Double graviton dressing and $\mathcal{O}(G^2)$ observables

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Outline

Soft eikonal operator with linear graviton modes

Generalization to quadratic graviton modes

Soft Observables

Conclusion

Introduction

Comprehensive review in [P. Di Vecchia, et. al. arXiv:2306.16488 [hep-th]]

- $2-2$ scattering of gravitationally interacting spinless fields
- Post-minkowski (PM) expansion: $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$; $\kappa^2 = 8\pi G_N$ and $\dfrac{G_N\sqrt{s}}{bc^2} \ll 1$ with

G_N : Newton's constant ; b : impact parameter

$s := E^2 = -(p_1 + p_2)^2$; com energy squared

$q = p_4 + p_1$; momentum exchange

- Eikonal approximation: $\dfrac{q^2}{s} \ll 1 \left(q \sim \dfrac{1}{b} \ll 1 \right) \Rightarrow$ all loop graviton exchanges can be resummed into a phase
- With the scattering operator $S = 1 + iT$, we have

$$\langle p'|S|p\rangle = (2\pi)^{(D-1)}\delta^{(D-1)}(\vec{p} - \vec{p}') + 2\pi i \frac{\delta(|\vec{p}| - |\vec{p}'|)}{4E|\vec{p}|}\mathcal{A}(s\,, -q^2)$$

Eikonal Ansatz

The Fourier transform ($D = 4 - 2\epsilon$)

$$\tilde{S} = \int \frac{d^{(D-1)}q}{(2\pi)^{(D-1)}} e^{i\vec{b}.\vec{q}} \langle p'|S|p\rangle$$

in the small q limit evaluates to

$$\tilde{S} = 1 + \tilde{A}(s,b) \; ; \qquad \tilde{A}(s,b) = e^{2i\delta}(1 + 2i\Delta)$$

with $\delta(\sigma\,, b)$ the eikonal phase and Δ a quantum remainder.

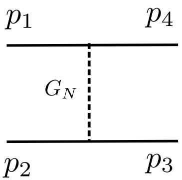
- $\tilde{A}$, δ and Δ expand in powers of G_N

$$i\tilde{A}_0 = 2i\delta_0$$

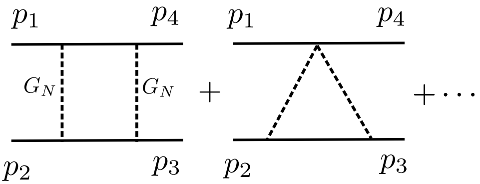
$$i\tilde{A}_1 = 2i\delta_1 + \frac{1}{2}(2i\delta_0)^2 + 2i\Delta_1 + \cdots$$

$$i\tilde{A}_2 = (2i\delta_2 + 2i\delta_0 2i\Delta_i) + \frac{1}{3!}(2i\delta_0)^3 + (2i\delta_0)(2i\delta_1) + \cdots$$

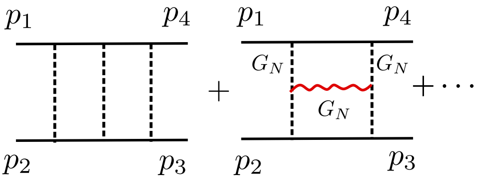
- Tree level



- 1-loop



- 2-loop



δ_2 properties

- Inelastic radiative exchanges from two and higher loop orders

$$\Rightarrow 2\delta_2 = \mathrm{Re}(2\delta_2) + i\mathrm{Im}(2\delta_2) \qquad \text{@ 3PM}$$

- $\mathrm{Re}(2\delta_2)$ provides the scattering angle via

$$\frac{1}{2} Q_\mu = \frac{\partial \mathrm{Re} 2\delta}{\partial b^\mu} \; ; \quad Q_\mu = 2 p_\mu \sin\left(\frac{\chi}{2}\right)$$

- $\mathrm{Re}(2\delta_2)$ also has a *dissipative* radiation reaction contribution; Related to the leading divergence of $\mathrm{Im}(2\delta_2)$ by analyticity

$$[\mathrm{Re}(2\delta_2)]_{RR} = \lim_{\epsilon \to 0} \left[-\pi\epsilon \mathrm{Im}(2\delta_2)|_{\epsilon^{-1}} \right] \quad \left(\epsilon = \frac{D-4}{2}\right)$$

- $\mathrm{Im}(2\delta_2)$ term that diverges when $\epsilon \to 0$ is related to Weinberg soft graviton factor [P. Di Vecchia et. al. PLB **818** (2021) 136379]

Soft factors $\leftrightarrow$ Radiation reaction @ 3PM

Soft eikonal operator for inelastic exchanges

[P. Di Vecchia et. al., JHEP **08** (2022) 172; JHEP **07** (2023) 039; PLB **843** (2023) 138049]

- *Eikonal operator* involving graviton modes:
  Unitarity and radiative observables

- Leading low frequency contribution: Weinberg soft graviton dressing

$$e^{2i\hat{\delta}_2} \xrightarrow[\omega\to 0]{} \exp[-\Delta_1^\kappa]\, e^{i\mathrm{Re}2\delta_2}$$

$$\Delta_1^\kappa = \frac{1}{\hbar}\int_{\vec{k}} d^3k \left(a_i(k) f_i^*(k) - a_i^\dagger(k) f_i(k)\right)$$

$$f_i(k) = \varepsilon^*_{i\,,\mu\nu}(k) F^{\mu\nu}(k)\ ; \qquad F^{\mu\nu}(k) = \sum_n \frac{\kappa p_n^\mu p_n^\nu}{k\cdot p_n - i0_k}$$

- Exp. values of graviton operators give radiative observables

$$\langle 0| e^{\Delta_1^\kappa} e^{-i\mathrm{Re}2\delta_2} \mathcal{O} e^{-\Delta_1^\kappa} e^{i\mathrm{Re}2\delta_2} |0\rangle = \langle 0| e^{\Delta_1^\kappa} \mathcal{O} e^{-\Delta_1^\kappa} |0\rangle := \langle \mathcal{O}\rangle_{\Delta^\kappa}$$

$$\langle 0| \exp[-\Delta_1^\kappa]\, e^{i\mathrm{Re}2\delta_2} |0\rangle = \exp[-\mathrm{Im}(2\delta_2)|_{\epsilon^{-1}}]\, e^{i\mathrm{Re}2\delta_2}$$

Outline

Schwinger formalism

Use $p_\mu = -i\partial_\mu$ in minimally coupled scalar field action to find [D. Bonocore et. al. JHEP **03** (2022) 147 ; C. White JHEP **05** (2011) 060]

$$S = -\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m^2\phi^*\phi\right) := -\int d^4x\phi^*\left(2\hat{H}\right)\phi$$

- Schwinger formalism: Identify $\left(\hat{H}(x\,,p) - i\epsilon\right)^{-1}$ with scalar field propagator (in background field). For soft factor, we consider transition from initial position to final momentum

$$\left\langle p_f\right|\left(\hat{H} - i\epsilon\right)^{-1}\left|x_i\right\rangle = \int_0^\infty dT\int_{x(0)=x_i}^{p(T)=p_f}\mathcal{D}p\mathcal{D}x$$
$$\exp\left[-\frac{i}{\hbar}p(T).x(T) + \frac{i}{\hbar}\int_0^T dt\left(p\dot{x} - \hat{H}(x\,,p) + i\epsilon\right)\right]\,,$$

- Evaluate for fluctuations about asymptotic trajectories

$$x(t) \to x_i + p_f t + x(t)\,, \qquad p(t) \to p_f + p(t)\,, \qquad x(0) = 0 = p(T)$$

Next-to-eikonal soft dressing

- Amputated propagator gives its dressing

$$\lim_{p_f^2\to -m^2} -i(p_f^2+m^2-i\epsilon)\left\langle p_f\right|\left(\hat{H}-i\epsilon\right)^{-1}\left|x_i\right\rangle = \lim_{T\to\infty} f(T) := f(\infty)$$

- $n=1,\cdots 4$ particle generalization gives amplitude soft factor dressing

$$f(\infty)=\exp[-\Delta_{p_f}]\to\exp\left[-\sum_n \Delta_{p_n}\right] := \exp[-\Delta]$$

$$\begin{aligned}\Delta_{p_n} &= -i\kappa\int_0^\infty dt\left[-p_n^\mu p_n^\nu + ip_n^{(\mu}\partial^{\nu)} - \frac{i}{2}\eta^{\mu\nu}p_n^\alpha\partial_\alpha + \frac{i}{2}tp_n^\mu p_n^\nu\partial^2\right]h_{\mu\nu}(p_n t)\\ &+2i\kappa^2\int_0^\infty dt\int_0^\infty ds\left[\frac{p_n^\mu p_n^\nu p_n^\rho p_n^\sigma}{4}\mathrm{min}(t,s)\partial_\alpha h_{\mu\nu}(p_n t)\partial^\alpha h_{\rho\sigma}(p_n s)\right.\\ &\left.+p_n^\mu p_n^\nu p_n^\rho\Theta(t-s)h_{\rho\sigma}(p_n s)\partial^\sigma h_{\mu\nu}(p_n t) + p_n^\nu p_n^\rho\delta(t-s)\eta^{\mu\sigma}h_{\rho\sigma}(p_n s)h_{\mu\nu}(p_n t)\right]\end{aligned}$$

Collinear double graviton dressing

- We substitute collinear graviton modes in $\exp[-\Delta]$

$$h_{\mu\nu}(p_n t) = \int_{\vec{k}} d^3k \left[a_i(k)\varepsilon_{i\,,\mu\nu}(k)e^{ik\cdot p_n t} + a_i^\dagger(k)\varepsilon^*_{i\,,\mu\nu}(k)e^{-ik\cdot p_n t}\right]$$

$$h_{\rho\sigma}(p_n s) = \int_{\vec{l}} d^3l \left[a_j(l)\varepsilon_{j\,,\rho\sigma}(l)e^{il\cdot p_n s} + a_j^\dagger(l)\varepsilon^*_{j\,,\rho\sigma}(l)e^{-il\cdot p_n s}\right](2\pi)^2\delta(\Omega_k\,,\Omega_l)$$

$$\left[a_i(k)\,,a_j^\dagger(k')\right] = 2\hbar\omega_k\delta_{ij}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')$$

- Defined for on-shell outgoing gravitons

$$k^\mu = \omega_k q_k^\mu \ \ ,\ q_k^\mu = (1\,,\hat{k})\ ; \quad l^\mu = \omega_l q_l^\mu \ \ ,\ q_l^\mu = (1\,,\hat{l})$$

- On-shell graviton measure over soft region (“UV” cut-off ω)

$$\int_{\vec{k}} d^3k \equiv \frac{1}{2(2\pi)^3}\int_0^\infty d\omega_k\omega_k \oint d\Omega_k\Theta(\omega-\omega_k)$$

- Main takeaways: Recovers a manifestly gauge invariant dressing;
  Two gravitons have different energies but same orientation

Collinear double graviton dressing: Result

$$\exp[-\Delta] = \exp[-\Delta_1^{\kappa} - \Delta_2^{\kappa^2}]$$

Δ_1^{κ} as before (*Coherent* Weinberg dressing), and

$$\Delta_2^{\kappa^2} = \frac{1}{2\hbar}\int_{\vec{k}} d^3k \int_{\vec{l}} d^3l \Big[a_i^{\dagger}(k)a_j^{\dagger}(l)A_{ij}(k,l) - a_i(k)a_j(l)A_{ij}^{*}(k,l) \\ + a_i^{\dagger}(k)a_j(l)B_{ij}^{*}(k,l) - a_j^{\dagger}(l)a_i(k)B_{ij}(k,l)\Big](2\pi)^2\delta(\Omega_k\,,\Omega_l)$$

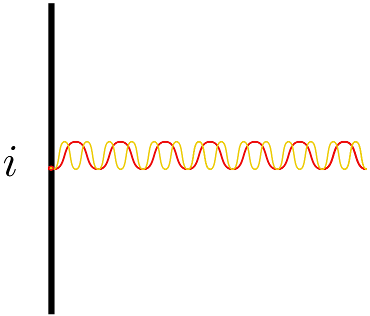
$$A_{ij}(k,l) = \varepsilon^{*}_{i\,,\mu\nu}(k)\tilde{C}_{+}^{\mu\nu\rho\sigma}\varepsilon^{*}_{j\,,\rho\sigma}(l)\ , \quad B_{ij}(k,l) = \varepsilon_{i\,,\mu\nu}(k)\tilde{C}_{-}^{\mu\nu\rho\sigma}\varepsilon^{*}_{j\,,\rho\sigma}(l)$$

$$C_{\pm}^{\mu\nu\rho\sigma}(k,l) = -\sum_n \frac{\kappa^2}{p_n\cdot(k\pm l)\mp i0_k \mp i0_l}\alpha_n^{\mu\nu\rho\sigma}$$

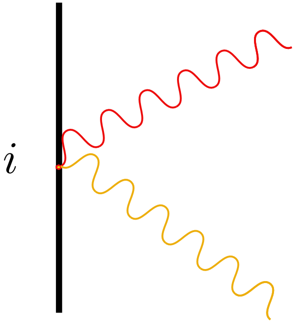
$$\alpha_n^{\mu\nu\rho\sigma} = p_n^{\rho}p_n^{\mu}\eta^{\nu\sigma} + p_n^{\rho}p_n^{\nu}\eta^{\mu\sigma} + p_n^{\sigma}p_n^{\mu}\eta^{\nu\rho} + p_n^{\sigma}p_n^{\nu}\eta^{\mu\rho}$$

Main takeaways: $\Delta_2^{\kappa^2}$ is for a collinear seagull vertex.

Recoil effects due to emission and absorption of soft gravitons



Collinear seagull vertex



Seagull vertex

Factorization

- Using the Baker-Campbell-Haussdorf formula, we find

$$\exp[-\Delta] = \exp[-\Delta_1]\exp[-\Delta_2^{\kappa^2}]\,,$$

$$\Delta_1 = \sum_{n=0}^{\infty} \Delta_1^{\kappa^{2n+1}} = \Delta_1^{\kappa} + \Delta_1^{\kappa^3} + \mathcal{O}(\Delta_1^{\kappa^5})$$

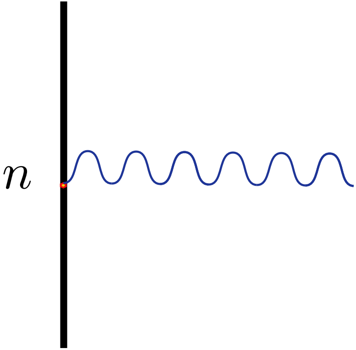
$$\Delta_1^{\kappa^3} = \frac{1}{2\hbar}\int_{\vec{k}} d^3k \int_{\vec{l}} d^3l \left[a_i^\dagger(k)\left(A_{ij}(k,l)f_j^*(l) + B_{ij}^*(k,l)f_j(l)\right)\right.$$
$$\left. - a_i(k)\left(A_{ij}^*(k,l)f_j(l) + B_{ij}(k,l)f_j^*(l)\right)\right](2\pi)^2\delta(\Omega_k\,,\Omega_l)$$

- As the dressing results from a soft factor, we consider this as a soft eikonal operator to three-loops

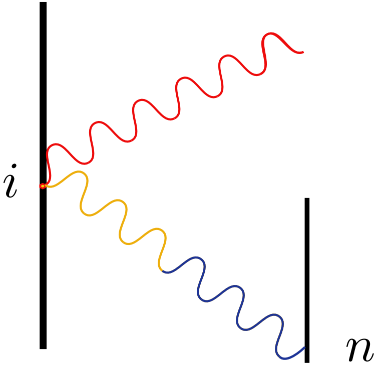
$$e^{2i\hat{\delta}} \xrightarrow[\omega\approx 0]{} \exp[-\Delta]\left(e^{i\mathrm{Re}2\delta_2} + \cdots\right)$$

- Radiative observables as before from

$$\langle 0|e^{\Delta_1}e^{\Delta_2^{\kappa^2}}\boldsymbol{\mathcal{O}}e^{-\Delta_2^{\kappa^2}}e^{-\Delta_1}|0\rangle := \langle\boldsymbol{\mathcal{O}}\rangle_\Delta$$



$$\Delta_1^{\kappa}$$



$$\Delta_1^{\kappa^3}$$

Expectation values and observables

$$\langle 0|e^{\Delta_1} e^{\Delta_2^{\kappa^2}} \mathcal{O} e^{-\Delta_2^{\kappa^2}} e^{-\Delta_1}|0\rangle := \langle \mathcal{O}\rangle_\Delta$$

- Results follow from canonical commutation relations $[a\,, a^\dagger] \sim \hbar$

  $\Rightarrow e^{\Delta_2^{\kappa^2}}$ has subleading $\hbar \to 0$ limit

  Does not contribute to classical observables.

- The coherent part of the dressing contributes

$$\langle \mathcal{O}\rangle_\Delta = \langle \mathcal{O}\rangle_{\Delta_1^\kappa} + \langle \mathcal{O}\rangle_{\Delta_1^{\kappa^3}}$$

$$\langle \exp[-\Delta_1]\rangle_\Delta = \exp[-\mathrm{Im}(2\delta_2)|_{\epsilon^{-1}}] + \exp[-\mathrm{Im}(2\delta_3)]$$

- Results involve functions of Lorentz invariant relative velocity $\sigma_{nm} = -\eta_n\eta_m \dfrac{p_n.p_m}{m_n m_m}$, which has an impulse PM expansion

  $\mathcal{O}(G)$ observables from $\langle \mathcal{O}\rangle_{\Delta_1^\kappa}$

  $\mathcal{O}(G^2)$ observables from $\langle \mathcal{O}\rangle_{\Delta_1^{\kappa^3}}$

Outline

I. Emitted energy spectrum : $\mathcal{O}(G)$ result

$$P^0(x) = \int_{\vec{k}} d^3k k^0 a_i^\dagger(k) a_i(k) \; ; \qquad \langle P^0 \rangle_\Delta = E^G + E^{G^2}$$

- $\dfrac{dE^G}{d\omega}$ gives zero frequency limit (ZFL) result

[P. Di Vecchia et. al., JHEP **07** (2022) 039]

$$\lim_{\omega\to 0} \frac{dE^G}{d\omega} = \frac{2G}{\pi} \sum_{n\,,m} \eta_n \eta_m m_m m_n \left(\sigma_{nm}^2 - \frac{1}{2} \right) F_{nm} = \lim_{\epsilon\to 0} \left[-4\hbar\epsilon \mathrm{Im} 2\delta_2|_{\epsilon^{-1}} \right]$$

$$F_{nm} = \frac{\operatorname{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}} \; ;$$

- $\dfrac{dE^{G^2}}{d\omega}$ and $\mathrm{Im}(2\delta_3)$ are not IR divergent in $\omega = \omega_k + \omega_l$

$$\frac{dE^{G^2}}{d\omega} = -\frac{G^2\omega}{\pi} \sum_{i,n,m} H_{inm} F_{inm} = 2\hbar \mathrm{Im}(2\delta_3)$$

- The above is a generalization to subleading soft frequencies

II. Memory effect : 1PM result

$$h_{\mu\nu}(x) = \int_{\vec{k}} d^3k \left[a_i(k)\varepsilon_{i\,,\mu\nu}(k)e^{ik.x} + a_i^\dagger(k)\varepsilon^*_{i\,,\mu\nu}(k)e^{-ik.x}\right]$$

- Waveform contribution : $2\kappa\langle h_{\mu\nu}\rangle_\Delta = W^{G}_{\mu\nu} + W^{G^2}_{\mu\nu}$
- The memory effect from the asymptotic difference in $u = t - r$

$$\Delta W^{\mu\nu}_{\mathsf{TT}} := W^{\mu\nu}_{\mathsf{TT}}(x)\Big|_{u>0} - W^{\mu\nu}_{\mathsf{TT}}(x)\Big|_{u<0} = \Delta W^{G\,;\mu\nu}_{\mathsf{TT}} + \Delta W^{G^2\,;\mu\nu}_{\mathsf{TT}}$$

- $\Delta W^{G\,;\mu\nu}_{\mathsf{TT}}$ gives the linear memory effect

[P. Di Vecchia et. al., JHEP **08** (2022) 172]

$$\Delta W^{G\,;\mu\nu}_{\mathsf{TT}} = \frac{2G_N}{R}\sum_n \frac{p_n^\mu p_n^\nu}{E_n - \vec{p}_n.\hat{x}}$$

R : Distance from source to detector ; $\hat{x}$: Orientation to detector

Equivalent to [I. Braginsky and K. Thorne, Nature **327** (1987) 123 - 125]

II. Memory effect : 2PM result

- In a 'self-correction' limit

$$\Delta W^{G^2;\alpha\beta}_{\mathsf{TT}}(x) = -\frac{8G^2\omega}{R}\sum_n \frac{m_n^2 p_n^\alpha p_n^\beta}{(E_n - \vec{p}_n\cdot\hat{x})^2}$$

- Related to nonlinear memory [D. Christodoulou, PRL **67** (1991) 1486]

$$\Delta W^{IJ}_{G;\text{non-linear}}(x) = \frac{4G}{R}\int d\Omega\, \frac{dE^{GW}}{d\Omega}\left(\frac{n^I n^J}{1 - n\cdot\hat{x}}\right),$$

- Ultrarelativistic limit of external particles

$$|\vec{v}_n| = \frac{|\vec{p}_n|}{E_n} = 1 - \xi\ ; \qquad m_n^2 = E_n^2(1 - \vec{v}_n^2) \simeq 2E_n^2\xi$$

- Energy distribution for nearly soft gravitons

$$\frac{dE^{GW}_{\text{soft}}}{d\Omega_k} = \frac{\omega G\xi}{2\pi^2}\sum_n m_n^2 \frac{1}{(1 - \vec{v}_n\cdot\hat{k})^2}$$

From E^G in [P. Di Vecchia et. al., JHEP **07** (2022) 039]

II. Memory effect : Relation with nonlinear memory

- Ultrarelativistic particle approximation for nonlinear memory

$$\Delta\widetilde{W}^{IJ}_{G;\text{non-linear}}(x) \sim \frac{4G}{R}\int d\Omega_k \frac{dE^{GW}_{\text{soft}}}{d\Omega_k}\left(\frac{\vec{v}^I_n\vec{v}^J_n}{1-\vec{v}_n\cdot\hat{k}}\right)$$

- Saddle approximation about detector orientation $\hat{x}$

$$\Delta\widetilde{W}^{IJ}_{G;\text{non-linear}}(x) \sim \frac{2G^2\omega\xi}{\pi R}\sum_n m_n^2\frac{\vec{v}^I_n\vec{v}^J_n}{(1-\vec{v}_n\cdot\hat{x})^2}$$

- Gives the relation

$$\lim_{\xi\to 0}\frac{4\pi}{\xi}\Delta\widetilde{W}^{IJ}_{G;\text{non-linear}}(x) = \Delta W^{G^2;IJ}_{\text{TT}}(x)$$

- $\Delta W^{G^2;IJ}_{\text{TT}}(x)$: memory following recoil from soft gravitons.

- Nonlinear memory effect in ultrarelativistic limit

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Summary & Future directions

- Inelastic eikonal amplitudes $\Rightarrow$ Dressed eikonal amplitudes
- Provides gravitational wave observables from expectation values.
- Generalized Wilson line: Next-to-eikonal fluctuations give multi-graviton corrections to leading single graviton dressing
- At NE order: factorized dressing in terms of the classically relevant coherent term, and squeeze-like operator ($\exp[-\Delta_2^{\kappa^2}]$)
- Derived 2PM contributions to memory; $\mathcal{O}(G^2)$ emitted momentum spectrum and angular momentum

1. Comparison with 2PM memory effect [G. Jakobsen et. al., PRL **126** (2021) 20, 201103] and 4PM contribution to emitted energy [C. Dlapa et. al., PRL **130** (2023) 10, 101401]
2. $\exp[-\Delta_2^{\kappa^2}]$: Entanglement constraints to binary scattering? [D. Carney et. al., PRL **119** (2017) 18, 180502]

Thank You