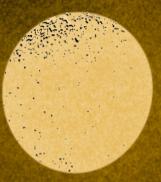
General Relativity from Scattering Amplitudes

Andrea Cristofoli, University of Edinburgh, 23rd February 2024

Yukawa Institute for Theoretical Physics, Gravity and Cosmology 2024



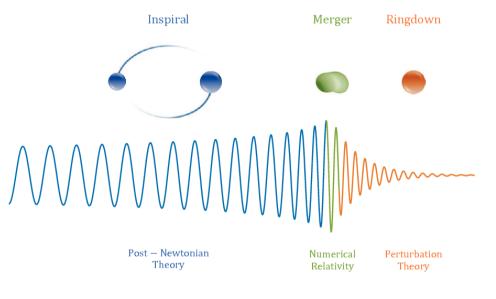
Motivation

• Gravitational waves carry fingerprints of a two-body dynamics

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G_N}{c^4}T_{\mu\nu} \quad , \quad \ddot{x}^{\mu}_{a} = -\Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}_{a}\dot{x}^{\beta}_{a}$$

... however, no exact solution is known!

• The Effective One Body approach (EOB) provides an accurate solution combining results from different regimes of motion

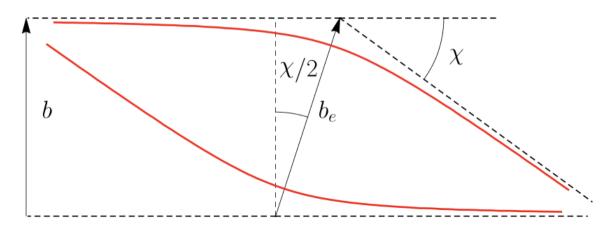


Credit: Antelis and Moreno, 1610.03567

Improving the EOB

Relativistic scattering observables can be used in the EOB to improve gravitational wave templates (Damour, 1609.00354)

• An example is the Post-Minkowskian (PM) scattering angle, computed by expanding in G_N while keeping all terms in v/c

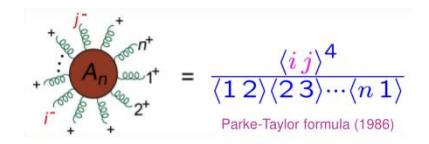


Credit: Bern et al. 2002.02459

State of the art till 2019 (Westpfhal, 1985)

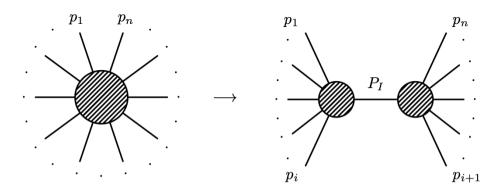
$$\chi_{2PM} = \frac{2G_N}{L} \frac{2(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} + \frac{G_N^2}{L^2} \frac{3\pi (m_1 + m_2)(5(p_1 \cdot p_2)^2 - m_1^2 m_2^2)}{4E}$$

• This simplicity is reminiscent of QFT calculations boiling down to simple answers. An example is the tree level MHV scattering of gluons proposed by Parke and Taylor in 1986



• This formula can be proved with spinor-helicity variables $p^{\mu} \sim \lambda_{\alpha} \tilde{\lambda}_{\dot{\alpha}}$ and on-shell recursion relations (BCFW, 2005).

$$\mathcal{M}_n(p_1,...,p_n) o \mathcal{M}_{i+1}^L(p_1,...,p_i,P_I) rac{1}{P_I^2} \mathcal{M}_{n-i+1}^R(-P_I,p_{i+1},...,p_n) \,,$$



Fundamental principles: unitarity, causality and locality.

Don't try this at home

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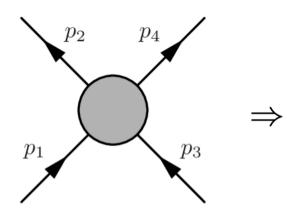
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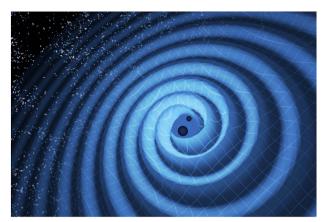
> x 100 pages ~ 220 Feynman diagrams...

 $k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5$

Paradigm shift

• Can we approach general relativity using the same fundamental principles we use in amplitude calculations?





Credit: Tim Pyle

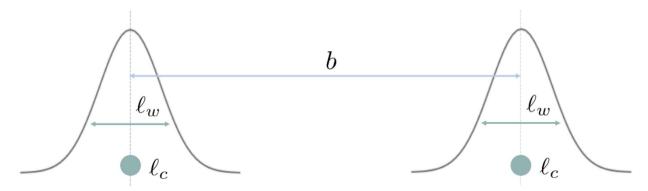
State of the art results (Bern et al., 2019) + (Veneziano et al., 2021) $\chi_{3PM} = -\frac{16m_1^3m_2^3\sigma^6G^3}{3L^3(\sigma^2 - 1)^{3/2}} + \frac{32m_1^4m_2^4\sigma^6G^3}{L^3(\sigma^2 - 1)(m_1^2 + m_2^2 + 2m_1m_2\sigma)}$ $-\frac{32m_1^4m_2^4\sigma^4G^3}{L^3(m_1^2 + m_2^2 + 2m_1m_2\sigma)} \left[1 - \frac{\sigma(\sigma^2 - 2)}{(\sigma^2 - 1)^{\frac{3}{2}}}\right] \cosh^{-1}(\sigma)$

The KMOC formalism

• Binary system as superposition of single particle states

$$|\psi\rangle = \int \underbrace{d\Phi(p_1) d\Phi(p_2)}_{\frac{d^4p}{(2\pi)^4}\theta(p_0)2\pi\delta(p^2 - m^2)} \phi_1(p_1) \phi_2(p_2) e^{\frac{ib \cdot p_1}{\hbar}} |p_1p_2\rangle$$

• Classical limit \leftrightarrow Goldilocks relations $\ell_c \ll \ell_w \ll b$



Credit: Ben Maybee, 2105.10268

• Classical observables from the *S*-matrix (1811.10950)

$$O = \lim_{\hbar \to 0} \langle \psi | S^{\dagger} \hat{\mathcal{O}} S | \psi \rangle$$

• Consider the scattering of two black holes separated by a large impact parameter *b*. The classical change in momentum is

$$\Delta p_2^{\mu} = \langle \psi | S^{\dagger} \mathbb{P}_2^{\mu} S | \psi \rangle - \langle \psi | \mathbb{P}_2^{\mu} | \psi \rangle$$

• We can write it in terms of scattering amplitudes only using S = 1 + iT. The general structure is $\Delta p_2^{\mu} = I_{(1)}^{\mu} + I_{(2)}^{\mu}$

$$I_{(1)}^{\mu} = \int d\Phi(p_1) d\Phi(p_2) \hat{d}^4 q \, \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \Theta(p_1^0 + q^0) \Theta(p_2^0 - q^0)$$

$$\phi_1(p_1) \qquad \phi_1^*(p_1 + q)$$

$$\times e^{-ib \cdot q} \, iq^{\mu} \times \qquad \phi_2(p_2) \qquad \phi_2^*(p_2 - q)$$

Credit: KMOC, 1811.10950

• The contribution to the impulse quadratic in T is

Credit: KMOC, 1811.10950

Similarities with Iwasaki's approach (1971)

 $I^{\mu}_{(2)}$ acts as a Born subtraction, removing divergences when $\hbar
ightarrow 0$

Progress of Theoretical Physics, Vol. 46, No. 5, November 1971

Quantum Theory of Gravitation vs. Classical Theory^{*)}

-----Fourth-Order Potential------

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Research Institute for Fundamental Physics, Kyoto University, Kyoto

(Received May 18, 1971)

The perihelion-motion of Mercury depends on the fourth-order potential in quantum field theory; it is a "Lamb shift". In spite of the unrenormalizability of the theory, we have extracted a finite and physically meaningful quantity, a fourth-order potential, from fourthorder graphs. We have also discussed briefly renormalization of the Newtonian potential in the fourth-order perturbation.

The Hamiltonian obtained is the same as the classical one and so it cannot explain the Dicke-Goldenberg experiment.

We have calculated fourth-order potential also in Q.E.D.

• In general relativity, the LO contribution in G_N is from $I_{(1)}^{\mu}$

$$\Delta p_2^{\mu,LO} = i \int d^4 q \,\delta \left(2p_1 \cdot q\right) \delta \left(2p_2 \cdot \bar{q}\right) e^{-ib \cdot q}$$
$$\times q^{\mu} \underbrace{\mathcal{M}_4 \left(p_1, p_2 \to p_1 + q, p_2 - q\right)}_{\Gamma_1 \to \Gamma_2}$$

causality+*locality*

$$\mathcal{M}_3(p,k^+) = -\kappa \left(p \cdot \varepsilon_+(k)\right)^2, \qquad \mathcal{M}_3(p,k^-) = -\kappa \left(p \cdot \varepsilon_-(k)\right)^2$$

• The change in momentum to this order is

$$\Delta p_{2}^{\mu,LO} = 2G_{N}m_{1}m_{2}\frac{2\left(p_{1}\cdot p_{2}\right)^{2}-m_{1}^{2}m_{2}^{2}}{\sqrt{\left(p_{1}\cdot p_{2}\right)^{2}-m_{1}^{2}m_{2}^{2}}}\frac{b^{\mu}}{b^{2}}$$

in agreement with purely classical methods (e.g. the geodesic equation and the Einstein field equations).

• We can also study radiative processes in KMOC, including gravitational waveforms (C. et al., 2107.10193)

$$\langle \psi | S^{\dagger} \mathbb{R}_{\mu \nu \rho \sigma}(u, r, \mathbf{n}) S | \psi \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} \frac{f_{\mu \nu \rho \sigma}(\omega, \mathbf{n})}{r}$$

• Equivalent to an on-shell integral over a 5-point amplitude

$$f_{\mu\nu\rho\sigma}(\omega, \boldsymbol{n}) = \frac{\kappa}{8\pi} \sum_{\eta=\pm} \iint d^4 q_1 d^4 q_2 \delta \left(2p_1 \cdot q_1\right) \delta \left(2p_2 \cdot q_2\right)$$
$$\delta^{(4)} \left(q_1 + q_2 - k\right) \times k_{[\mu} k_{[\sigma} \varepsilon_{\nu]\rho]}^{-\eta}(k) e^{iq_1 \cdot b}$$
$$\underbrace{\mathcal{M}_5 \left(p_1 p_2 \to p'_1 p'_2 k^{\eta}\right)}_{\text{causality+locality}}|_{k=\omega(1,\boldsymbol{n})}$$

• Gravitational waves are made by a large number of gravitons. How can we describe them only from a single emission?

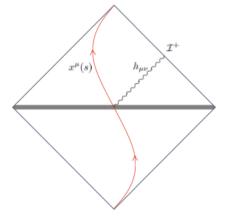
$$S|\psi\rangle = \int d\Phi (p_1, p_2) \int d^4x \, d^4x_1 \, d^4x_2 \phi (x_1, x_2) \, e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)/\hbar} \\ \times \int d^4q \, \exp\left[i \left[q \cdot (x + b - x_1 + x_2)/\hbar + \chi \left(x_{\perp}; s\right)\right]/\hbar\right] \\ \xrightarrow{eikonal \, exponentiation \, \sim \, exp[\mathcal{M}_4]} \\ \times \exp\left[\frac{1}{\hbar^{3/2}} \sum_{\eta} \int d\Phi(k) \alpha^{(\eta)} \left(k, x_1, x_2\right) a^{\dagger}_{\eta}(k)\right] \, |p_1, p_2\rangle \\ \xrightarrow{coherent \, state \, \sim \, exp[\mathcal{M}_5]}$$

 The final state is an exponential of a 5-point and a 4-point. The classical agreement from a single emission₀ follows from the properties of coherent states (C. et al., 2112.07556)

Cosmology

• Can we include cosmological effects to gravitational waveforms while still using on-shell amplitudes?

$$egin{aligned} &ds^2 = a^2(\eta) \left(d\eta^2 - dx^i dx^j \delta_{ij}
ight) \ &\lim_{\eta o -\infty} a(\eta) = 1 \quad , \quad \lim_{\eta o +\infty} a(\eta) = a_\infty \in \mathbb{R}^+ \end{aligned}$$



Aoki and C., 2402.06555

• The waveform operator on \mathcal{I}^+ is

$$\hat{\mathbb{H}}_{ij}(r, u, \hat{\mathbf{x}}) = -\frac{i\kappa}{4\pi r} \sum_{\eta} \int_{0}^{+\infty} \hat{d}\omega \left(\alpha e^{-i\omega u} +\beta e^{+i\omega u}\right) \hat{a}_{\mathbf{k},\eta} \varepsilon_{ij}^{-\eta}(\hat{\mathbf{x}}) + h.c.$$

non vanishing function of the Bogoliubov coefficients α and β

• The leading waveform due to geodesic motion is captured by an on-shell 3-point amplitude without conservation of energy

$$\langle \psi | \, \mathcal{S}^{\dagger} \hat{a}_{\mathbf{k},\eta} \mathcal{S} \, | \psi \rangle = \underbrace{\mathcal{M}_{3}(\omega, \mathbf{\hat{x}}, p)}_{3-point}$$

$$\times \int_{-\infty}^{+\infty} d\eta \, \Psi^*(\eta) \frac{e^{-i \int_{-\infty}^{\eta} d\eta' \frac{\mathbf{p} \cdot \bar{\mathbf{k}}}{E_p(\eta')}}}{E_p(\eta)} + \mathcal{O}(\hbar) \,, \ E_p(\eta) := \sqrt{p^2 + m^2 a^2(\eta)}$$

• Closed expression for the waveform in an impulsive FRW

$$h_{ij}(r, u, \hat{\mathbf{x}})|_{\mathcal{I}^{+}} = -\frac{4G[\mathbf{pp}]_{ij}^{TT}}{r} \left[\theta(-u) \left(\frac{\alpha}{E_{\rho}(-\infty) - \mathbf{p} \cdot \hat{\mathbf{x}}} + \frac{\alpha\beta}{E_{\rho}(+\infty) - \mathbf{p} \cdot \hat{\mathbf{x}}} + \frac{\alpha\beta}{E_{\rho}(+\infty) + \mathbf{p} \cdot \hat{\mathbf{x}}} + \frac{\alpha\beta}{E_{\rho}(+\infty) - \mathbf{p} \cdot \hat{\mathbf{x}}} + \frac{\alpha\beta^{2}}{E_{\rho}(+\infty) - \mathbf{p} \cdot \hat{\mathbf{x}}} + \frac{\beta^{2}}{E_{\rho}(+\infty) - \mathbf{p} \cdot \hat{\mathbf{x}}} + \frac{\beta^{2}}{E_{\rho}(+\infty) + \mathbf{p} \cdot \hat{\mathbf{x}}} \right)$$

• Agreement with classical methods (Aoki and C., 2402.06555)

Outlook

- New collaborations between the general relativity and high energy physics community + annual meetings (e.g. QCD meets gravity 2024 in Taipei and Amplitudes 2024 in Princeton)
- High precision physics and use of new ideas from particle physics (e.g. double copy) [Snowmass White Paper: Gravitational Waves and Scattering Amplitudes (Buonanno et al.)]
- The field of scattering amplitude is becoming an active research field. It will continue to grow with applications to general relativity!

