General Relativity from Scattering Amplitudes

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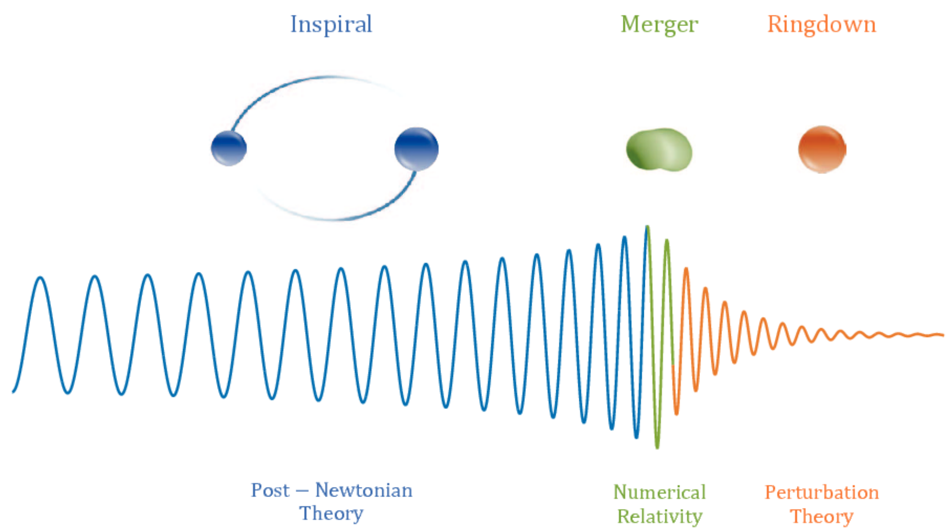
Motivation

- Gravitational waves carry fingerprints of a two-body dynamics

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu} \quad , \quad \ddot{x}_a^\mu = -\Gamma^\mu_{\alpha\beta} \dot{x}_a^\alpha \dot{x}_a^\beta$$

  ... however, no exact solution is known!

- The Effective One Body approach (EOB) provides an accurate solution combining results from different regimes of motion

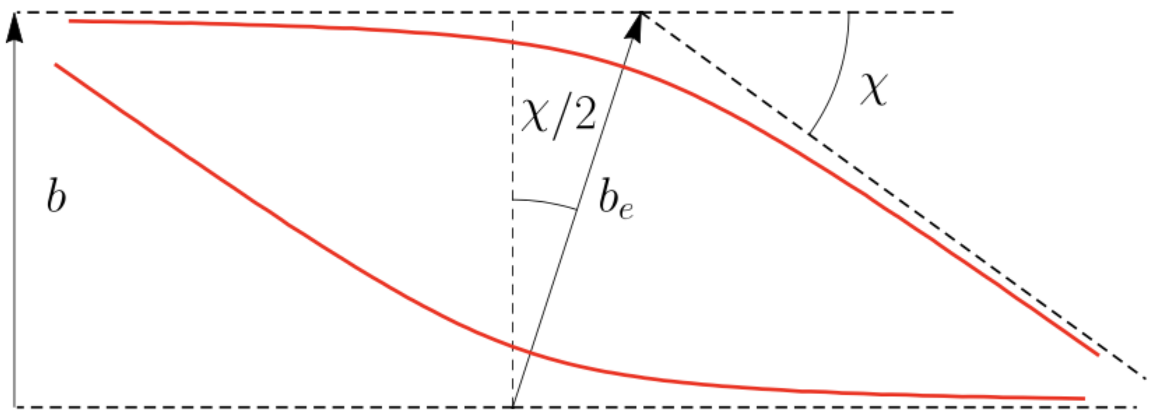


Credit: Antelis and Moreno, 1610.03567

Improving the EOB

Relativistic scattering observables can be used in the EOB to improve gravitational wave templates (Damour, 1609.00354)

- An example is the Post-Minkowskian (PM) scattering angle, computed by expanding in G_N while keeping all terms in v/c

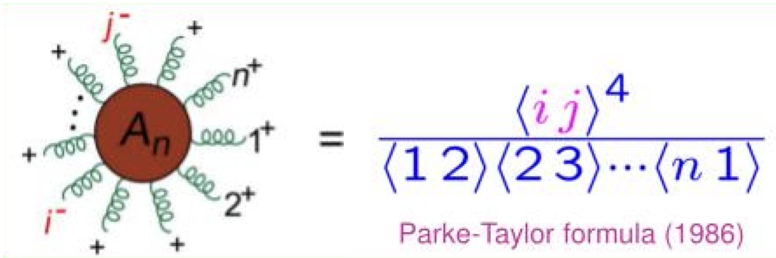


Credit: Bern et al. 2002.02459

State of the art till 2019 (Westpfhal, 1985)

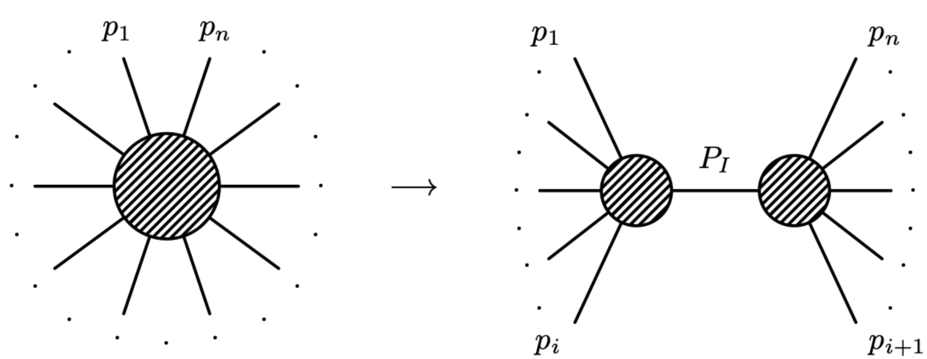
$$\chi_{2PM} = \frac{2G_N}{L}\frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2}{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} + \frac{G_N^2}{L^2}\frac{3\pi(m_1+m_2)(5(p_1\cdot p_2)^2 - m_1^2 m_2^2)}{4E}$$

- This simplicity is reminiscent of QFT calculations boiling down to simple answers. An example is the tree level MHV scattering of gluons proposed by Parke and Taylor in 1986



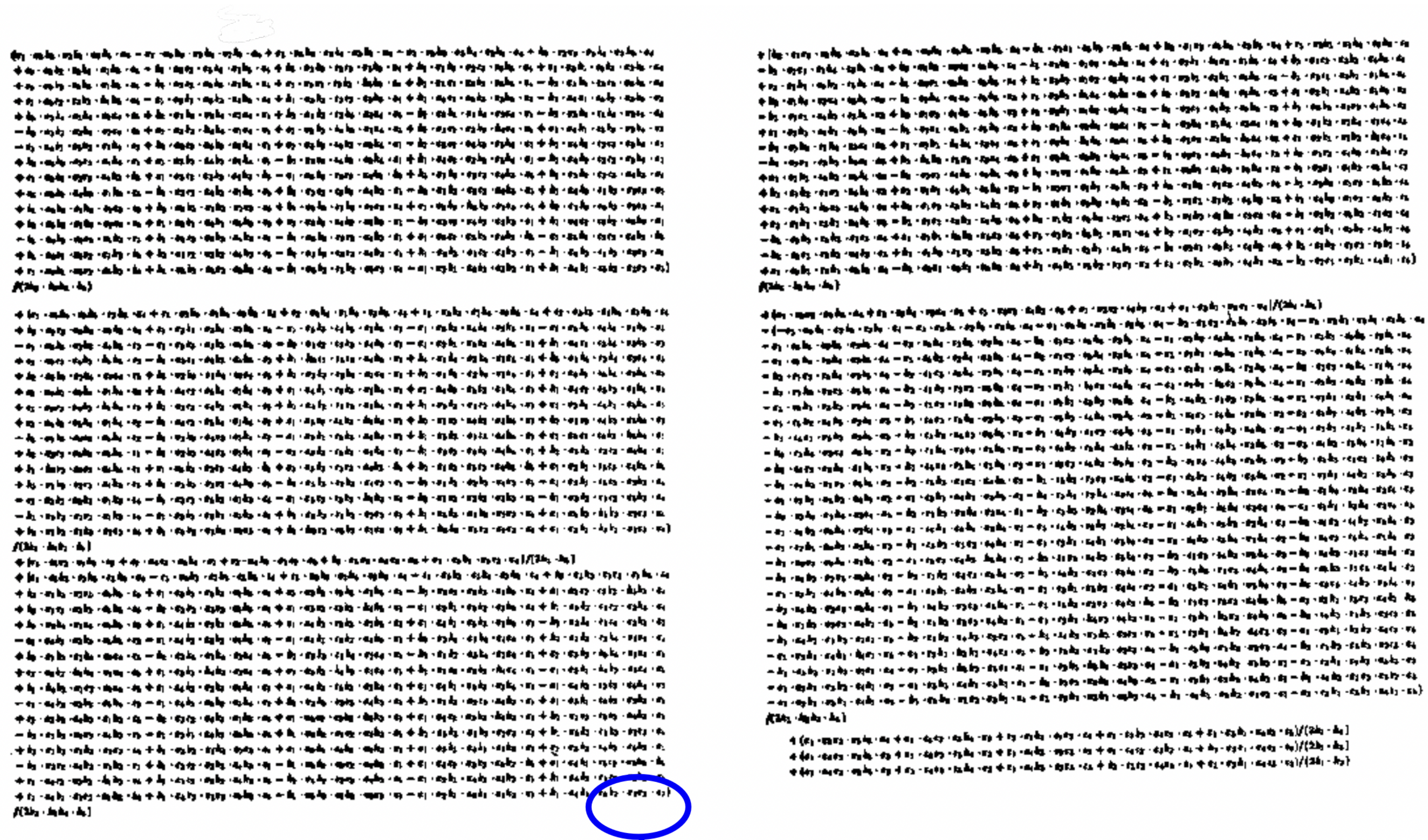
- This formula can be proved with spinor-helicity variables $p^\mu \sim \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ and on-shell recursion relations (BCFW, 2005).

$$\mathcal{M}_n(p_1, ..., p_n) \rightarrow \mathcal{M}^L_{i+1}(p_1, ..., p_i, P_I) \frac{1}{P_I^2} \mathcal{M}^R_{n-i+1}(-P_I, p_{i+1}, ..., p_n)\,,$$



Fundamental principles: unitarity, causality and locality.

Don't try this at home

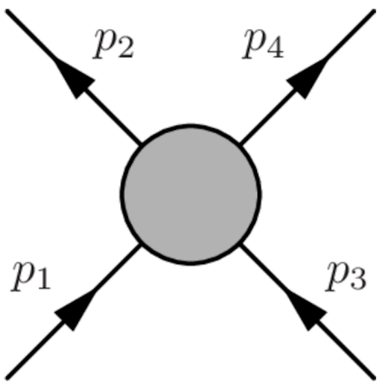


$$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$$

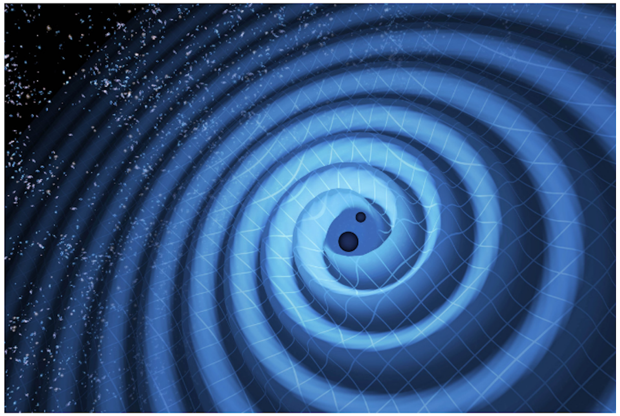
x 100 pages ~ 220 Feynman diagrams…

Paradigm shift

- Can we approach general relativity using the same fundamental principles we use in amplitude calculations?



$\Rightarrow$



Credit: Tim Pyle

State of the art results (Bern et al., 2019) + (Veneziano et al., 2021)

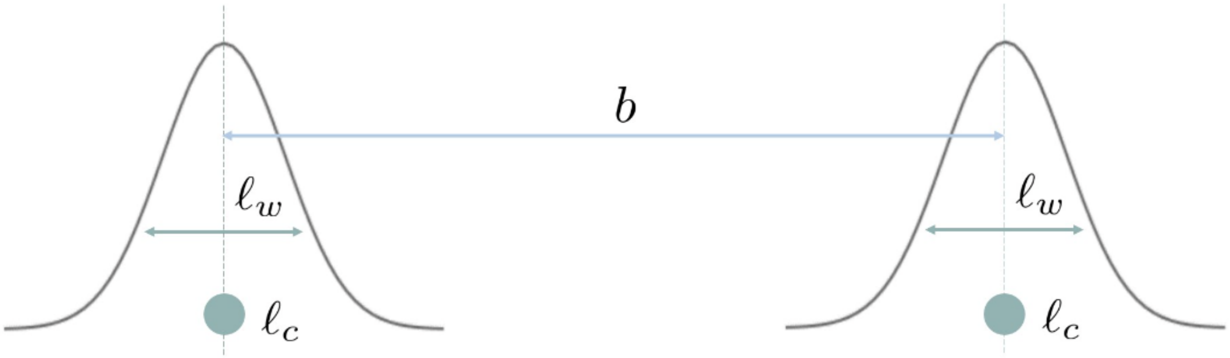
$$\chi_{3\mathrm{PM}} = -\frac{16 m_1^3 m_2^3 \sigma^6 G^3}{3L^3\left(\sigma^2-1\right)^{3/2}} + \frac{32 m_1^4 m_2^4 \sigma^6 G^3}{L^3\left(\sigma^2-1\right)\left(m_1^2+m_2^2+2m_1 m_2\sigma\right)}$$
$$-\frac{32 m_1^4 m_2^4 \sigma^4 G^3}{L^3\left(m_1^2+m_2^2+2m_1 m_2\sigma\right)}\left[1-\frac{\sigma\left(\sigma^2-2\right)}{\left(\sigma^2-1\right)^{\frac{3}{2}}}\right]\cosh^{-1}(\sigma)$$

The KMOC formalism

- Binary system as superposition of single particle states

$$|\psi\rangle = \int \underbrace{d\Phi(p_1)\, d\Phi(p_2)}_{\frac{d^4p}{(2\pi)^4}\theta(p_0)2\pi\delta(p^2-m^2)} \phi_1(p_1)\,\phi_2(p_2)\, e^{\frac{ib\cdot p_1}{\hbar}} |p_1 p_2\rangle$$

- Classical limit $\leftrightarrow$ Goldilocks relations $\ell_c \ll \ell_w \ll b$



Credit: Ben Maybee, 2105.10268

- Classical observables from the S-matrix (1811.10950)

$$O = \lim_{\hbar \to 0} \langle\psi| S^\dagger \hat{O} S |\psi\rangle$$

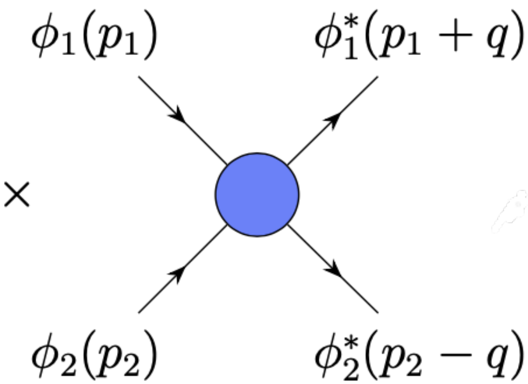
- Consider the scattering of two black holes separated by a large impact parameter b. The classical change in momentum is

$$\Delta p_2^\mu = \langle \psi | S^\dagger \mathbb{P}_2^\mu S | \psi \rangle - \langle \psi | \mathbb{P}_2^\mu | \psi \rangle$$

- We can write it in terms of scattering amplitudes only using $S = 1 + iT$. The general structure is $\underbrace{\Delta p_2^\mu = I_{(1)}^\mu + I_{(2)}^\mu}_{\textit{valid to all orders}}$

$$I_{(1)}^\mu = \int d\Phi(p_1) d\Phi(p_2) \hat{d}^4 q\, \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \Theta(p_1^0 + q^0) \Theta(p_2^0 - q^0)$$

$$\times\, e^{-ib\cdot q}\, iq^\mu \,\times$$



Credit: KMOC, 1811.10950

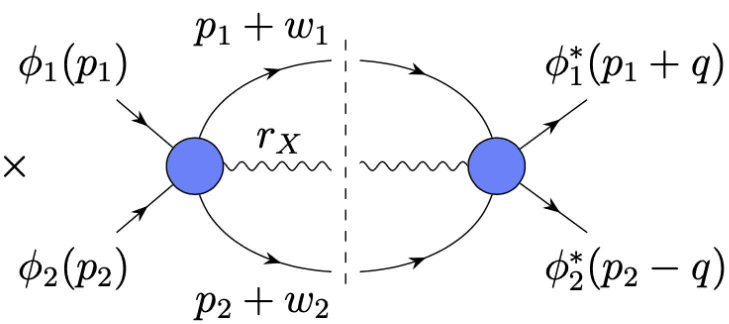
- The contribution to the impulse quadratic in T is

$$I^{\mu}_{(2)} = \sum_X \int \prod_{i=1,2} d\Phi(p_i)\hat{d}^4 w_i \hat{d}^4 q\, \hat{\delta}(2p_i \cdot w_i + w_i^2)\Theta(p_i^0 + w_i^0)$$

$$\times\, \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\Theta(p_1^0 + q^0)\Theta(p_2^0 - q^0)$$

$$\times\;\, e^{-ib\cdot q/\hbar}\, w_1^{\mu}\, \hat{\delta}^{(4)}(w_1 + w_2 + r_X)$$

$\times$



Credit: KMOC, 1811.10950

Similarities with Iwasaki's approach (1971)

$I^{\mu}_{(2)}$ acts as a Born subtraction, removing divergences when $\hbar \to 0$

Progress of Theoretical Physics, Vol. 46, No. 5, November 1971

Quantum Theory of Gravitation vs. Classical Theory*)

——*Fourth-Order Potential*——

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(Received May 18, 1971)

The perihelion-motion of Mercury depends on the fourth-order potential in quantum field theory; it is a "Lamb shift". In spite of the unrenormalizability of the theory, we have extracted a finite and physically meaningful quantity, a fourth-order potential, from fourth-order graphs. We have also discussed briefly renormalization of the Newtonian potential in the fourth-order perturbation.

The Hamiltonian obtained is the same as the classical one and so it cannot explain the Dicke-Goldenberg experiment.

We have calculated fourth-order potential also in Q.E.D.

- In general relativity, the LO contribution in G_N is from $I^{\mu}_{(1)}$

$$\Delta p_2^{\mu,LO} = i \int d^4q \, \delta(2p_1 \cdot q) \, \delta(2p_2 \cdot \bar{q}) \, e^{-ib\cdot q} \times q^{\mu} \underbrace{\mathcal{M}_4(p_1, p_2 \to p_1 + q, p_2 - q)}_{causality+locality}$$

$$\mathcal{M}_3(p, k^+) = -\kappa \, (p \cdot \varepsilon_+(k))^2 \, , \qquad \mathcal{M}_3(p, k^-) = -\kappa \, (p \cdot \varepsilon_-(k))^2$$

- The change in momentum to this order is

$$\Delta p_2^{\mu,LO} = 2 G_N m_1 m_2 \frac{2\,(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{b^{\mu}}{b^2}$$

in agreement with purely classical methods (e.g. the geodesic equation and the Einstein field equations).

Waveforms

- We can also study radiative processes in KMOC, including gravitational waveforms (C. et al., 2107.10193)

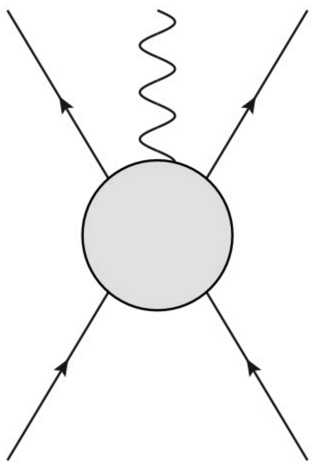
$$\langle\psi| S^\dagger \mathbb{R}_{\mu\nu\rho\sigma}(u, r, \mathbf{n}) S |\psi\rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} \frac{f_{\mu\nu\rho\sigma}(\omega, \boldsymbol{n})}{r}$$

- Equivalent to an on-shell integral over a 5-point amplitude

$$f_{\mu\nu\rho\sigma}(\omega, \boldsymbol{n}) = \frac{\kappa}{8\pi} \sum_{\eta=\pm} \iint d^4 q_1 d^4 q_2 \delta\left(2p_1 \cdot q_1\right) \delta\left(2p_2 \cdot q_2\right)$$

$$\delta^{(4)}\left(q_1 + q_2 - k\right) \times k_{[\mu} k_{[\sigma} \varepsilon^{-\eta}_{\nu]\rho]}(k) e^{iq_1 \cdot b}$$

$$\underbrace{\mathcal{M}_5\left(p_1 p_2 \to p_1' p_2' k^\eta\right)}_{causality+locality} |_{k=\omega(1,\boldsymbol{n})}$$



- Gravitational waves are made by a large number of gravitons. How can we describe them only from a single emission?

$$S|\psi\rangle = \int \mathrm{d}\Phi\left(p_1, p_2\right) \int \mathrm{d}^4 x \, \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \phi\left(x_1, x_2\right) e^{i\left(p_1 \cdot x_1 + p_2 \cdot x_2\right)/\hbar}$$

$$\times \underbrace{\int \mathrm{d}^4 q \, \exp\left[i\left[q \cdot \left(x + b - x_1 + x_2\right)/\hbar + \chi\left(x_\perp; s\right)\right]/\hbar\right]}_{\textit{eikonal exponentiation} \sim \exp[\mathcal{M}_4]}$$

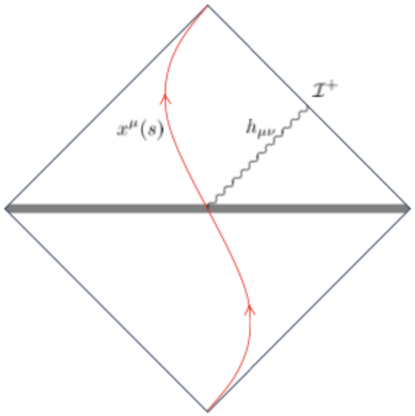
$$\times \underbrace{\exp\left[\frac{1}{\hbar^{3/2}} \sum_\eta \int \mathrm{d}\Phi(k) \alpha^{(\eta)}\left(k, x_1, x_2\right) a_\eta^\dagger(k)\right]}_{\textit{coherent state} \sim \exp[\mathcal{M}_5]} |p_1, p_2\rangle$$

- The final state is an exponential of a 5-point and a 4-point. The classical agreement from a single emission follows from the properties of coherent states (C. et al., 2112.07556)

Cosmology

- Can we include cosmological effects to gravitational waveforms while still using on-shell amplitudes?

$$ds^2 = a^2(\eta)\left(d\eta^2 - dx^i dx^j \delta_{ij}\right)$$
$$\lim_{\eta\to-\infty} a(\eta) = 1 \quad , \quad \lim_{\eta\to+\infty} a(\eta) = a_\infty \in \mathbb{R}^+$$



Aoki and C., 2402.06555

- The waveform operator on $\mathcal{I}^+$ is

$$\hat{\mathbb{H}}_{ij}(r,u,\hat{\mathbf{x}}) = -\frac{i\kappa}{4\pi r}\sum_\eta \int_0^{+\infty} \hat{d}\omega\,(\alpha e^{-i\omega u}$$
$$+\beta e^{+i\omega u})\hat{a}_{\mathbf{k},\eta}\varepsilon_{ij}^{-\eta}(\hat{\mathbf{x}}) + h.c.$$

non vanishing function of the Bogoliubov coefficients α and β

- The leading waveform due to geodesic motion is captured by an on-shell 3-point amplitude without conservation of energy

$$\langle\psi|\,\mathcal{S}^\dagger \hat{a}_{\mathbf{k},\eta}\mathcal{S}\,|\psi\rangle = \underbrace{\mathcal{M}_3(\omega,\hat{\mathbf{x}},p)}_{3-point}$$

$$\times\int_{-\infty}^{+\infty} d\eta\,\Psi^*(\eta)\frac{e^{-i\int_{-\infty}^{\eta}d\eta'\,\frac{\mathbf{p}\cdot\bar{\mathbf{k}}}{E_p(\eta')}}}{E_p(\eta)} + \mathcal{O}(\hbar)\,,\; E_p(\eta) := \sqrt{p^2+m^2a^2(\eta)}$$

- Closed expression for the waveform in an impulsive FRW

$$h_{ij}(r,u,\hat{\mathbf{x}})|_{\mathcal{I}^+} = -\frac{4G[\mathbf{pp}]_{ij}^{TT}}{r}\Bigg[\theta(-u)\Bigg(\frac{\alpha}{E_p(-\infty)-\mathbf{p}\cdot\hat{\mathbf{x}}} + \frac{\alpha\beta}{E_p(+\infty)-\mathbf{p}\cdot\hat{\mathbf{x}}} + \frac{\alpha\beta}{E_p(+\infty)+\mathbf{p}\cdot\hat{\mathbf{x}}}$$

$$+\theta(u)\Bigg(\frac{\beta}{E_p(-\infty)-\mathbf{p}\cdot\hat{\mathbf{x}}} + \frac{\alpha^2}{E_p(+\infty)-\mathbf{p}\cdot\hat{\mathbf{x}}} + \frac{\beta^2}{E_p(+\infty)+\mathbf{p}\cdot\hat{\mathbf{x}}}\Bigg)$$

- Agreement with classical methods (Aoki and C., 2402.06555)

Outlook

- New collaborations between the general relativity and high energy physics community + annual meetings (e.g. QCD meets gravity 2024 in Taipei and Amplitudes 2024 in Princeton)
- High precision physics and use of new ideas from particle physics (e.g. double copy) [Snowmass White Paper: Gravitational Waves and Scattering Amplitudes (Buonanno et al.)]
- The field of scattering amplitude is becoming an active research field. It will continue to grow with applications to general relativity!