Gravity and Cosmology 2024, YITP

Pseudospectrum of black hoels and horizonless black hole mimickers

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Phys.Rev.D 107 (2023) 6, 064012;
2312.11998 (also to appear in PRD)

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Talk outline:

- QNM spectral instability and pseudospectrum
- Numerical convergence of the pseudospectrum, and what it can and cannot be used for
- QNMs in AdS: overtones and regularity class
- Dynamical instabilities: exotic compact objects

QNMs as eigenvalues

Linear QNMs in spherical symmetry:

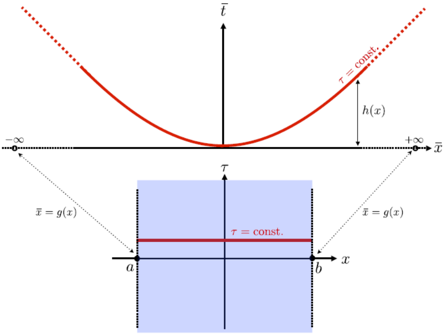
$$-\partial_{tt}\psi + \partial_{r^* r^*}\psi - V(r)\psi = 0. \qquad (1)$$

Hyperboloidal coordinate system $\{\tau, \chi\}$, order-reduction in time, Fourier transform: **eigenvalue problem**

$$L \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \omega \begin{pmatrix} \psi \\ \phi \end{pmatrix}, \qquad (2)$$

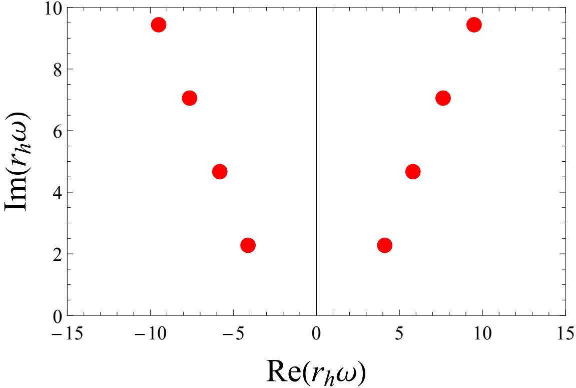
with $\phi = \partial_\tau \psi$, and

$$L = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ L_1(\chi, \partial_\chi) & L_2(\chi, \partial_\chi) \end{pmatrix}. \qquad (3)$$



From Jaramillo et al., *Phys. Rev. X 11, 031003* (2021).

QNMs: eigenvalues of L



QNM spectrum for a scalar field, $\ell = 2$, Schwarzchild-Anti-de Sitter (SAdS) with $r_h = R_{AdS}$.

Perturbations to L

If we perturb L, the eigenvalues change.[1]

$$L \to \tilde{L} = L + \delta L,$$

Size of δL: energy norm.[2]

$$\langle u_1, u_2 \rangle_E = \frac{1}{2}\int_a^b \left(w(\chi)\bar{\phi}_1\phi_2 + p(\chi)\partial_\chi\bar{\psi}_1\partial_\chi\psi_2 + q(\chi)\bar{\psi}_1\psi_2\right) d\chi.$$

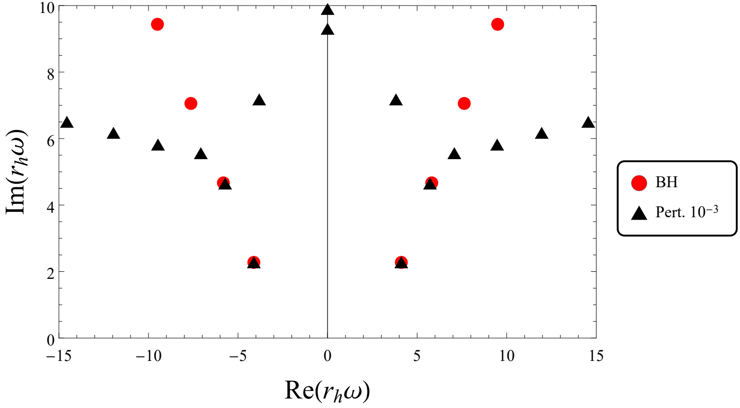
For $\|\delta L\| = \epsilon$, if the migration of any of the QNMs $|\tilde{\omega}_n - \omega_n|$ is greater than ϵ, then the operator L is **spectrally unstable** in this norm.

We consider a "physical" perturbation one which affects only the potential V (without changing its boundary behaviour).

[1] Jaramillo et al., *Phys. Rev. X 11, 031003* (2021).

[2] Gasperin & Jaramillo, *Class. Quantum Grav. 39 115010* (2022).

QNM instability



Scalar field, $\ell = 2$, Schwarzchild-Anti-de Sitter (SAdS) with $r_h = R_{AdS}$.
Spectrum before and after a perturbation $\delta\tilde{V} = \varepsilon \sin(2\pi k\chi)$.

Pseudospectrum

The QNM spectrum is given by

$$\sigma(L) = \{\omega \in \mathbb{C} : |L - \omega\mathbb{I}| = 0\}. \tag{4}$$

The ϵ-pseudospectrum is

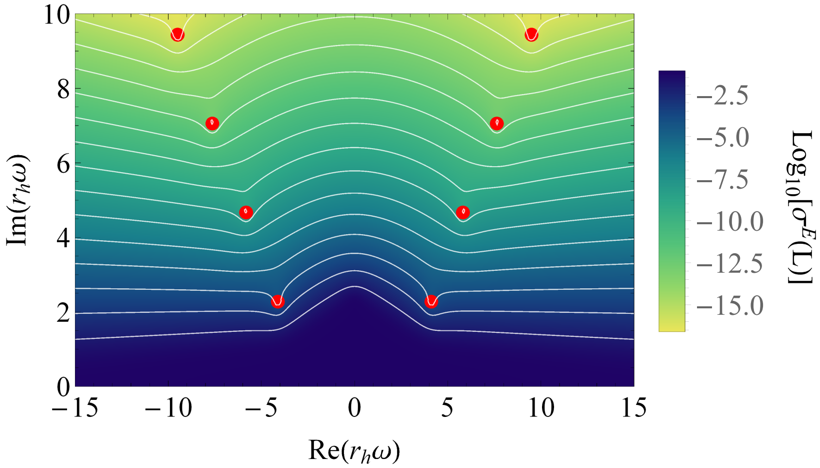
$$\sigma^{\epsilon}(L) = \{\lambda \in \mathbb{C} : \|R_L(\lambda)\| = \|(L - \lambda\mathbb{I})^{-1}\| > 1/\epsilon\}, \tag{5}$$

where $R_L(\lambda)$ is called the *resolvent*. An equivalent definition is

$$\sigma^{\epsilon}(L) = \{\lambda \in \mathbb{C}, \exists\delta L, \|\delta L\| < \epsilon : \lambda \in \sigma(L + \delta L)\}, \tag{6}$$

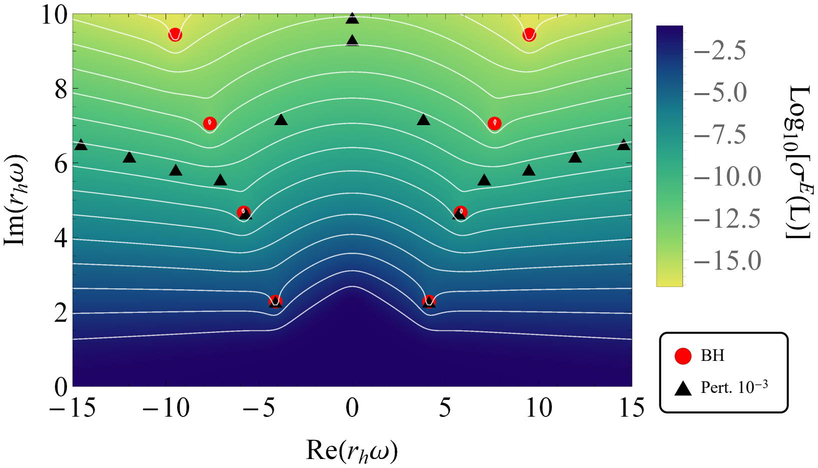
where δL is any perturbation to L.

Pseudospectrum



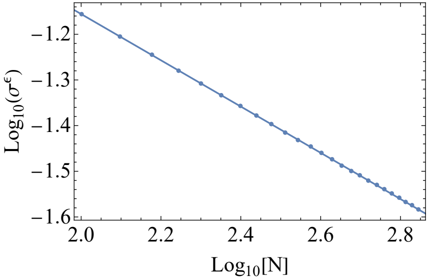
Pseudospectrum of the same case as above, calculated with $\|R_L(\lambda)\|$.

Pseudospectrum



Pseudospectrum calculated with $\|R_L(\lambda)\|$, and perturbed modes after δL.

(Non-)convergence of the pseudospectrum



Energy norm of (inverse of) resolvent operator at $\lambda = 2 + 0.5i$ (for same SAdS problem as above) as function of numerical resolution N. Similar whenever $\mathrm{Im}(\lambda) > 0$.

However, qualitative aspects of the QNM are captured at finite N.

→ Problem approximating L with finite-rank matrices?

→ All points λ where $\|R_L(\lambda)\|$ tends to diverge are QNMs?

Way forward

- Analyse convergence with the numerical gridpoint number N.
- Understand the reason behind non-convergence.
- Determine when and how quantitative results can be obtained.

[e.g. Boyanov, Cardoso, Destounis, Jaramillo, Panosso-Macedo: arXiv:2312.11998]

- Analyse qualitative aspects and robustness of the non-convergent result.
- Obtain quantitative results from the $\mathrm{Im}(\lambda) < 0$ region (early-time non-modal dynamical effects, nonlinear instabilities).

[e.g. Boyanov, Cardoso, Destounis, Jaramillo, Panosso-Macedo: Phys.Rev.D 107 (2023) 6, 064012]

AdS QNMs: a precise definition

- In a Sobolev functional space H^k, the spectrum of L (for the SAdS BH) is only discrete in the region[3]

$$\mathrm{Im}(\lambda) < a + \kappa(k - \frac{1}{2}), \tag{7}$$

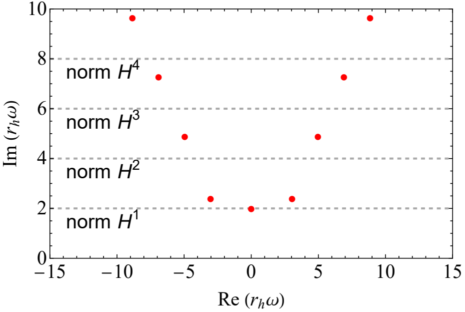
where κ is the surface gravity of the BH horizon, and a is a constant.

- Above this band, all points are part of the spectrum.
- Because part of the spectrum is always continuous, L in H^k is never a compact operator.
- L might not be well approximated by finite-rank matrices.
- Still, improved convergence behaviour might be found in some region of $\mathbb{C}$ if we impose higher regularity.

[3] C. Warnick, *Commun. Math. Phys. 333, 959–1035* (2015).

H^k norms and convergence

- In practice, the way to ensure the function space is H^k is through the norm, increasing the number of derivatives.



Boyanov, Cardoso, Destounis, Jaramillo, Panosso-Macedo, arXiv:2312.11998

Stability analysis

On the other hand, we can obtain results from the region in $\mathbb{C}$ where the resolvent norm does converge $(\mathrm{Im}(\lambda) < 0)$.

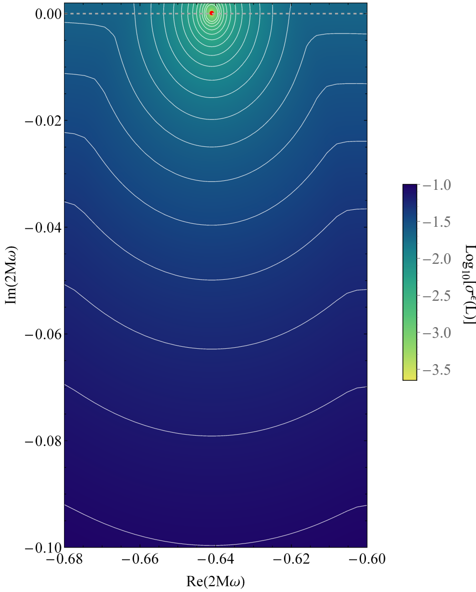
- Systems prone to instabilities tend have long-lived modes.
- The evolution of ψ dictated by L is not a superposition of QNMs.
- Non-modal transient effects are captured by the pseudospectrum.[4]

[4] Jaramillo, *Class.Quant.Grav.* 39 (2022) 21, 217002

Exotic compact objects

Example: ultracompact object (reflective surface at $(1+10^{-3})2M$, asymptotically flat).

- Pseudospectral abscissa and transient growth (no).
- Large $\|R_L\|$ on the real axis: pseudoresonances.



Conclusions

- QNMs are unstable to perturbative changes of the background.
- The pseudospectrum results capture this instability qualitatively, but the quantitative aspects still need to be refined.
- The pseudospectrum can also be used to study transient non-modal aspects of evolution.

Thank you for your attention!