BH AND NS SOLUTIONS IN $f(\mathbb{Q})$ GRAVITY

- OR WHAT NOT TO DO WHEN LOOKING FOR BEYOND GR RESULTS -

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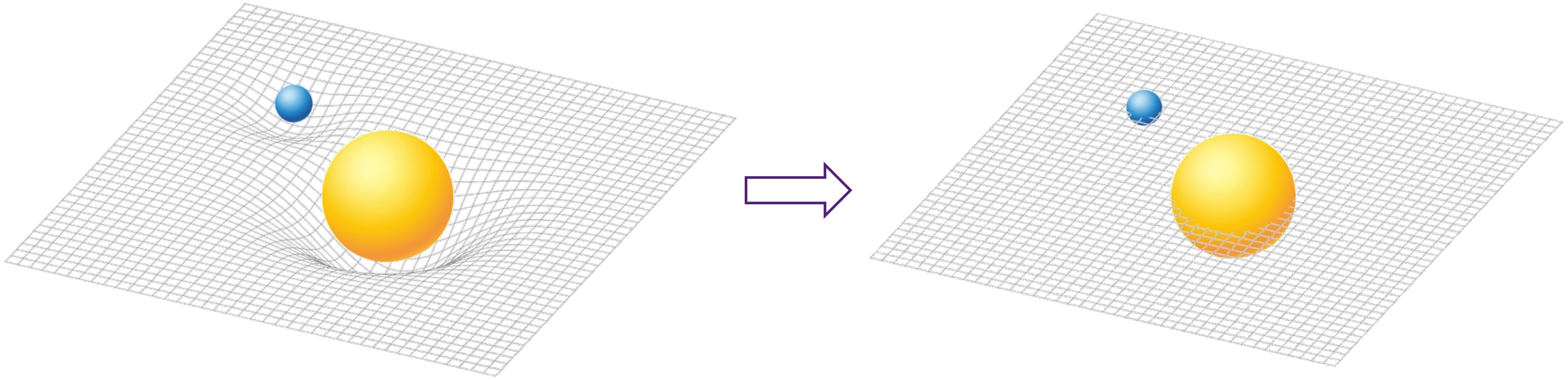
Symmetric Teleparallel Gravity?

✓ Set of gravity theories that build on a geometry with **no curvature** and **no torsion**, where **only the non-metricity** is non-vanishing.



What is non-metricity?

How are these two descriptions equivalent?

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Our geometrical model

A manifold $\mathcal{M}$:

Points on the manifold, paths ($\gamma : [0,1] \to \mathcal{M}$), scalar ($\Phi : \mathcal{M} \to \mathbb{R}$), vector ($V : \mathcal{M} \to T\mathcal{M}$), tensor fields...

A metric tensor $g : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$:

Length of paths, area of regions, geodesics...

A connection $\Gamma^{\rho}_{\mu\nu}$:

Parallel transport, covariant derivative.

$\Longrightarrow$ $\boxed{(\mathcal{M}, g, \Gamma)}$ METRIC-AFFINE GEOMETRY

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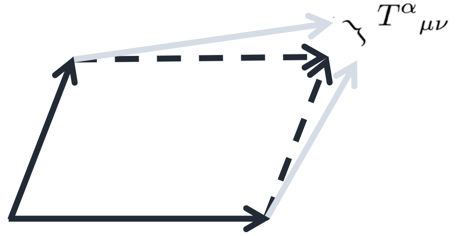
Curvature, torsion & non-metricity

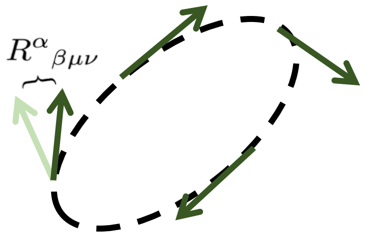
Torsion $(\mathcal{M}, g, \Gamma)$: $T^{\alpha}{}_{\mu\nu} \coloneqq \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$

✓ Torsion affects **how parallelograms** formed by two vectors **close** when parallel transported along each other.



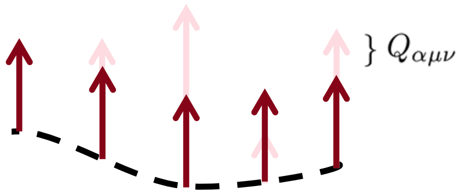
Curvature $(\mathcal{M}, g, \Gamma)$: $R^{\alpha}{}_{\mu\nu\rho} \coloneqq 2\partial_{[\mu}\Gamma^{\alpha}{}_{\nu]\beta} + 2\Gamma^{\alpha}{}_{[\mu|\lambda}\Gamma^{\lambda}{}_{\nu]\beta}$

✓ Curvature **modifies the direction** of a vector parallel transported along a closed path.



Non-metricity $(\mathcal{M}, g, \Gamma)$: $Q_{\alpha\mu\nu} \coloneqq \nabla_{\alpha} g_{\mu\nu}$

✓ Non-metricity quantifies **how the norm** of a vector **changes** in this transport.



Figs. adapted from J. B. Jimenez, L. Heisenberg, and T. S. Koivisto, The geometrical trinity of gravity.

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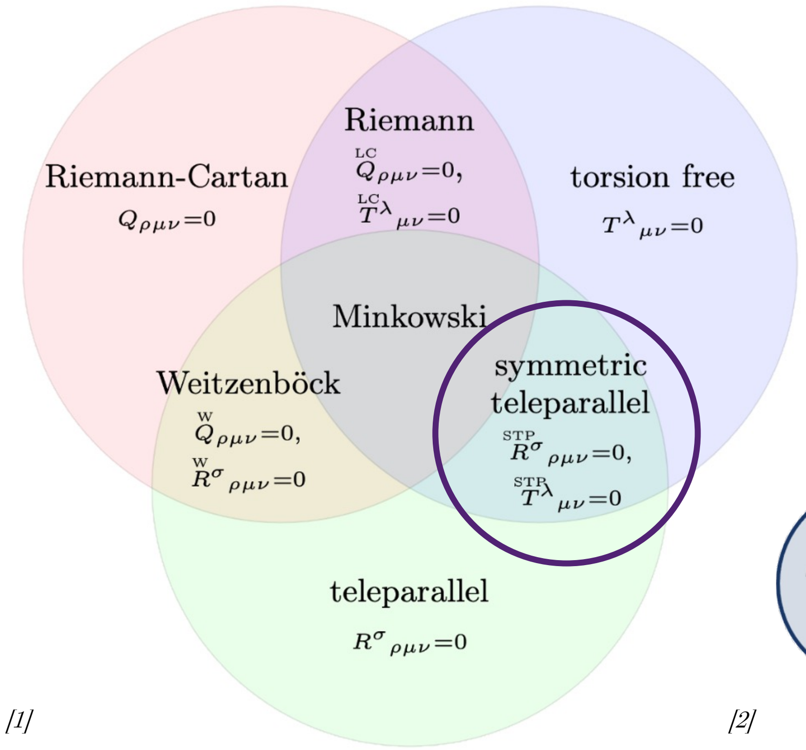
Possible metric-affine geometries



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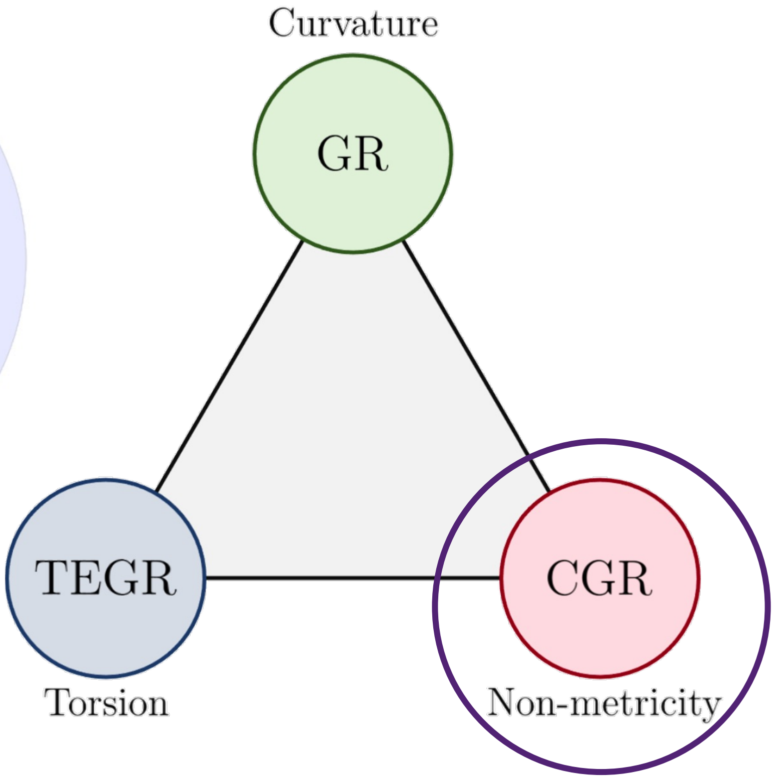

[2]

[1] Borrowed from L. Järv, M. Rünkla, M. Saal, and O. Vilson, "Nonmetricity formulation of general relativity and its scalar-tensor extension".
[2] Adapted from L. Heisenberg, "A systematic approach to generalisations of general relativity and their cosmological implications".

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A theory of gravity based on non-metricity

Most general second order quadratic form of the non-metricity: **non-metricity scalar**:

$$\mathbb{Q} := c_1 Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + c_2 Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + c_3 Q_\alpha Q^\alpha + c_4 \tilde{Q}_\alpha \tilde{Q}^\alpha + c_5 Q_\alpha \tilde{Q}^\alpha .$$

Most general action:

Lagrange multipliers

$$\mathcal{S}_{\mathbb{Q}}[g,\Gamma;\lambda,\rho] := \int_{\mathcal{M}} d^4x \left(\frac{1}{2\kappa}\sqrt{-g}\,\mathbb{Q} + \lambda_\alpha{}^{\beta\mu\nu} R^\alpha{}_{\beta\mu\nu} + \rho^\alpha{}_{\mu\nu} T^\alpha{}_{\mu\nu} \right) + \mathcal{S}_{\text{matter}}$$

The variation w.r.t. the Lagrange multipliers imposes:

1. Vanishing torsion
2. Vanishing curvature

$$\Gamma^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\lambda}\partial_\mu \partial_\nu \xi^\lambda$$

✓ The connection can be **aribitrarily chosen** and there exists a special gauge choice, the so-called ***coincident gauge*** $\xi^\alpha = x^\alpha$, in which the it becomes trivial.

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Symmetric Teleparallel Equivalent of GR

Most general second order quadratic form of the non-metricity: **non-metricity scalar**:

$$\mathbb{Q} := c_1 Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + c_2 Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + c_3 Q_\alpha Q^\alpha + c_4 \tilde{Q}_\alpha \tilde{Q}^\alpha + c_5 Q_\alpha \tilde{Q}^\alpha,$$

but to recover GR limit (STEGR) :

$$\{c_1, c_2, c_3, c_4, c_5\} = \left\{-\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{2}\right\} \Rightarrow \mathring{\mathbb{Q}} = -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_\alpha Q^\alpha - \frac{1}{2}Q_\alpha \tilde{Q}^\alpha.$$

How is that?

$$\Rightarrow \boxed{\mathcal{R} = -\mathring{\mathbb{Q}} - \mathcal{D}_\alpha(Q^\alpha - \tilde{Q}^\alpha)}$$

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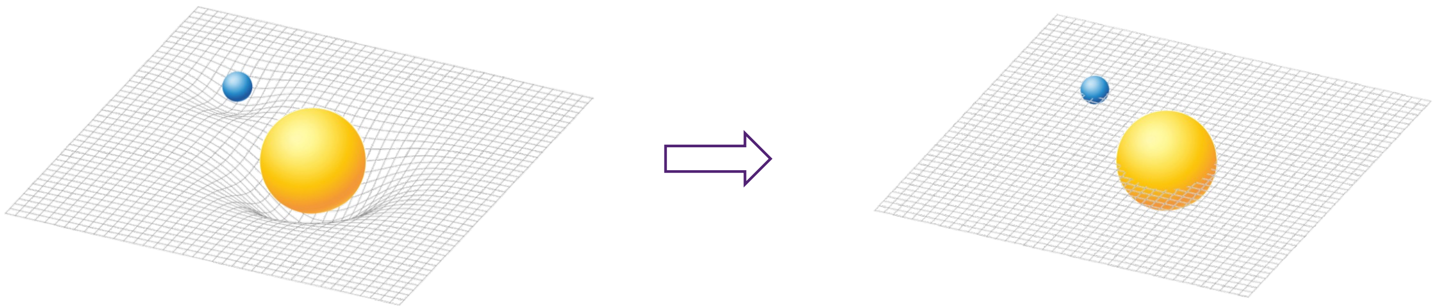
Symmetric Teleparallel Equivalent of GR

STEGR describes the same physics as GR:

$$\mathcal{S}_{\rm GR}[g] := \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}\,\mathcal{R} \quad \Longrightarrow \quad \mathcal{S}_{\rm STEGR}[g,\Gamma] = -\frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}\,\mathring{\mathbb{Q}}$$



$$\mathcal{R}^{\alpha}{}_{\beta\mu\nu} \stackrel{!}{=} 0 \qquad\qquad R^{\alpha}{}_{\beta\mu\nu} \stackrel{!}{=} 0$$

$$\boxed{\mathcal{R} = -\mathring{\mathbb{Q}} - \mathcal{D}_\alpha(Q^\alpha - \tilde{Q}^\alpha)}$$

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$f(\mathbb{Q})$ gravity

Leap to $f(\mathbb{Q})$ gravity:

$$\mathcal{S}_{f(\mathbb{Q})}[g,\Gamma;\lambda,\rho] := \int_{\mathcal{M}} d^4x \left(\frac{1}{2}\sqrt{-g}\, f(\mathbb{Q}) + \lambda_\alpha{}^{\beta\mu\nu} R^\alpha{}_{\beta\mu\nu} + \rho^\alpha{}_{\mu\nu} T^\alpha{}_{\mu\nu}\right) + \mathcal{S}_{\text{matter}}$$

Equations of Motion:

Variation w.r.t. the metric:

$$f'(\mathbb{Q})\mathcal{G}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\left[f(\mathbb{Q}) - f'(\mathbb{Q})\mathbb{Q}\right] + 2f''(\mathbb{Q}) P^\alpha{}_{\mu\nu}\left(\partial_\alpha \mathbb{Q}\right) = \kappa \mathcal{T}_{\mu\nu}$$

Variation w.r.t. the connection:

$$\mathcal{C}_\alpha := \nabla_\mu \nabla_\nu \left(\sqrt{-g}\, f'(\mathbb{Q}) P^{\mu\nu}{}_\alpha\right) = 0$$

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Spherically symmetric & stationary

Assume both the components of the metric and the affine connection can be expressed in the chart $(t, r, \theta, \phi) \in \mathbb{R} \times \mathbb{R}_{>0} \times [0, \pi] \times [0, 2\pi)$. Stationarity makes them ro be time-independent.

Metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} g_{tt}(r) & 0 & 0 & 0 \\ 0 & g_{rr}(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Contents

Spherically symmetric & stationary

Connection components:

Independent components	All connection components can be expressed in terms of an arbitrary constant $\Gamma^t_{\theta\theta}$, the two functions $\Gamma^r_{rr}(r), \Gamma^r_{\theta\theta}(r)$, with $\Gamma^r_{\theta\theta}(r) \neq 0$; and trigonometric functions.
Non-zero components	There are 10 non-zero components: the three independent components $\Gamma^t_{\theta\theta} = \text{const.}$, $\Gamma^r_{rr}(r)$, $\Gamma^r_{\theta\theta}(r)$; and $\Gamma^t_{rr} = -\frac{\Gamma^t_{\theta\theta}}{(\Gamma^r_{\theta\theta})^2}$, $\Gamma^\theta_{r\theta} = -\frac{1}{\Gamma^r_{\theta\theta}}$, $\Gamma^\phi_{r\phi} = -\frac{1}{\Gamma^r_{\theta\theta}}$, $\Gamma^t_{\phi\phi} = \sin^2\theta\,\Gamma^t_{\theta\theta}$, $\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta$, $\Gamma^\phi_{\theta\phi} = \cot\theta$, $\Gamma^r_{\phi\phi} = \sin^2\theta\,\Gamma^r_{\theta\theta}$.
Derivatives of independent components	Of the two independent functions, the r-derivative of $\Gamma^r_{\theta\theta}$ can be expressed as $\partial_r\Gamma^r_{\theta\theta} = -1 - \Gamma^r_{rr}\Gamma^r_{\theta\theta}$, while $\partial_r\Gamma^r_{rr}$ cannot be expressed in terms of other components.

Adapted from Table 2 and Table 3 in F. D'Ambrosio, S. D. B. Fell, L. Heisenberg, and S. Kuhn, "Black holes in $f(\mathbb{Q})$ gravity".

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Spherically symmetric & stationary

Connection components:

Independent components	All connection components can be expressed in terms of an arbitrary constant $\Gamma^t_{\theta\theta}$, the two functions $\Gamma^r_{rr}(r), \Gamma^r_{\theta\theta}(r)$, with $\Gamma^r_{\theta\theta}(r) \neq 0$; and trigonometric functions.
Non-zero components	There are 10 non-zero components: the three independent components $\Gamma^t_{\theta\theta}$ = const., $\Gamma^r_{rr}(r)$, $\Gamma^r_{\theta\theta}(r)$; and $\Gamma^t_{rr} =$ [illegible], $\Gamma^\theta_{r\theta} = -\frac{1}{\Gamma^r_{\theta\theta}}$, $\Gamma^\phi_{r\phi} = -\frac{1}{\Gamma^r_{\theta\theta}}$, $\Gamma^t_{\phi\phi} = \sin^2\theta\,\Gamma^t_{\theta\theta}$, $\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta$, $\Gamma^\phi_{\theta\phi} = \cot\theta$, $\Gamma^r_{\phi\phi} = \sin^2\theta\,\Gamma^r_{\theta\theta}$.
Derivatives of independent components	Of the two independent functions, the r-derivative of $\Gamma^r_{\theta\theta}$ can be expressed as $\partial_r\Gamma^r_{\theta\theta} = -1 - \Gamma^r_{rr}\Gamma^r_{\theta\theta}$, while $\partial_r\Gamma^r_{rr}$ cannot be expressed in terms of other components.

The background connection has one non-trivial component

Adapted from Table 2 and Table 3 in F. D'Ambrosio, S. D. B. Fell, L. Heisenberg, and S. Kuhn, "Black holes in $f(\mathbb{Q})$ gravity".

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Counting degrees of freedom

From the metric: $\{g_{tt}, g_{rr}\}$.

From the connection: $\Gamma^r_{rr}(r)$ or $\Gamma^r_{\theta\theta}(r)$.

Three d.o.f. instead of two.

Another posible reparametrization: to use the non-metricity scalar and solve for the connection

$$\mathbb{Q} = \frac{1}{r^2 g_{rr}^2 g_{tt} (\Gamma^r_{\theta\theta})^2} \Big\{ \Gamma^r_{\theta\theta} g_{rr} \big[(\Gamma^r_{\theta\theta})^2 g_{rr} + 2r\Gamma^r_{\theta\theta} + r^2 \big] (\partial_r g_{tt}) + g_{tt} \big[\Gamma^r_{\theta\theta} \big((\Gamma^r_{\theta\theta})^2 g_{rr} - r^2 \big) (\partial_r g_{rr}) + 2 g_{rr} \big((\Gamma^r_{\theta\theta})^2 - g_{rr} \Gamma^r_{rr} (\Gamma^r_{\theta\theta})^3 + r^2 + r\Gamma^r_{\theta\theta} (r\Gamma^r_{rr} + 2) \big) \big] \Big\} .$$

✓ We need to solve the theory for 3 d.o.f.: $\{g_{tt}, g_{rr}, \Gamma^r_{\theta\theta}\} \leftrightarrow \{g_{tt}, g_{rr}, \Gamma^r_{rr}\} \leftrightarrow \{g_{tt}, g_{rr}, \mathbb{Q}\}$.

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Further simplifications

New structure of the EoMs:

$$\begin{pmatrix} \mathcal{M}_{tt} & 0 & 0 & 0 \\ 0 & \mathcal{M}_{rr} & 0 & 0 \\ 0 & 0 & \cancel{\mathcal{M}_{\theta\theta}} & 0 \\ 0 & 0 & 0 & \cancel{\mathcal{M}_{\theta\theta}} \sin^2\theta \end{pmatrix} , \quad \begin{pmatrix} 0 \\ \mathcal{C}_r \\ 0 \\ 0 \end{pmatrix} .$$

Simplifying a bit:

Consider an **ideal fluid** and introduce a reparametrization of the metric components:

$$\mathcal{T}_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$$

$$ds^2 = -e^{\xi(r)} dt^2 + e^{\zeta(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) , \quad \zeta(r) \coloneqq \log\left(1 - \frac{2m(r)}{r}\right)^{-1} .$$

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Recovering GR from $f(\mathbb{Q}) = \mathbb{Q}$

Remember the STEGR limit of the action and covariant form of the EoMs in $f(\mathbb{Q})$:

$$\mathcal{S}_{f(\mathbb{Q})}[g,\Gamma;\lambda,\rho] := \int_{\mathcal{M}} d^4x \left(\frac{1}{2}\sqrt{-g}\,f(\mathbb{Q}) + \lambda_\alpha{}^{\beta\mu\nu} R^\alpha{}_{\beta\mu\nu} + \rho^\alpha{}_{\mu\nu} T^\alpha{}_{\mu\nu}\right) + \mathcal{S}_{\text{matter}}$$

→ STEGR action

$$f'(\mathbb{Q})\mathcal{G}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left[f(\mathbb{Q}) - f'(\mathbb{Q})\mathbb{Q}\right] + 2f''(\mathbb{Q})P^\alpha{}_{\mu\nu}(\partial_\alpha\mathbb{Q}) = \kappa\mathcal{T}_{\mu\nu}$$

→ Einstein eqs.

In our metric EoMs:

$$\frac{dp}{dr} = -\frac{(\rho+p)\left[m + (\kappa/2)pr^3\right]}{r(r-2m)}$$

$$\frac{dm}{dr} = \frac{\kappa}{2}r^2\rho$$

TOV and **mass** equations from GR with no extra assumptions.

- ✓ The GR limit is recovered in a general way for the choice $f(\mathbb{Q}) = \mathbb{Q}$.
- ✓ Eqs. to be solved for $\{m, p, \rho, \xi\}$, together with the conservation equation and the EoS.

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Recovering GR from the connection

Ansatz for the connection:

Fix the free connection component to be $\Gamma^r_{\theta\theta} = -r$. This automatically leads to:

$$\mathbb{Q}(r) = -\frac{2\kappa m(p+\rho)}{f'(\mathbb{Q})(r-2m)}\,, \qquad \{\Gamma^r_{\theta\theta}, \Gamma^r_{rr}\} = \{-r, 0\}\,.$$

Using the equation that allowed us to trade $\Gamma^r_{rr} \leftrightarrow \mathbb{Q}$, one obtains $\Gamma^r_{rr} = 0$.

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Recovering GR from the connection

Exterior solution:

$$p = \rho = 0 \Rightarrow \boxed{\mathbb{Q} = 0} \longrightarrow \partial_r \Gamma^r_{\theta\theta} = -1$$

Solving the remaining EoMs (metric):

$$\Rightarrow \boxed{\begin{aligned} g_{tt} &= c_2 + \frac{c_1 c_2}{r} + \frac{c_2}{6}\frac{f(0)}{f'(0)} r^2 \equiv c_2 + \frac{c_1 c_2}{r} + \frac{c_2 \Lambda_{\text{eff}}}{3} r^2 \\ g_{rr} &= \frac{c_2}{g_{tt}}; \end{aligned}}$$

Setting $c_2 \equiv 1$ (asymptotic flatness) and $c_1 \equiv -r_s$, being $r_s = 2G$ (with $f'(0) \neq 0$):

$\longrightarrow$ (de Sitter-) Schwarzschild.

- ✓ For exterior solutions, the use of the spherical connection automatically implies $\mathbb{Q} = 0$ and therefore **any attempt to use it to obtain beyond GR solutions is doomed to failure**.
- ✓ For interior solutions, this choice of connection is **incompatible** with the system of equations

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Let's go to beyond-GR cases

Beyond GR scenario $f(\mathbb{Q}) = \mathbb{Q} + \alpha\mathbb{Q}^2$ **:**

Now we do have a **dynamical connection**, and hence the EoM for the connection and the constraint equation.

System of equations to be solved

Metric EoMs:

$$\frac{dm}{dr} = \frac{1}{2(1+2\alpha\mathbb{Q})}\left\{\frac{\alpha}{2\Gamma^r_{\theta\theta}}\left\{r^2\mathbb{Q}^2\Gamma^r_{\theta\theta} + 4(\partial_r\mathbb{Q})\left[(\Gamma^r_{\theta\theta}+r)^2 - 2m(2\Gamma^r_{\theta\theta}+r)\right]\right\} + \kappa r^2\rho\right\}$$

$$\frac{dp}{dr} = \frac{(\rho+p)}{2(1+2\alpha\mathbb{Q})(r-2m)}\left\{\frac{\alpha r^3\mathbb{Q}^2 - 4m(1+2\alpha\mathbb{Q})}{2r} + \frac{2\alpha\left[(\Gamma^r_{\theta\theta})^2 + 2rm - r^2\right](\partial_r\mathbb{Q})}{\Gamma^r_{\theta\theta}} - \kappa r^2 p\right\}$$

Conservation:

$$\frac{d\xi}{dr} = -\frac{2}{p+\rho}\frac{dp}{dr}$$

EoS:

$$f(p,\rho) = 0$$

Connection EoM:

$$\partial_r^2\mathbb{Q} = -\frac{r^2\kappa(\partial_r\mathbb{Q})}{2(1+2\alpha\mathbb{Q})(r-2m)}\left\{p + \frac{(\Gamma^r_{\theta\theta})^2 - 2rm + r^2}{(\Gamma^r_{\theta\theta})^2 + 2rm - r^2}\rho\right\} + \frac{(\partial_r\mathbb{Q})}{1+2\alpha\mathbb{Q}}\left\{\frac{2\left[r - m + \Gamma^r_{\theta\theta}(-\partial_r\Gamma^r_{\theta\theta})\right]}{(\Gamma^r_{\theta\theta})^2 + 2rm - r^2}\right.$$
$$\left. - \alpha\frac{\left[\mathbb{Q}^2 r^3 - 8\mathbb{Q}\left(r - m + \Gamma^r_{\theta\theta}(-\partial_r\Gamma^r_{\theta\theta})\right) + 4(\partial_r\mathbb{Q})\left((\Gamma^r_{\theta\theta}+r)^2 - 2rm\right)\right]}{2\left[(\Gamma^r_{\theta\theta})^2 + 2rm - r^2\right]}\right\}$$

Constraint equation:

$$\partial_r\Gamma^r_{\theta\theta} = -\frac{\Gamma^r_{\theta\theta}\left\{4(r-m) + \Gamma^r_{\theta\theta}(4 - r^2\mathbb{Q}) + \alpha\mathbb{Q}\left[8(r - m + \Gamma^r_{\theta\theta}) - r^2\mathbb{Q}\left(3\Gamma^r_{\theta\theta} + r\right)\right]\right\}}{2(1+2\alpha\mathbb{Q})\left[(\Gamma^r_{\theta\theta})^2 + 2rm - r^2\right]}$$
$$- \frac{r^2\kappa\Gamma^r_{\theta\theta}}{2(r-2m)(1+2\alpha\mathbb{Q})}\left\{\rho + \frac{(\Gamma^r_{\theta\theta}+r)^2 - 2m(2\Gamma^r_{\theta\theta}+r)}{(\Gamma^r_{\theta\theta})^2 + 2rm - r^2}p\right\}.$$

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Looking for analytical results

Two options for a perturbative analysis:

1. Perturbation in α:
   - ✓ Very transparent. Helps us to understand **how the beyond GR effects appear** and can provide us with the physical intuition we lack.
   - ✓ **How much we must deviate** from GR to appreciate the modified gravity effects and even obtain a scale at which this occurs.
   - ✓ Not very good for initial conditions unless we accept to **restrict** ourselves to a certain order in α.

2. Perturbation in r:
   - ✓ How our equations behave in **certain ranges of r**.
   - ✓ Can be used as "fixed zones" in our numerical analysis and integrate only in the intermediate region where the expansion fails and obtain a **semi-analytical solution**.
   - ✓ **Assumes a concrete dependency** of the solutions on r.

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Perturbation in α (exterior)

The result:

The deviations from GR appear to second order for the metric variables (first in the connection):

$$\Rightarrow \quad \begin{aligned} g_{tt}(r) &= -\left(1 - \frac{2M_{\text{ren}}}{r}\right) + \alpha^2 \frac{\mu}{r} \log\left(\frac{r}{r^*}\right) + \mathcal{O}(\alpha^3) \\ g_{rr}(r) &= r\left[(r - 2M_{\text{ren}}) - \alpha^2 \mu \log\left(\frac{r}{r^*}\right)\right]^{-1} + \mathcal{O}(\alpha^3) \\ \Gamma^r_{\theta\theta}(r) &= -r + \alpha r\left(c_5 + c_6 r^3 + r^3 c_7 \log r\right) + \mathcal{O}(\alpha^2) \end{aligned}$$

$\longleftarrow \quad M(r) \approx M_{\text{ren}} + \frac{\alpha^2 \mu}{2} \log(r/r^*)$

The associated non-metricity scalar reads:

$$\mathbb{Q}(r) = -\frac{4\alpha M^2 (c_7 + 3c_6 + 3c_7 \log r)}{\alpha\left[c_2 - r\left(c_6 r^3 + c_5\right)\right] + M\left[2\alpha\left(c_6 r^3 + c_5\right) - 1\right] + (2M - r) r^3 \alpha c_7 \log(r)}$$

- ✓ Physically well-behaved metric and $\mathbb{Q}$.
- ✓ Connection diverges at infinity.
- ✓ Coupling metric-connection as a logarithm.
- ❖ **Mass divergent at infinity.**

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Perturbation in α (interior)

The expansions:

$$
\begin{aligned}
p(r) &= p^{(0)}(r) + p^{(1)}(r)\,\alpha + p^{(2)}(r)\,\alpha^2 + \mathcal{O}(\alpha^3),\\
\rho(r) &= \rho^{(0)}(r) + \rho^{(1)}(r)\,\alpha + \rho^{(2)}(r)\,\alpha^2 + \mathcal{O}(\alpha^3),\\
m(r) &= m^{(0)}(r) + m^{(1)}(r)\,\alpha + m^{(2)}(r)\,\alpha^2 + \mathcal{O}(\alpha^3),\\
\xi(r) &= \xi^{(0)}(r) + \xi^{(1)}(r)\,\alpha + \xi^{(2)}(r)\,\alpha^2 + \mathcal{O}(\alpha^3),\\
\Gamma^r_{\theta\theta}(r) &= {\Gamma^r_{\theta\theta}}^{(0)}(r) + {\Gamma^r_{\theta\theta}}^{(1)}(r)\,\alpha + {\Gamma^r_{\theta\theta}}^{(2)}(r)\,\alpha^2 + \mathcal{O}(\alpha^3).
\end{aligned}
$$

One big difference and one issue:

The choice $\{\Gamma^r_{\theta\theta}, \Gamma^r_{rr}\} = \{-r, 0\}$ now does not trivialize the connection EoM and leads to:

$$\mathbb{Q}(r) = -\frac{1}{4\alpha}\left[1 \pm \sqrt{1 - \frac{16\kappa\alpha m\,(p+\rho)}{r-2m}}\right].$$

We also need to opt for an Ansatz for ${\Gamma^r_{\theta\theta}}^{(0)}(r)$ that, if it does not imply $\mathbb{Q} = 0$ or $\mathbb{Q} = \text{const.}$ at $\mathcal{O}(\alpha^0)$, at least simplifies the expressions as much as possible. ✗

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Perturbation in r (exterior)

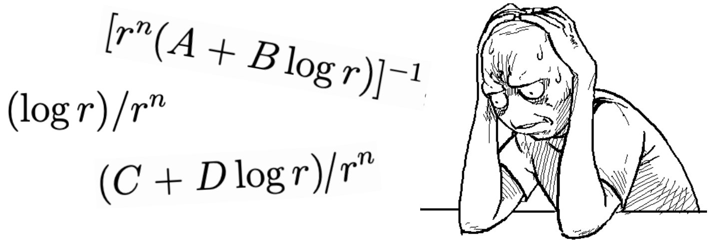
The expansions:

$$\left.\begin{aligned} m(r) &= m^{(0)} + m^{(1)}r^{-1} + m^{(2)}r^{-2} + \mathcal{O}(r^{-3}), \\ \xi(r) &= \xi^{(0)} + \xi^{(1)}r^{-1} + \xi^{(2)}r^{-2} + \mathcal{O}(r^{-3}), \\ \Gamma^r_{\theta\theta}(r) &= {\Gamma^r_{\theta\theta}}^{(0)} + {\Gamma^r_{\theta\theta}}^{(1)}r^{-1} + {\Gamma^r_{\theta\theta}}^{(2)}r^{-2} + \mathcal{O}(r^{-3}), \\ \mathbb{Q}(r) &= \mathbb{Q}^{(0)} + \mathbb{Q}^{(1)}r^{-1} + \mathbb{Q}^{(2)}r^{-2} + \mathcal{O}(r^{-3}); \end{aligned}\right\} \quad \text{with } \xi^{(0)} = 0 \text{ and } \mathbb{Q}^{(0)} = 0\,.$$

The result:

The only solution compatible with these asymptotic conditions and the functional dependency in r above is the one for which $\mathbb{Q}(r) = 0$.

✓ Among the different possibilities for an exterior solution, there is **none** that behaves as a power series in $1/r$ and we will **only obtain GR solutions** with such an expansion.



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Perturbation in r (interior)

	Beyond GR $f(\mathbb{Q}) = \mathbb{Q} + \alpha\mathbb{Q}^2$
To determine	$m^{(0)}$, $p^{(0)}$, $\xi^{(0)}$, ${\Gamma^r_{\theta\theta}}^{(0)}$, $\mathbb{Q}^{(0)}$, $\mathbb{Q}^{(1)}$, ($\rho^{(0)}$ using EoS)
Common initial conditions	$m^{(0)} = 0$, $p^{(0)} = \text{const.}$, $\xi^{(0)} = \text{const.}$, $\xi(r \to \infty) = 0$
Common expansions at $r \approx 0$ (truncated)	$m \approx \frac{(\mathbb{Q}^{(0)})^2\alpha + 2\rho^{(0)}\kappa}{6(4\alpha\mathbb{Q}^{(0)} + 2)} r^3$ $p \approx p^{(0)} + \frac{\left(\rho^{(0)} + p^{(0)}\right)\left[(\mathbb{Q}^{(0)})^2\alpha - \kappa\left(\rho^{(0)} + 3p^{(0)}\right)\right]}{4(6\alpha\mathbb{Q}^{(0)} + 3)} r^2$ $\xi \approx \xi^{(0)} - \frac{(\mathbb{Q}^{(0)})^2\alpha - \kappa\left(\rho^{(0)} + 3p^{(0)}\right)}{4(6\alpha\mathbb{Q}^{(0)} + 3)} r^2$

	Set 1	Set 2	Set 3
Initial conditions	${\Gamma^r_{\theta\theta}}^{(0)} = \text{const.}$ $\mathbb{Q}^{(0)} = \text{const.}$ $\mathbb{Q}^{(1)} = 0$	${\Gamma^r_{\theta\theta}}^{(0)} = 0$ $\mathbb{Q}^{(0)} = \text{const.}$ $\mathbb{Q}^{(1)} = 0$	${\Gamma^r_{\theta\theta}}^{(0)} = 0$ $\mathbb{Q}^{(0)} = 0$ $\mathbb{Q}^{(1)} = 0$
Expansions at $r \approx 0$ (truncated)	$\mathbb{Q} \approx \mathbb{Q}^{(0)}$ $\Gamma^r_{\theta\theta} \approx {\Gamma^r_{\theta\theta}}^{(0)} - 2r$	$\mathbb{Q} \approx \mathbb{Q}^{(0)}$ $\Gamma^r_{\theta\theta} \approx {\Gamma^r_{\theta\theta}}^{(2)} r^2 + {\Gamma^r_{\theta\theta}}^{(3)} r^3$	$\mathbb{Q} \approx 0$ $\Gamma^r_{\theta\theta} \approx -r + {\Gamma^r_{\theta\theta}}^{(2)} r^2$

Contents

Perturbation in r (interior)

	Beyond GR $f(\mathbb{Q}) = \mathbb{Q} + \alpha\mathbb{Q}^2$
To determine	$m^{(0)}, p^{(0)}, \xi^{(0)}, \Gamma^{r\,(0)}_{\theta\theta}, \mathbb{Q}^{(0)}, \mathbb{Q}^{(1)}, (\rho^{(0)}$ using EoS)
Common initial conditions	$m^{(0)} = 0,\ p^{(0)} = \text{const.},\ \xi^{(0)} = \text{const.},\ \xi(r \to \infty) = 0$
Common expansions at $r \approx 0$ (truncated)	$m \approx \frac{(\mathbb{Q}^{(0)})^2\alpha + 2\rho^{(0)}\kappa}{6(4\alpha\mathbb{Q}^{(0)} + 2)} r^3$ $p \approx p^{(0)} + \frac{(\rho^{(0)} + p^{(0)})\left[(\mathbb{Q}^{(0)})^2\alpha - \kappa\left(\rho^{(0)} + 3p^{(0)}\right)\right]}{4(6\alpha\mathbb{Q}^{(0)} + 3)} r^2$ $\xi \approx \xi^{(0)} - \frac{(\mathbb{Q}^{(0)})^2\alpha - \kappa\left(\rho^{(0)} + 3p^{(0)}\right)}{4(6\alpha\mathbb{Q}^{(0)} + 3)} r^2$

	Set 1	Set 2	Set 3
Initial conditions	$\Gamma^{r\,(0)}_{\theta\theta} = \text{const.}$ $\mathbb{Q}^{(0)} = \text{const.}$ $\mathbb{Q}^{(1)} = 0$	$\Gamma^{r\,(0)}_{\theta\theta} = 0$ $\mathbb{Q}^{(0)} = \text{const.}$ $\mathbb{Q}^{(1)} = 0$	$\Gamma^{r\,(0)}_{\theta\theta} = 0$ $\mathbb{Q}^{(0)} = 0$ $\mathbb{Q}^{(1)} = 0$
Expansions at $r \approx 0$ (truncated)	$\mathbb{Q} \approx \mathbb{Q}^{(0)}$ $\Gamma^r_{\theta\theta} \approx \Gamma^{r\,(0)}_{\theta\theta} - 2r$	$\mathbb{Q} \approx \mathbb{Q}^{(0)}$ $\Gamma^r_{\theta\theta} \approx \Gamma^{r\,(2)}_{\theta\theta} r^2 + \Gamma^{r\,(3)}_{\theta\theta} r^3$	$\mathbb{Q} \approx 0$ $\Gamma^r_{\theta\theta} \approx -r + \Gamma^{r\,(2)}_{\theta\theta} r^2$

✓ Any attempt to obtain a solution that deviates from GR using as Ansatz a solution that admits a **Taylor expansion** around $r = 0$, is doomed to failure and will inevitably **lead to GR solutions**.

Contents

What follows?

Numerics! To be careful:

1. Zero/Constant non-metricity scalar is a solution of the connection EoM. Be careful with vanishing derivatives.
2. Outside we know that by taking $\Gamma^r_{\theta\theta} = -r$, we should automatically obtain $\mathbb{Q} = 0$ and vice-versa. For the inside we do not have such a reference.
3. No analogy/solid criterion to stablish our initial conditions. And numerical shooting is more complicated because some of them are fixed at the origin and some at infinity.
4. From 3 to 5 coupled differential equations, one of them of the second order.

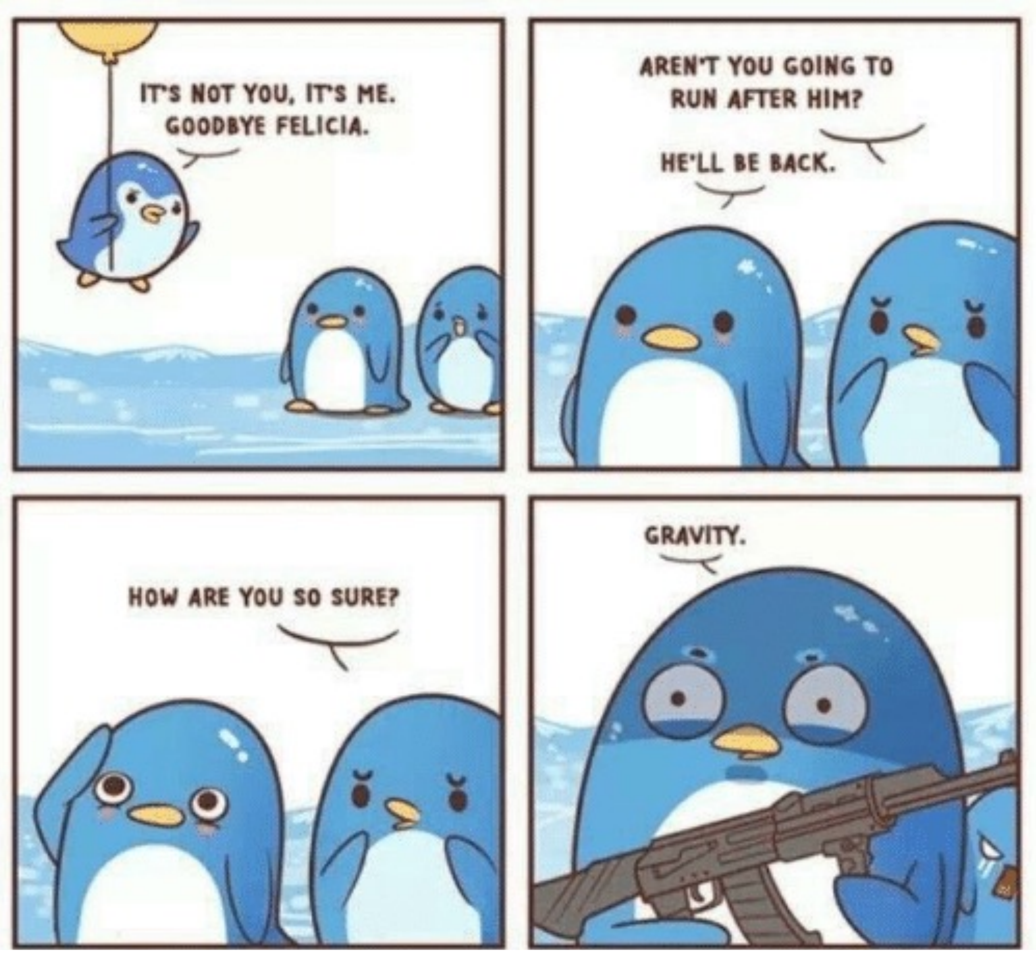
Alternative approaches:

- Work on the more restricted case of a **Weyl connection**, i.e., $Q_{\alpha\mu\nu} := \nabla_\alpha g_{\mu\nu}$, with $\nabla_\alpha g_{\mu\nu} = \Sigma_\alpha g_{\mu\nu}$; instead of the most general one above, to better understand the behavior of the non-metricity scalar, the role of the non-metricity and to study how to impose spherical symmetry and stationarity conditions in a different and covariant way.
- Study the presence or absence of hair.
- Study how to compute the ADM mass in these theories.

Contents

To sum up

- ✓ In STEGR the connection can be gauged away and plays no role in the physics.
- ✓ In $f(\mathbb{Q})$, however, it becomes dynamical and has its own (second order EoM), providing one extra d.o.f..
- ✓ The choice of the spherical connection outside a NS for a generic $f(\mathbb{Q})$, inevitably leads to GR results. Inside, this does not work and one has to set $f(\mathbb{Q}) = \mathbb{Q}$.
- ✓ In GR, a solution to the EoMs admits a Taylor expansion in a neighborhood of $r = 0$.
- ✓ The behavior of beyond GR solutions, if they happen to exist, at infinity and at the center of the NS, does not admit a series expansion in powers of $1/r$ or r, respectively.



THANK YOU FOR YOUR ATTENDANCE!

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