha}}|^{-\frac{1}{3}}, |\bar{\mathcal{R}}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma};\hat{\alpha};\hat{\beta}}|^{-\frac{1}{4}}\right]$$

minimum curvature radius

Lagrangian up to 0.5 PN

$$\mathcal{L}_{\rm binary} = \mathcal{L}_{\rm N} + \mathcal{L}_{1/2}$$

$$\mathcal{L}_{\rm N} \equiv \frac{1}{2}\sum_{I=1}^{2} m_I \dot{\boldsymbol{x}}_I^2 + \frac{G m_1 m_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} + \mathcal{L}_a + \mathcal{L}_\omega + \mathcal{L}_{\bar{\mathcal{R}}}$$

$$\mathcal{L}_a = -\sum_{I=1}^{2} m_I a_{\hat{k}} x_I^{\hat{k}},$$

$$\mathcal{L}_\omega = -\sum_{I=1}^{2} m_I \left[\epsilon_{\hat{j}\hat{k}\hat{\ell}}\omega^{\hat{\ell}} x_I^{\hat{k}} \dot{x}_I^{\hat{j}} - \frac{1}{2}\left(\boldsymbol{\omega}^2 \boldsymbol{x}_I^2 - (\boldsymbol{\omega}\cdot\boldsymbol{x}_I)^2\right)\right]$$

$$\mathcal{L}_{\bar{\mathcal{R}}} = -\frac{1}{2}\sum_{I=1}^{2} m_I \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{0}\hat{\ell}} x_I^{\hat{k}} x_I^{\hat{\ell}}$$

$$\mathcal{L}_{1/2} \equiv -\frac{2}{3}\sum_{I=1}^{2} m_I c^2 \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} x_I^{\hat{k}} x_I^{\hat{\ell}} \frac{\dot{x}_I^{\hat{j}}}{c}$$

A binary system ⇒ center of mass $\boldsymbol{R}$ and relative coordinates $\boldsymbol{r}$

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{x}_1 + m_2 \boldsymbol{x}_2}{m_1 + m_2} \qquad \boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$$

"Newtonian" Lagrangian

$$\mathcal{L}_{\mathrm{N}} = \mathcal{L}_{\mathrm{CM}}(\boldsymbol{R}, \dot{\boldsymbol{R}}) + \mathcal{L}_{\mathrm{rel}}(\boldsymbol{r}, \dot{\boldsymbol{r}})$$

$$\mathcal{L}_{\mathrm{CM}}(\boldsymbol{R}, \dot{\boldsymbol{R}}) = \frac{1}{2}(m_1 + m_2)\dot{\boldsymbol{R}}^2 + \mathcal{L}_{\mathrm{CM}\text{-}a} + \mathcal{L}_{\mathrm{CM}\text{-}\omega} + \mathcal{L}_{\mathrm{CM}\text{-}\bar{\mathcal{R}}}$$

$$\mathcal{L}_{\mathrm{CM}\text{-}a} = -(m_1 + m_2)\boldsymbol{a} \cdot \boldsymbol{R}$$

$$\mathcal{L}_{\mathrm{CM}\text{-}\omega} = -(m_1 + m_2)\left[\epsilon_{\hat{j}\hat{k}\hat{\ell}}\,\omega^{\hat{\ell}} R^{\hat{k}} \dot{R}^{\hat{j}} - \frac{1}{2}\left(\boldsymbol{\omega}^2 \boldsymbol{R}^2 - (\boldsymbol{\omega} \cdot \boldsymbol{R})^2\right)\right]$$

$$\mathcal{L}_{\mathrm{CM}\text{-}\bar{\mathcal{R}}} = -\frac{1}{2}(m_1 + m_2)\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{0}\hat{\ell}} R^{\hat{k}} R^{\hat{\ell}}$$

$$\mathcal{L}_{\mathrm{rel}}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \frac{1}{2}\mu \dot{\boldsymbol{r}}^2 + \frac{G m_1 m_2}{r} + \mathcal{L}_{\mathrm{rel}\text{-}\omega} + \mathcal{L}_{\mathrm{rel}\text{-}\bar{\mathcal{R}}}$$

$$\mathcal{L}_{\mathrm{rel}\text{-}\omega} = -\mu\left[\epsilon_{\hat{j}\hat{k}\hat{\ell}}\,\omega^{\hat{\ell}} r^{\hat{k}} \dot{r}^{\hat{j}} - \frac{1}{2}\left(\boldsymbol{\omega}^2 \boldsymbol{r}^2 - (\boldsymbol{\omega} \cdot \boldsymbol{r})^2\right)\right],$$

$$\mathcal{L}_{\mathrm{rel}\text{-}\bar{\mathcal{R}}} = -\frac{1}{2}\mu \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{0}\hat{\ell}} r^{\hat{k}} r^{\hat{\ell}}$$

decoupled equations for $\boldsymbol{R}$ and $\boldsymbol{r}$

$\boldsymbol{R} = 0$ is a solution if $\boldsymbol{a} = 0$

"0.5 PN" Lagrangian

Choice of the origin

$$\mathcal{L}_{1/2} = \mathcal{L}_{1/2\text{-CM}}(\boldsymbol{R}, \dot{\boldsymbol{R}}) + \mathcal{L}_{1/2\text{-rel}}(\boldsymbol{r}, \dot{\boldsymbol{r}}) + \mathcal{L}_{1/2\text{-int}}(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{r}, \dot{\boldsymbol{r}}),$$

$$\mathcal{L}_{1/2\text{-CM}}(\boldsymbol{R}, \dot{\boldsymbol{R}}) = -\frac{2}{3}(m_1 + m_2)\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} R^{\hat{k}} R^{\hat{\ell}} \dot{R}^{\hat{j}}$$

$$\mathcal{L}_{1/2\text{-rel}}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = -\frac{2}{3}\mu \frac{(m_1 - m_2)}{(m_1 + m_2)} \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} r^{\hat{k}} r^{\hat{\ell}} \dot{r}^{\hat{j}}$$

$$\mathcal{L}_{1/2\text{-int}}(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{r}, \dot{\boldsymbol{r}}) = -\frac{2}{3}\mu \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} \left[r^{\hat{k}} r^{\hat{\ell}} \dot{R}^{\hat{j}} + \left(R^{\hat{k}} r^{\hat{\ell}} + r^{\hat{k}} R^{\hat{\ell}} \right) \dot{r}^{\hat{j}} \right]$$

Coupling between $\boldsymbol{R}$ and $\boldsymbol{r}$

integration by part

$$\mathcal{L}_{1/2\text{-int}}(\boldsymbol{R}, \dot{\boldsymbol{R}}, \boldsymbol{r}, \dot{\boldsymbol{r}}) = 2\mu \left[\frac{1}{3} \frac{d\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}}}{d\tau} r^{\hat{k}} r^{\hat{\ell}} + \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} r^{\hat{k}} \dot{r}^{\hat{\ell}} \right] R^{\hat{j}}$$

$$\mathcal{L}_{\text{CM-}a} = -(m_1 + m_2)\boldsymbol{a} \cdot \boldsymbol{R}$$

Interaction terms disappear if $a_{\hat{j}} = \dfrac{2\mu}{m_1 + m_2} \left[\dfrac{1}{3} \dfrac{d\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}}}{d\tau} r^{\hat{k}} r^{\hat{\ell}} + \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} r^{\hat{k}} \dot{r}^{\hat{\ell}} \right]$

$\boldsymbol{R} = 0$ is a solution

The CM follows the observer's orbit

The observer's orbit is no longer a geodesic motion

$$\frac{Du^{\mu}_{\rm CM}}{d\tau} = a^{\mu} = \frac{2\mu}{m_1+m_2} e^{\mu\hat{j}} \left[\frac{1}{3} \frac{d\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}}}{d\tau} r^{\hat{k}} r^{\hat{\ell}} + \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} r^{\hat{k}} \dot{r}^{\hat{\ell}} \right]$$

$$\frac{Dp^{\mu}_{\rm CM}}{d\tau} = e^{\mu\hat{j}} \left[\frac{1}{2} \bar{\mathcal{R}}_{\hat{0}\hat{j}\hat{k}\hat{\ell}} L^{\hat{k}\hat{\ell}} + \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}} \frac{dQ^{\hat{k}\hat{\ell}}}{d\tau} + \frac{2}{3} \frac{d\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}}}{d\tau} Q^{\hat{k}\hat{\ell}} \right]$$

$$L^{\hat{k}\hat{\ell}} \equiv r^{\hat{k}} p^{\hat{\ell}} - r^{\hat{\ell}} p^{\hat{k}}$$ angular momentum of a binary

$$Q^{\hat{k}\hat{\ell}} \equiv r^{\hat{k}} r^{\hat{\ell}} - \frac{1}{3} r^2 \delta^{\hat{k}\hat{\ell}}$$ mass quadrupole moment

The first term in r.h.s.:

Mathisson-Papapetrou-Dixon equation for a spinning particle $\left(L^{\hat{k}\hat{\ell}} \to S^{\hat{k}\hat{\ell}} \right)$

① Solve the EOM for the relative coordinate $\boldsymbol{r}$

$$\mathcal{L}_{\rm rel}(\boldsymbol{r},\dot{\boldsymbol{r}}) = \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 + \frac{Gm_1m_2}{r} + \mathcal{L}_{\text{rel-}\omega} + \mathcal{L}_{\text{rel-}\bar{\mathcal{R}}}$$

$$\mathcal{L}_{\text{rel-}\omega} = -\mu\left[\epsilon_{\hat{j}\hat{k}\hat{\ell}}\omega^{\hat{\ell}}r^{\hat{k}}\dot{r}^{\hat{j}} - \frac{1}{2}\left(\boldsymbol{\omega}^2\boldsymbol{r}^2 - (\boldsymbol{\omega}\cdot\boldsymbol{r})^2\right)\right],$$

$$\mathcal{L}_{\text{rel-}\bar{\mathcal{R}}} = -\frac{1}{2}\mu\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{0}\hat{\ell}}r^{\hat{k}}r^{\hat{\ell}}$$

⇨ $\boldsymbol{r} = \boldsymbol{r}(\tau)$

② Solve the EOM for the CM

$$\frac{Du^\mu_{\rm CM}}{d\tau} \quad \frac{2\mu}{m_1+m_2}e^{\mu\hat{j}}\left[\frac{1}{3}\frac{d\bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}}}{d\tau}r^{\hat{k}}r^{\hat{\ell}} + \bar{\mathcal{R}}_{\hat{0}\hat{k}\hat{j}\hat{\ell}}r^{\hat{k}}\dot{r}^{\hat{\ell}}\right]$$

⇨ $x^\mu_{\rm CM} = x^\mu_{\rm CM}(\tau)$

⇨ $x^\mu_1 = x^\mu_1(\tau)\,,\ x^\mu_2 = x^\mu_2(\tau)$ which are measured from γ

a binary motion in a background SMBH spacetime

Binary Motion in Kerr Spacetime:

Kerr metric in Boyer-Lindquist coordinates

$$d\bar{s}^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2\sin^2\theta\,, \;\; \Delta = r^2 - 2Mr + a^2$$

a circular geodesic with the radius r_0 on the equatorial plane of a test particle (observer) with a unit mass

Energy $$E = \frac{r_0^2 - 2Mr_0 + a\sigma\sqrt{Mr_0}}{r_0 F_0}$$

Angular momentum $$L_z = \frac{\sigma\sqrt{Mr_0}\left(r_0^2 + a^2 - 2a\sigma\sqrt{Mr_0}\right)}{r_0 F_0}$$

$$F_0 \equiv \left(r_0^2 - 3Mr_0 + 2a\sigma\sqrt{Mr_0}\right)^{1/2}$$

$\sigma = \pm 1$ prograde orbit / retrograde orbit non-rotating inertial frame $(\tau, \mathsf{x}, \mathsf{y}, \mathsf{z})$ $\omega^{\mu} = 0$

$$\mathcal{L}_{\rm rel} = \frac{1}{2}\mu \left(\frac{d\mathbf{r}}{d\tau}\right)^2 + \frac{Gm_1 m_2}{\mathsf{r}} + \mathcal{L}_{\text{rel-}\bar{\mathcal{R}}}(\mathbf{r}, \tau)$$

$$\mathcal{L}_{\text{rel-}\bar{\mathcal{R}}}(\mathbf{r}, \tau) = -\frac{\mu M}{2r_0^3}\left[\mathsf{r}^2 + \frac{3}{F_0^2}\left(-\Delta(r_0)\left(\mathsf{x}\cos\omega_{\rm R}\tau + \mathsf{y}\sin\omega_{\rm R}\tau\right)^2 + (\sigma\sqrt{Mr_0} - a)^2\mathsf{z}^2\right)\right]$$

: tidal force by SMBH

time dependent

$$\omega_{\rm R} = \frac{M^{1/2}}{r_0^{3/2}}$$ angular frequency

The binary motion may be close to elliptic orbit in hierarchical triplet

⇨ orbital parameters a , e , I , ω , Ω , f

a: semi-major axis

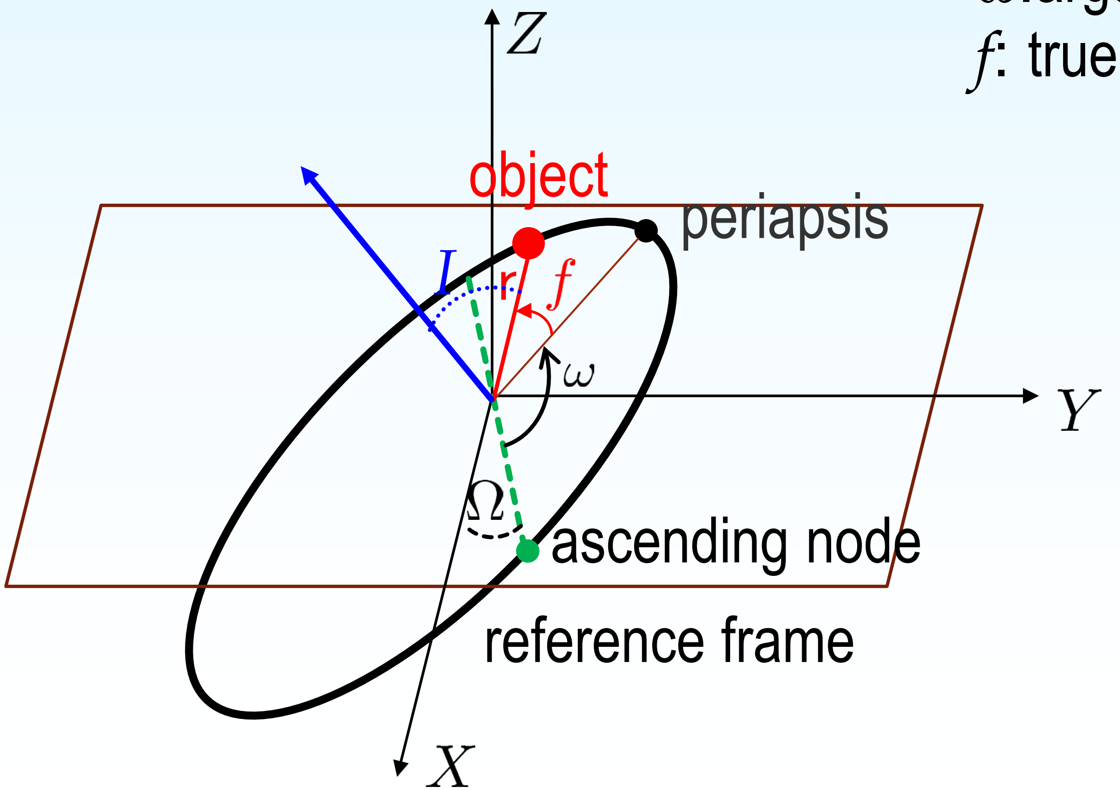
e: eccentricity

I: inclination

Ω:longitude of the ascending node

ω:argument of periapsis

f: true anomaly



numerical results

Model	a/M	a_0/M	r_0/M	e_0	I_0	ω_0	Ω_0
I	0.9	0.005	10	0.01	$85°$	$60°$	$30°$
II	0.9	0.005	2.9	0.01	$60°$	$60°$	$30°$
III	0.9	0.005	3.2	0.01	$85°$	$60°$	$30°$
IV	0.9	0.015	10	0.01	$85°$	$60°$	$30°$

Model I $r_0 \gg r_{0(\mathrm{cr})}$ $I_0 = 85°$ regular vZLK oscillations

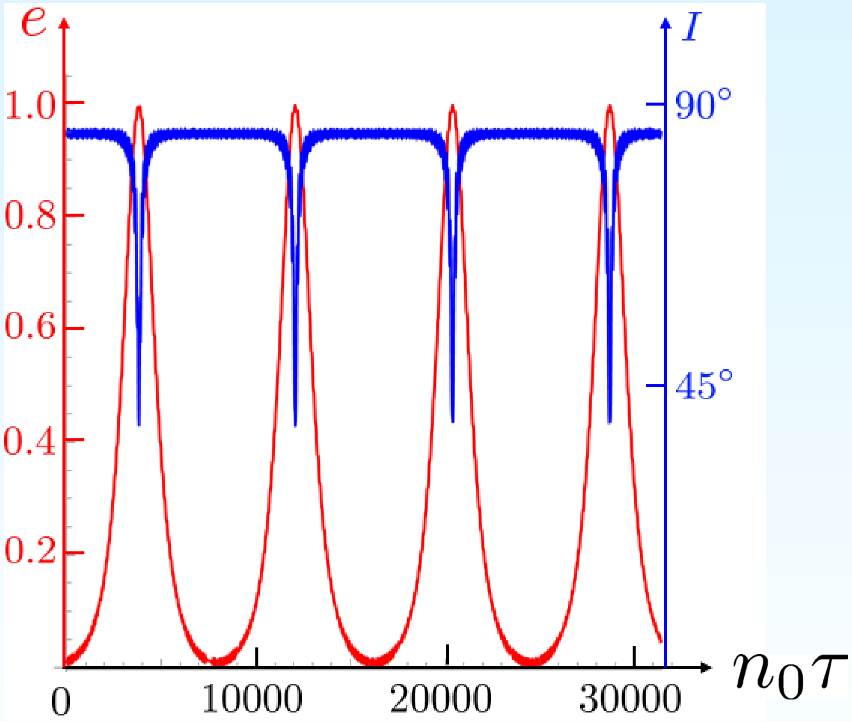
Model II $r_0 = r_{0(\mathrm{cr})}$ $I_0 = 60°$

Model III $r_0 = r_{0(\mathrm{cr})}$ $I_0 = 85°$

Model IV $r_0 \sim r_{0(\mathrm{cr})}$ $I_0 = 85°$

(Models II–IV): chaotic vZLK oscillations

Model I



$$a = 0.9M$$

$$r_0 = 10M$$

Initial orbital parameters

$$a_0 = 0.005M \qquad \omega_0 = 60^\circ$$

$$e_0 = 0.01 \qquad \Omega_0 = 30^\circ$$

$$I_0 = 85^\circ$$

$$n_0 = \sqrt{\frac{m_1 + m_2}{a_0^3}}$$ mean motion

$$P_0 = \frac{2\pi}{n_0}$$ Initial period of a binary

regular oscillations between the eccentricity and inclination

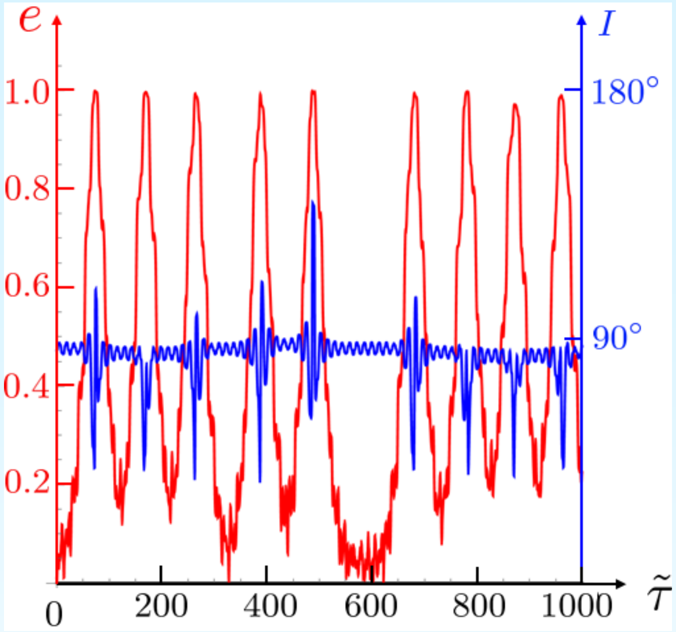
$a = 0.9M$

$a_0 = 0.005M$

$e_0 = 0.01$

$\omega_0 = 60^\circ$

$\Omega_0 = 30^\circ$



Model II

$r_0 = 2.9M$

$(= r_{0(\mathrm{cr})})$

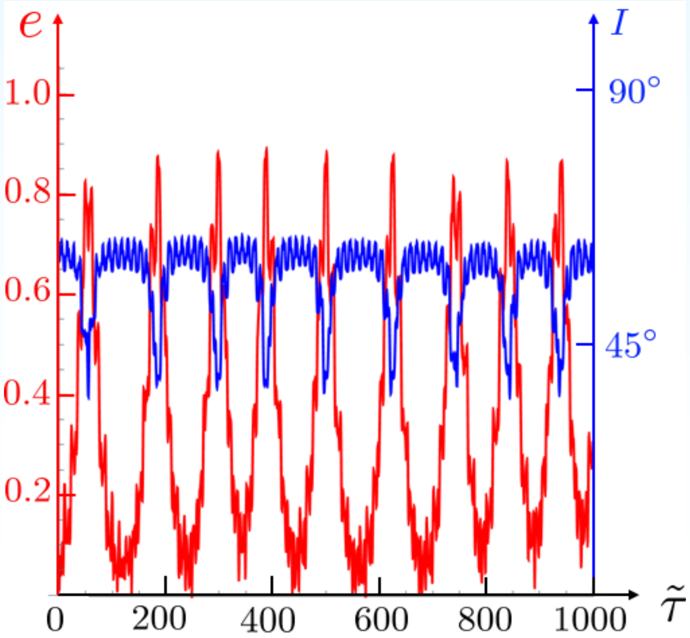
$I_0 = 85^\circ$

Choatic vZLK oscillations

Irregular oscillation period

Irregular oscillation amplitude

Orbital flip



Model III

$r_0 = 3.2M$

$(= r_{0(\mathrm{cr})})$

$I_0 = 60^\circ$

Choatic vZLK oscillations

Irregular oscillation period

Irregular oscillation amplitude

Model IV

$a = 0.9M$

$a_0 = 0.015M$

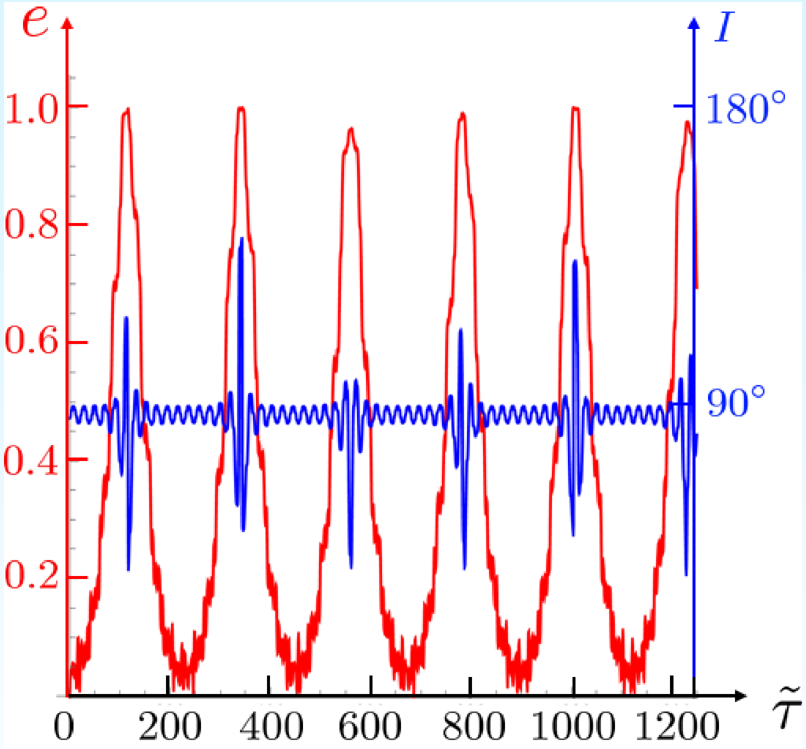
$r_0 = 10M$
$(\sim r_{0(\mathrm{cr})})$

$I_0 = 85^\circ$

$e_0 = 0.01$

$\omega_0 = 60^\circ$

$\Omega_0 = 30^\circ$



Choatic vZLK oscillations

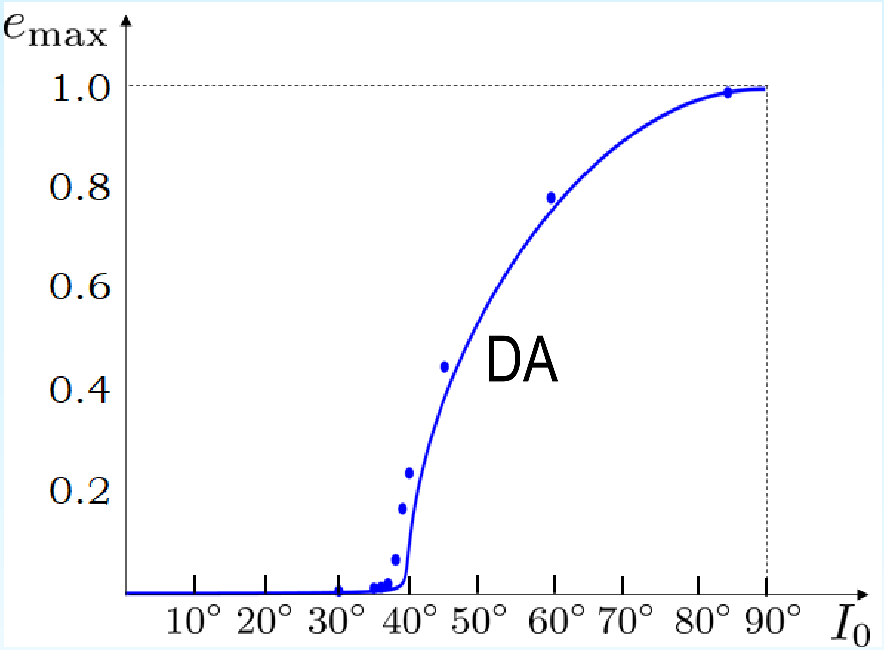
Irregular oscillation amplitude

Orbital flip

Critical inclination

$$I_{0(\mathrm{cr})} \sim 40^\circ$$

almost universal



$a = 0.9M$

$r_0 = 10M$

$a_0 = 0.005M$

$e_0 = 0.01$

$\omega_0 = 60^\circ$

$\Omega_0 = 30^\circ$

rotation dependence is small when we fix r_0 and a_0 (compactness)

The curvature components on the equatorial plane are the same as those in Schwarzschild case

The difference may be found in
how closely a binary system can approach a black hole

$$r_{\mathrm{ISCO}} : 6M(a=0) - M(a=M)$$

Motion of the CM of a binary and its stability

Observer (CM)

$$z^\mu = z^\mu_{(0)} + z^\mu_{(1)}$$
$$u^\mu = u^\mu_{(0)} + u^\mu_{(1)}$$

$$z^\mu_{(0)} = \left(\frac{r_0^2 + a\sigma\sqrt{M r_0}}{r_0 F_\sigma(r_0)}\tau \,, r_0 \,, \frac{\pi}{2} \,, \frac{\sigma\sqrt{M r_0}}{r_0 F_0}\tau \right)$$

: circular motion

$$z^\mu_{(1)} \equiv e^\mu{}_{\hat\ell} R^{\hat\ell} = (t_{(1)}, r_{(1)}, \theta_{(1)}, \varphi_{(1)})$$

EOM of CM

$$\frac{Du^\mu}{d\tau} = a^\mu$$

$z^\mu_{(1)} \,, u^\mu_{(1)}$ perturbations

$$\frac{du^\mu_{(1)}}{d\tau} + 2\Gamma^\mu{}_{\rho\sigma}(r_0) u^\rho_{(0)} u^\sigma_{(1)} + \frac{\partial \Gamma^\mu{}_{\rho\sigma}}{\partial x^\alpha}(r_0)\, z^\alpha_{(1)}\, u^\rho_{(0)} u^\sigma_{(0)} = a^\mu,$$

$$a^\mu = \frac{6\mu}{m_1+m_2} \frac{\sqrt{\Delta}(\sigma\sqrt{M r_0} - a)}{F_\sigma^2(r_0)} \frac{M}{r_0^3} \Big[\delta^\mu_1 \frac{\sqrt{\Delta}}{r_0} \dot{y} x + \delta^\mu_2 \frac{1}{r_0} \dot{y} z$$
$$+ \frac{1}{r_0 F_\sigma(r_0)\sqrt{\Delta}} \Big(\delta^\mu_0 \sigma\sqrt{M r_0}(r_0^2 + a^2 - 2a\sigma\sqrt{M r_0})$$
$$+ \delta^\mu_3 (r_0^2 - 2M r_0 + a\sigma\sqrt{M r_0}) \Big) (-\dot{x} x + \dot{z} z) \Big]$$

$$\frac{d^2 r_{(1)}}{d\tau^2} + k_r^2 r_{(1)} + A\left(x^2 - z^2\right) + B\,\dot{y}x = 0$$

$$k_r^2 \equiv \frac{M}{r_0^3 F_0^2}\left(r_0^2 - 6Mr_0 - 3a^2 + 8a\sigma\sqrt{Mr_0}\right)$$

$$\frac{d^2 \theta_{(1)}}{d\tau^2} + k_\theta^2\,\theta_{(1)} + B\,\dot{y}z = 0\,,$$

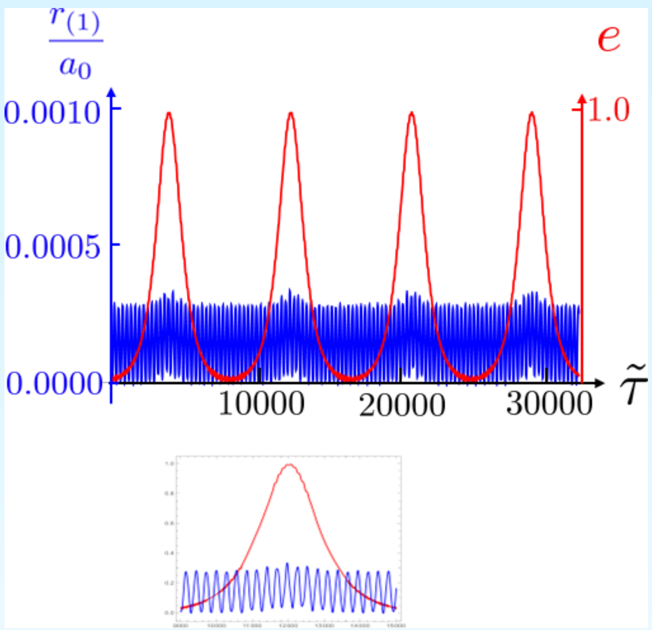
$$k_\theta^2 \equiv \frac{M}{r_0^3 F_0^2}\left(r_0^2 + 3a^2 - 4a\sigma\sqrt{Mr_0}\right)$$

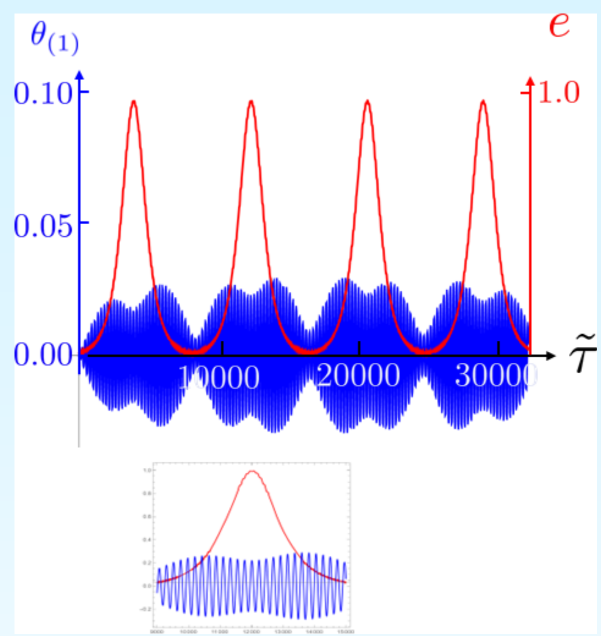
$$A \equiv \frac{6\mu M\Delta}{(m_1+m_2)r_0^5 F_0^4}\left(r_0^2 - 3Mr_0 - 2a^2\right)$$

$$B \equiv -\frac{6\mu M\Delta}{(m_1+m_2)r_0^4 F_0^2}\left(\sigma\sqrt{Mr_0} - a\right)\,.$$

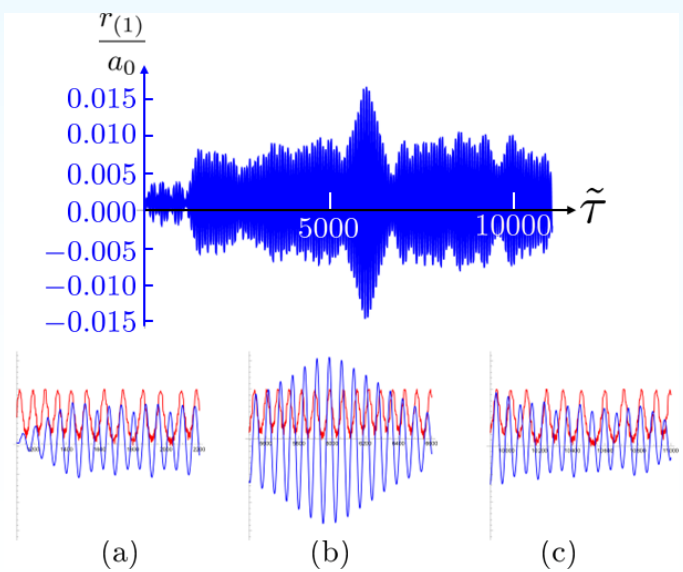
- homogeneous parts — harmonic oscillations if $k_r^2\,,\,k_\theta^2 > 0$ $\leftrightarrow r_0 > r_{\rm ISCO}$
- inhomogeneous parts ⇐ binary motion $(x(\tau), y(\tau), z(\tau))$

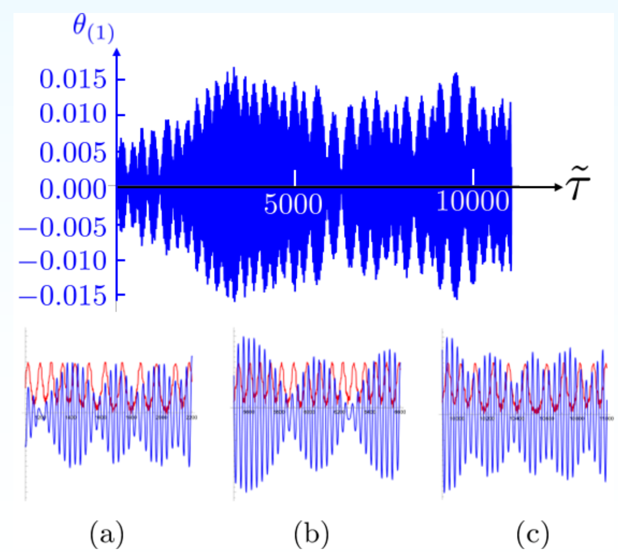
Model I





Model III

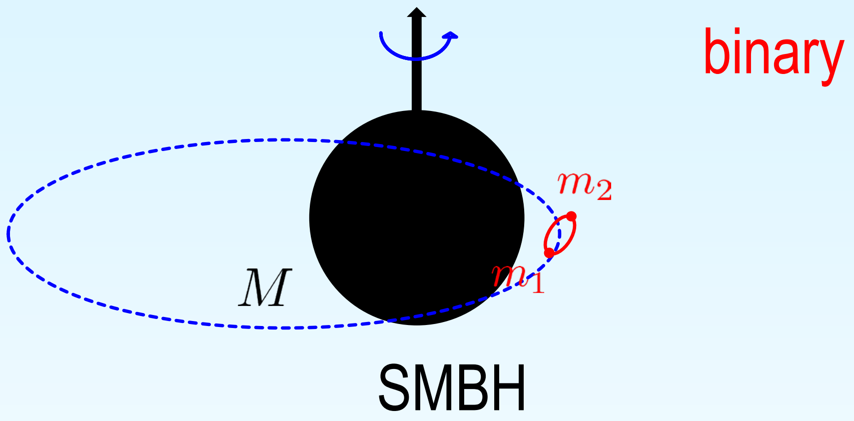




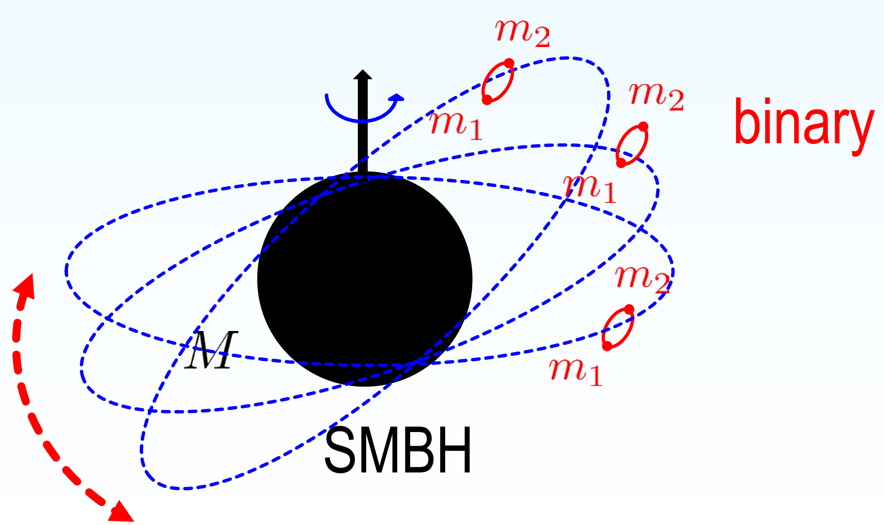
◆ generic orbits of CM

can be described by elliptic functions

➢ eccentric orbit



➢ spherical orbit

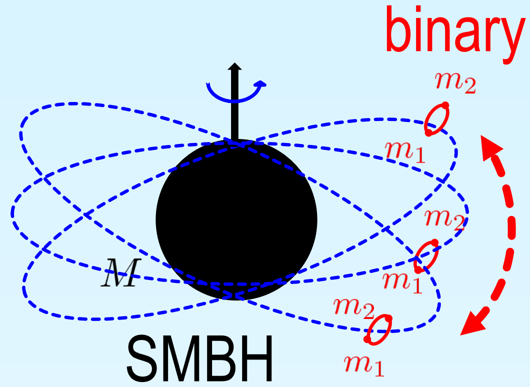


➢ spherical orbit

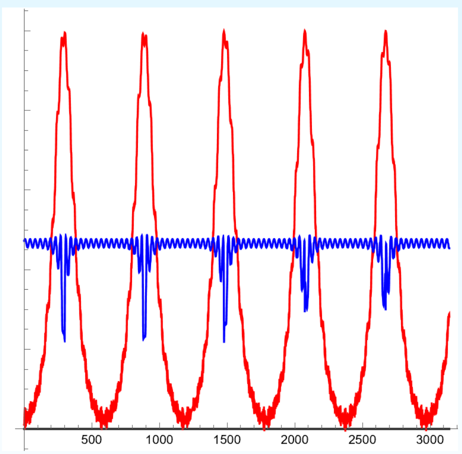
preliminary

$a = 0.9M$

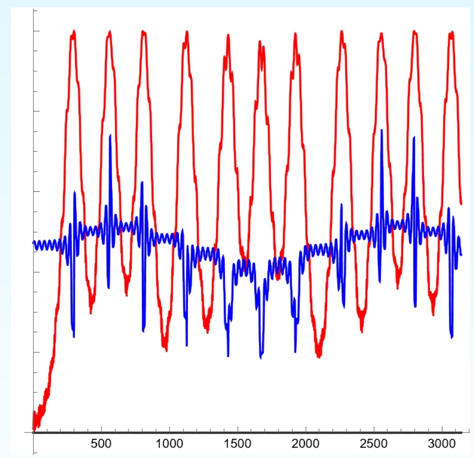
$r_0 = 9M$



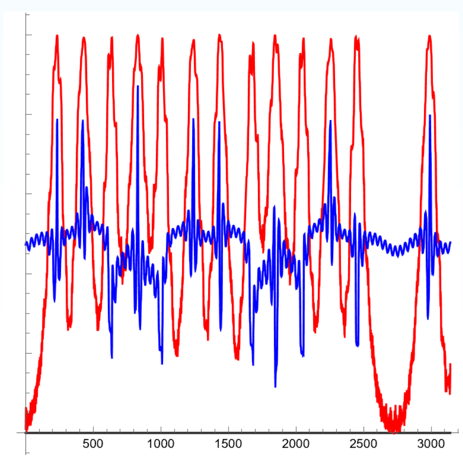
$\theta_{\rm osc}$



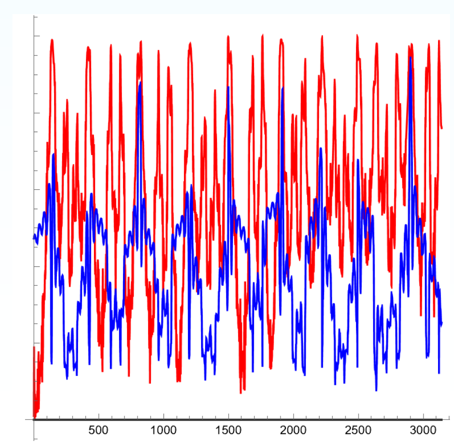
$\theta_{\rm osc} = 0^\circ$



$\theta_{\rm osc} = 11.5^\circ$



$\theta_{\rm osc} = 36.9^\circ$



$\theta_{\rm osc} = 53.1^\circ$

Future Issues :

◆ gravitational waves

- Quadrupole formula may not be applicable

- A binary motion in a background SMBH spacetime

$$x_1^\mu = x_1^\mu(\tau)\,,\;\; x_2^\mu = x_2^\mu(\tau)$$

⇨ BH perturbation method

Their motions are quite complicated

✓ outer binary + inner binary

✓ time-domain perturbation

Summary

- We discuss vZLK oscillations in hierarchical triple system
  - Newtoninan/1PN
    - Indirect observation of GWs
    - Direct observation of GWs
  - A binary system around SMBH
    - Local Inertial Frame and Binary Motion
    - Chaotic vZLK Oscillations near ISCO

Thank you for your attention