EFT of Black Hole Perturbations with Time-like Scalar Profile: Recent Progress

Vicharit Yingcharoenrat

Kavli IPMU, The University of Tokyo

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In Collaboration with
S. Mukohyama, K. Takahashi, K. Tomikawa, K. Aoki, M. A. Gorji, N. Oshita, Z. Wang, H. Kobayashi, E. Seraille and C. G. A. Barura

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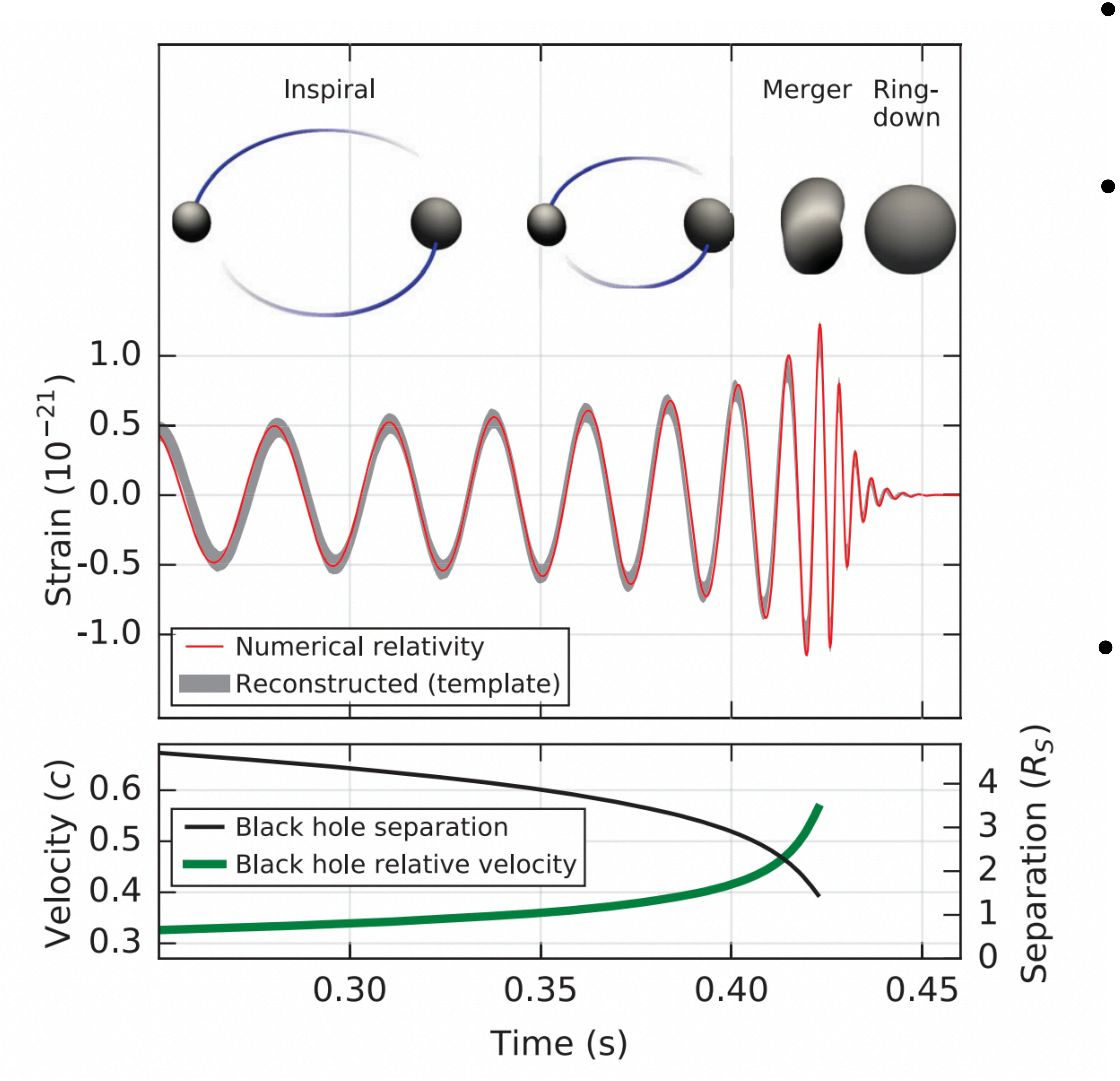
Outline

- Introduction
- Formulation of the EFT
- Applications and Recent Progress
- Conclusions/Future Directions

Introduction



LIGO/Virgo 2016

- We have detected many mergers e.g. GW150914, GW170817, GW190424 etc.
- Each phase can be described by:

  Inspiral: Post-Newtonian method

  Merger: Numerical relativity

  Ring-down: Black hole perturbation theory

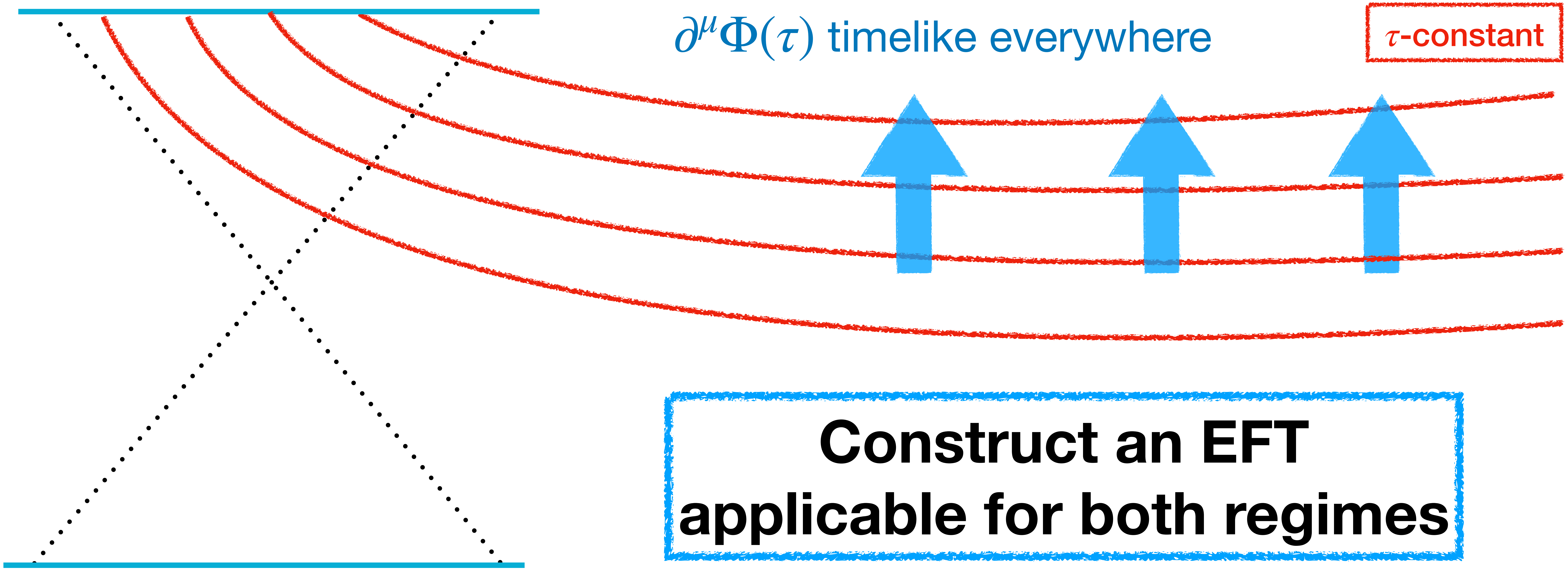
- Formulate the Effective Field Theory

  ⇒Test of GR and modified gravity (model-indep.)

BH Phenomenology from EFT of BH

e.g. QNM, Love number

Construction of the EFT



Construct an EFT applicable for both regimes

Black Hole regime

Cosmological regime

(EFT of Inflation/DE)

Cheung et al. 08, Gubitosi et al. 12, +++

Construction of the EFT

- A model-indep. way to study a perturbation around fixed background
- Examples: EFT of Inflation/DE on FRW metric Cheung et al. 08, Gubitosi et al. 12, +++

  ⇔ Scalar-tensor theories e.g. Horndeski and beyond

  Horndeski 74, Deffayet et al. 11, Zumalacárregui and García-Bellido 14, Gleyzes et al. 14, Langlois and Noui 15, Langlois 17, +++

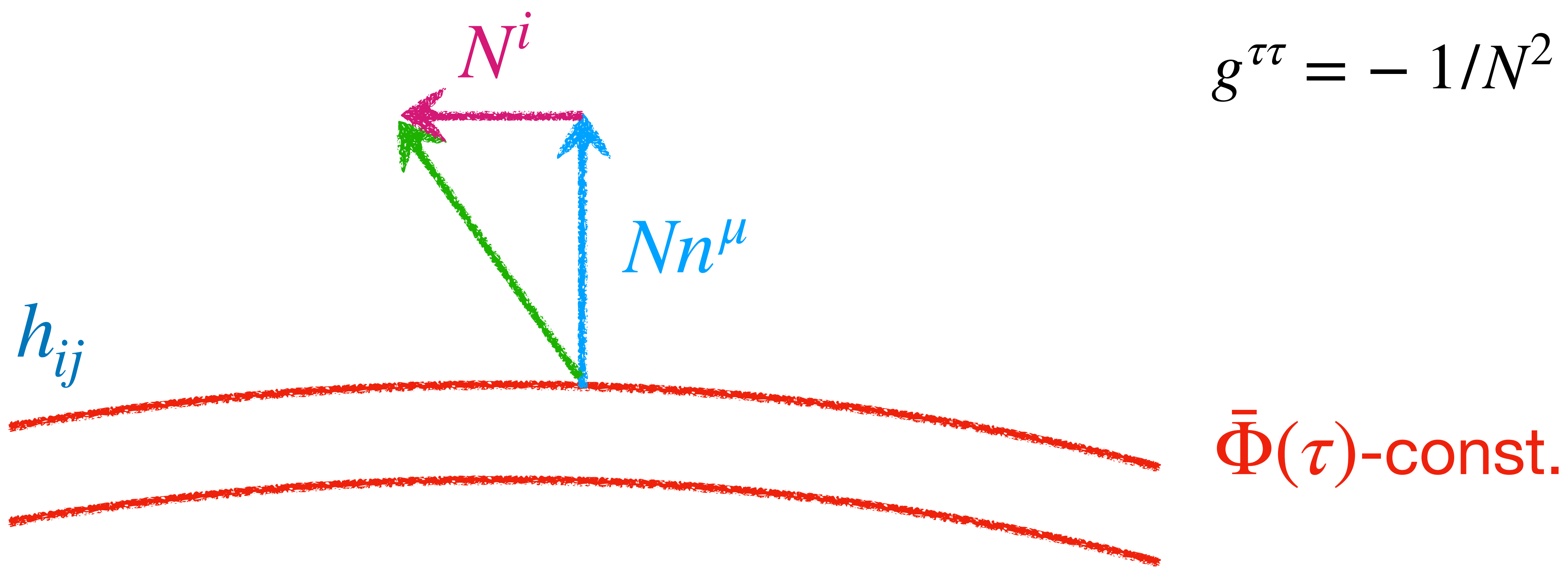
- $\bar{\Phi}(\tau)$ breaks τ-diffeo. spontaneously

$$ds^2 = -N^2 d\tau^2 + h_{ij}(dx^i + N^i d\tau)(dx^j + N^j d\tau)$$

- EFT action in unitary gauge: $\delta\Phi = 0$

$$g^{\tau\tau} = -1/N^2$$

$$S = \int d^4x \sqrt{-g}\; F(R_{\mu\nu\alpha\beta}, g^{\tau\tau}, K_{\mu\nu}, \nabla_\nu, \tau)$$



This action is invariant under 3d diffeo and valid for generic background geometries

Construction of the EFT

- Expand the EFT action around an inhomogeneous background: $y = \{\tau, x_i\}$

$$S = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}f(y)R - \Lambda(y) - c(y)g^{\tau\tau} - \beta(y)K + \frac{1}{2}m_2^4(y)(\delta g^{\tau\tau})^2 + \frac{1}{2}M_1^3(y)\delta g^{\tau\tau}\delta K + \cdots\right]$$

⇔ Scalar-tensor theories e.g. Horndeski and beyond

$$\delta g^{\tau\tau} \equiv g^{\tau\tau} - \bar{g}^{\tau\tau}(\tau, x_i) \qquad \delta K^\mu_\nu = K^\mu_\nu - \bar{K}^\mu_\nu(\tau, x_i)$$

Horndeski 74, Deffayet et al. 11, Zumalacárregui & García-Bellido 14, Langlois 17, +++

The backgrounds of the building blocks break 3d diffeo. explicitly

- The consistency relations ensures the 3d diffeo. invariance of the EFT

Chain rule in x^i-directions

$$\partial_i\Lambda + \bar{g}^{\tau\tau}\partial_i c - \frac{1}{2}M_\star^2{}^{(3)}\bar{R}\partial_i f + \frac{1}{6}\bar{K}(M_\star^2\bar{K}\partial_i f + 6\partial_i\beta) - \frac{1}{2}M_\star^2\bar{K}^\mu_\nu\bar{K}^\nu_\mu\partial_i f \simeq 0$$

$$\partial_i c + m_2^4\partial_i\bar{g}^{\tau\tau} + \frac{1}{2}M_1^3\partial_i\bar{K} \simeq 0$$

⋮

Dynamics in odd-parity sector

- Background metric: $ds^2 = -d\tau^2 + [1 - A(r)]d\rho^2 + r^2 d\Omega^2$ $\Leftrightarrow$ $ds^2 = -A(r)dt^2 + \dfrac{dr^2}{B(r)} + r^2 d\Omega^2$

- The odd EFT action up to second order:

$$S_{\rm odd} = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}R - \Lambda(y) - c(y)g^{\tau\tau} - \tilde{\beta}(y)K - \alpha(\tau)\bar{K}^\mu_\nu K^\nu_\mu + \frac{1}{2}M_3^2(y)\delta K^\mu_\nu \delta K^\nu_\mu\right]$$

Modifies speed of GW

- Quadratic Lagrangian with $A(r) \neq B(r)$:

$$\mathscr{L}_2 = a_1(\partial_t\chi)^2 - a_2(\partial_r\chi)^2 - a_4\chi^2$$

Function of $M_\star^2, M_3^2, \alpha, A(r)$ and $B(r)$

$$\Psi = (a_1 a_2)^{1/4}\chi$$

$$r_* \equiv \int\sqrt{\frac{a_1}{a_2}}dr$$

$$\frac{\partial^2\Psi}{\partial r_*^2} - \frac{\partial^2\Psi}{\partial t^2} - V_{\rm eff}(r)\Psi = 0$$

Generalized Regge-Wheeler

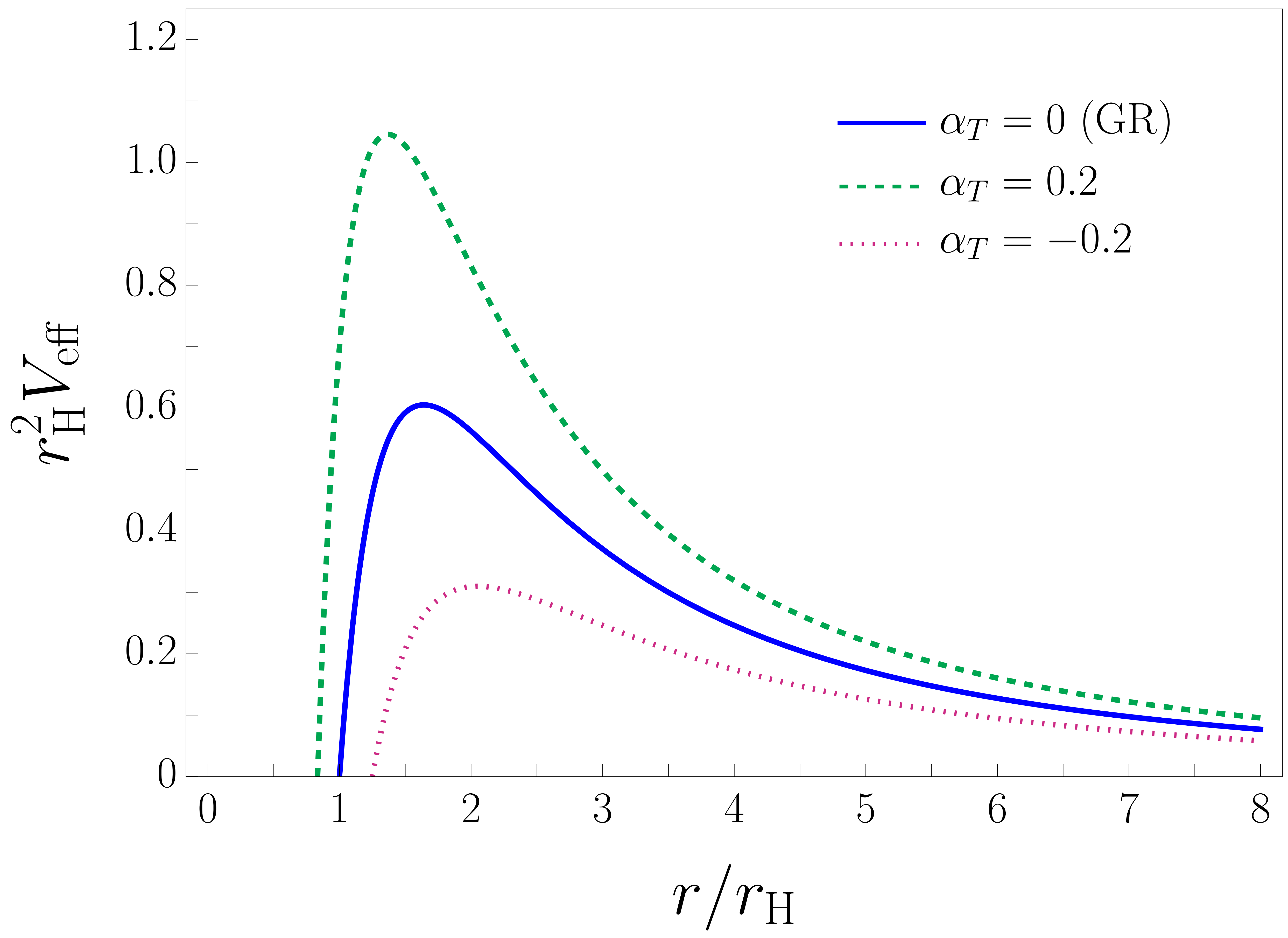
$$V_{\rm eff}(r) \equiv \frac{a_4}{a_1} + \frac{1}{2\sqrt{a_1a_2}}\frac{d^2\sqrt{a_1a_2}}{dr_*^2} - \frac{1}{4a_1a_2}\left(\frac{d\sqrt{a_1a_2}}{dr_*}\right)^2$$

We want to solve this diff. Eq.!!

Stealth Schwarzschild solution

We choose: $A(r) = B(r) = 1 - \frac{r_H}{r}$ $\qquad \alpha_T(r) \equiv c_T^2 - 1 = -\frac{M_3(r)^2}{M_\star^2 + M_3(r)^2}$

$\alpha_T = const.$



$$V_{\rm eff}(r) = (1+\alpha_T) f(r) \left[\frac{\ell(\ell+1)}{r^2} - \frac{3r_g}{r^3}\right]$$

$f(r) \equiv 1 - r_g/r$

Graviton horizon: $r_g \equiv r_H/(1+\alpha_T)$

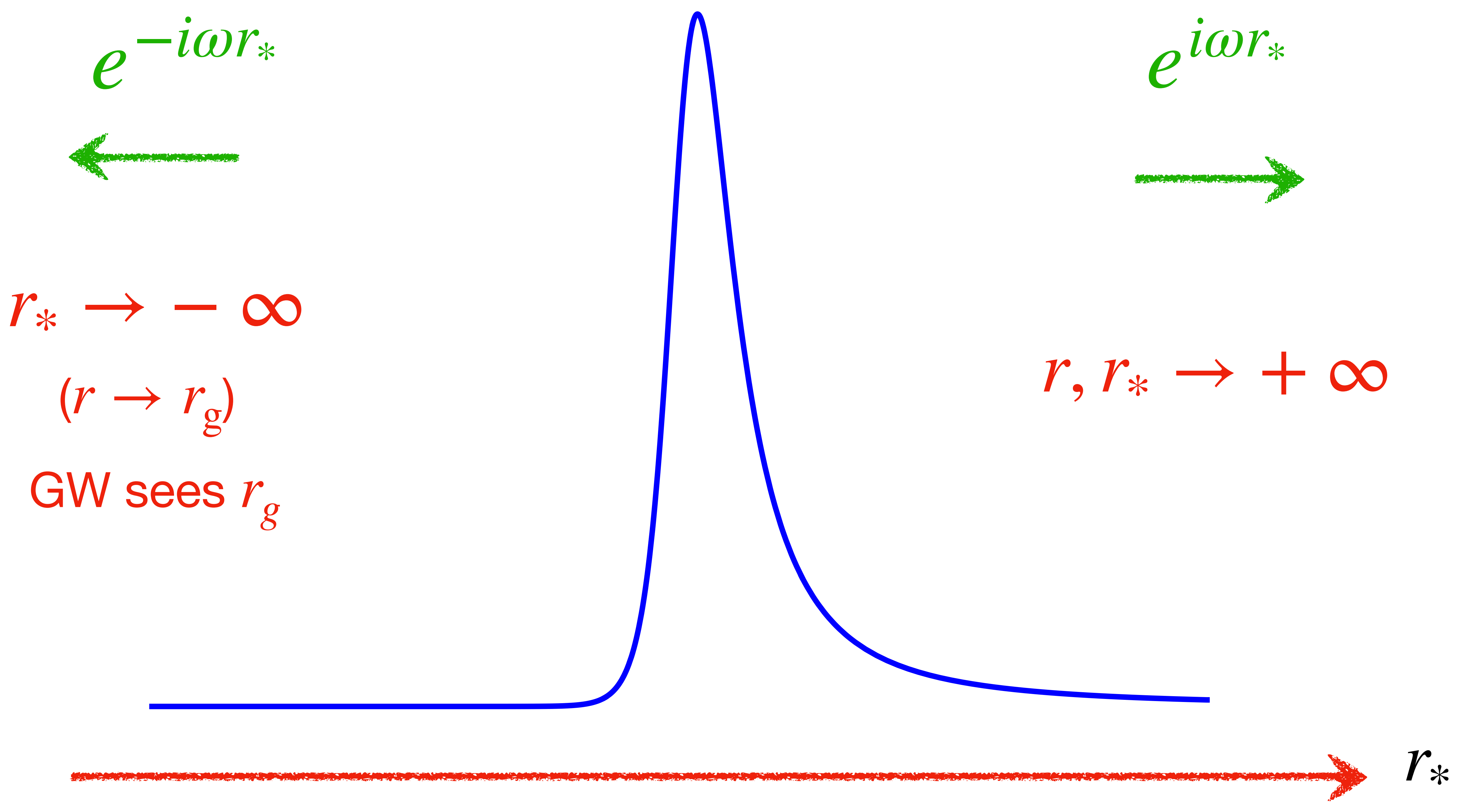
- $\alpha_T > 0 \Rightarrow r_g < r_H$
- $\alpha_T < 0 \Rightarrow r_g > r_H$

The presence of α_T changes the height of $V_{\rm eff}$

$$r_*(r) = (1+\alpha_T)^{-1/2}\left[r + r_g \log\left|\frac{r}{r_g} - 1\right|\right]$$

QNM calculation

$$\Psi(t, r_*) = Q(r_*)e^{-i\omega t}$$



$$\frac{d^2}{dr_*^2}Q(r_*) + (\omega^2 - V_{\rm eff})Q(r_*) = 0$$

1d time-indep. Schrödinger eq.

A specific frequency that satisfies the two BCs is the quasinormal frequency

Review: Hatsuda and Kimura 21

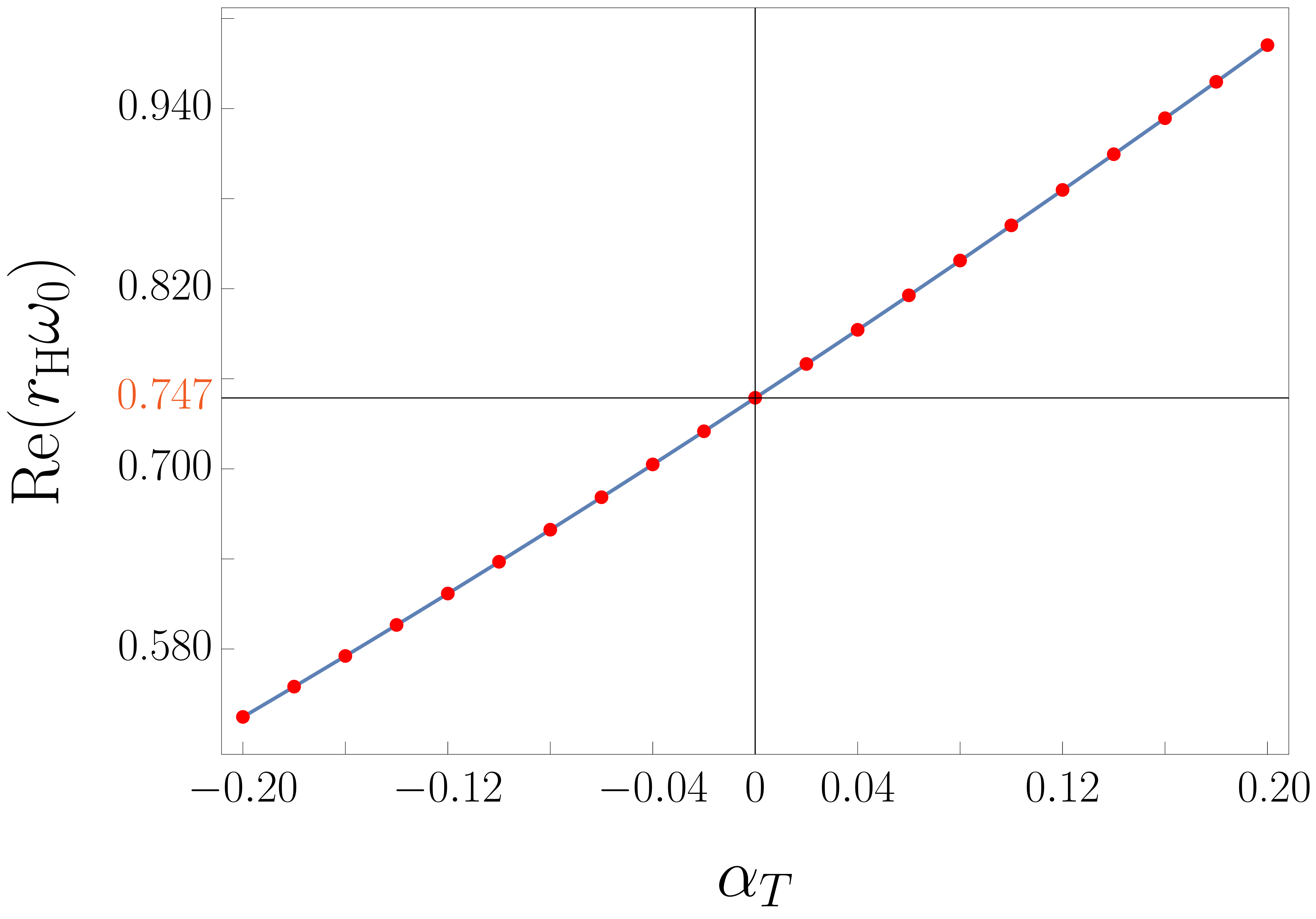
- ω is complex number $\Rightarrow$ the Im part describes the damping of the GW signal
- Other methods: WKB method, Leaver's method, parametrized QNM method

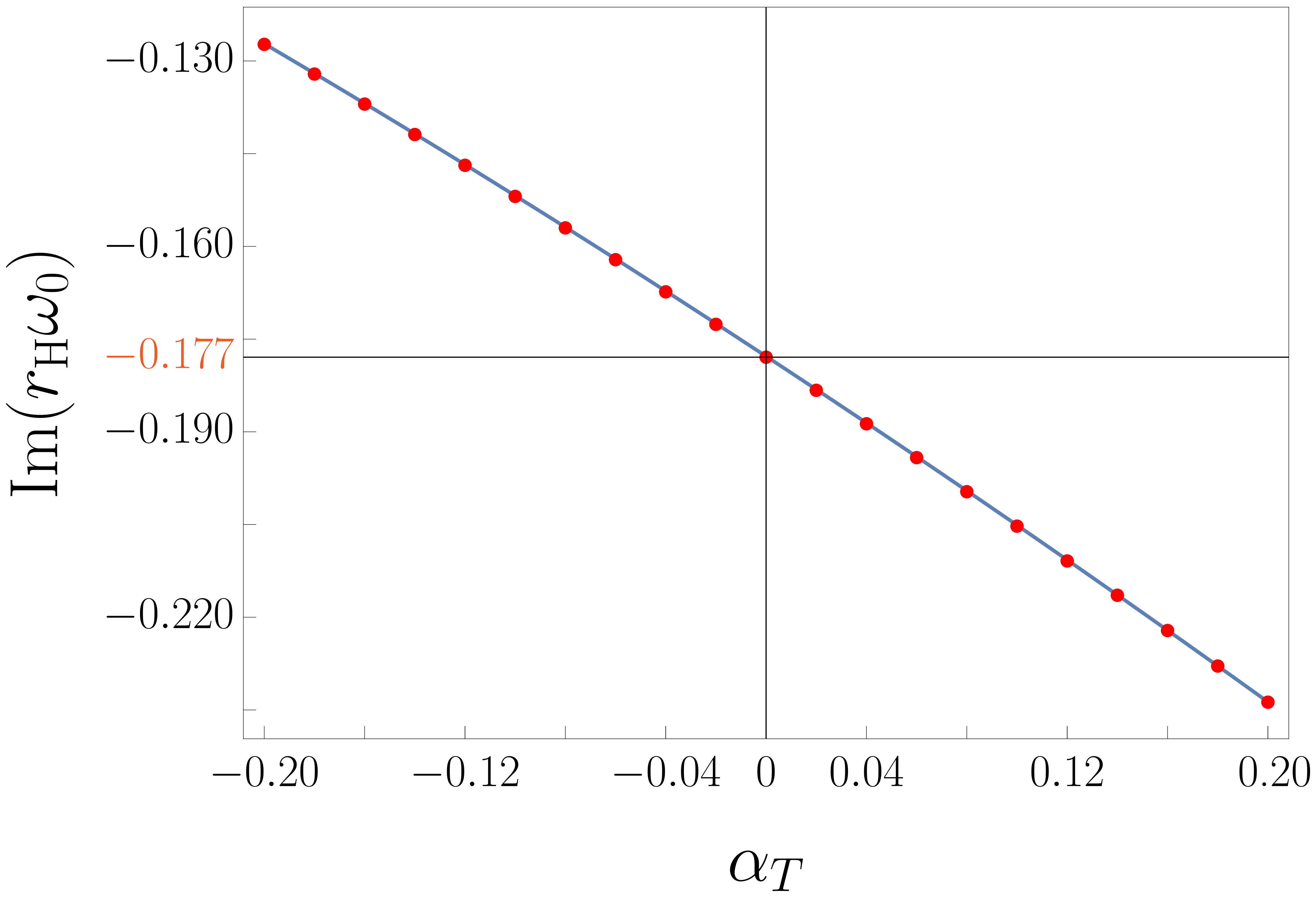
Chandrasekhar and Detweiler 75, Iyer and Will 87, Konoplya 03
Leaver 85, Schutz and Will 85, Nollert 93, McManus et al. 19, Kimura 20, +++

QNM in Stealth Schwarzschild case

- The fundamental QNM frequencies can be obtained by a simple rescaling from $\omega_{\rm GR}$

$$A(r) = B(r) = 1 - \frac{r_{\rm H}}{r} \qquad r_{\rm H}\omega = r_{\rm H}\omega_{\rm GR}(1+\alpha_T)^{3/2}$$





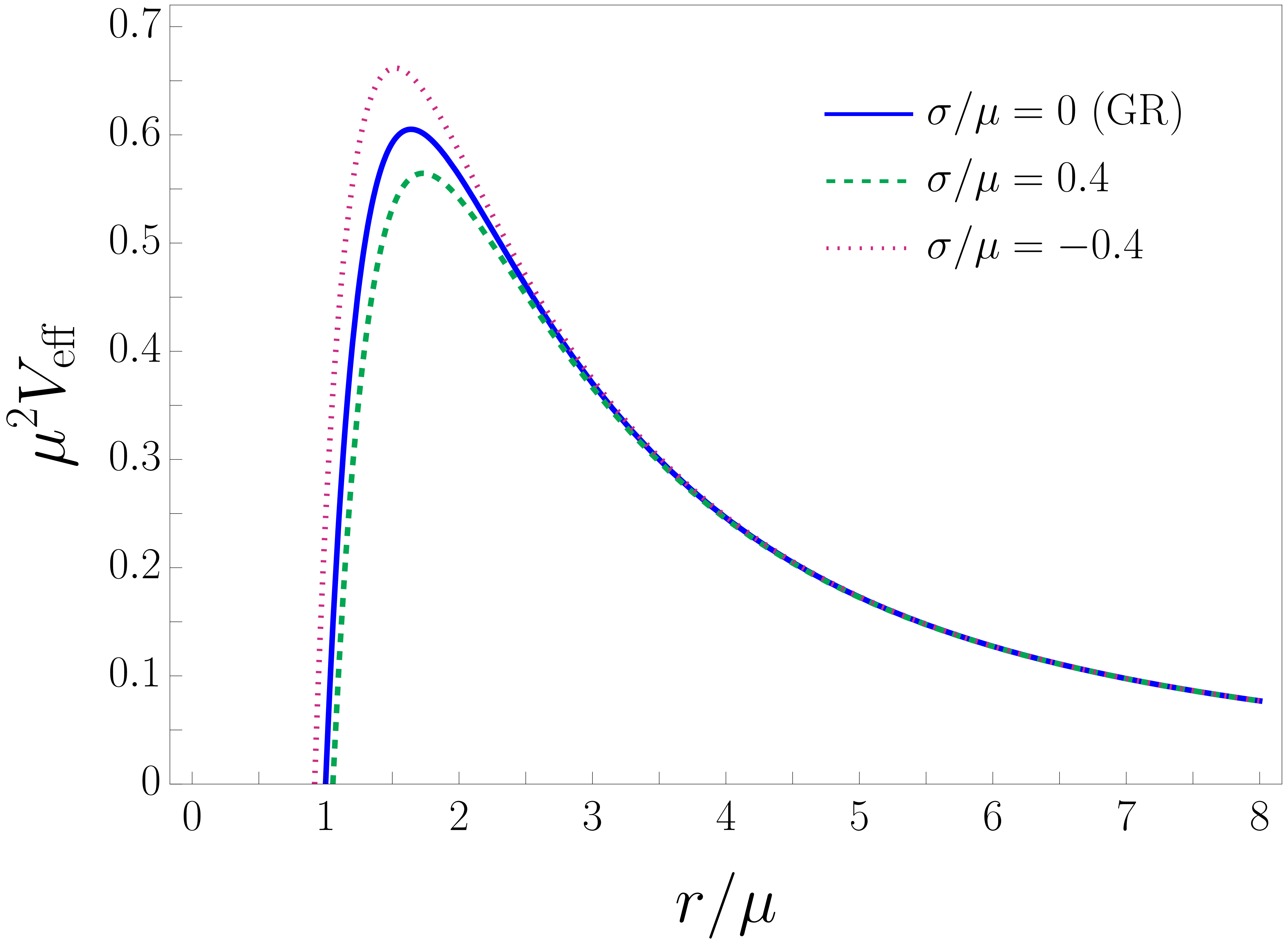
Red data points are QNFs from direct integration method

The Hayward potential

- We choose: $A(r) = B(r) = 1 - \dfrac{\mu r^2}{r^3 + \sigma^3}$ dS core as $r \to 0$ & Schwarzschild limit as $r \to \infty$

$$\alpha_T(r) = -\frac{\sigma^3(2r^3 + \sigma^3)}{(r^3 + \sigma^3)^2}$$

$\alpha_T(r) \sim 1/r^3$ as $r \to \infty$ (LIGO bound)



$$\mu^2 V_{\rm eff}(\tilde r) = \left[1 - \frac{\tilde r^3 + \tilde\sigma^3}{\tilde r^4}\right] \times$$
$$\times \left\{ \frac{\ell(\ell+1)\tilde r^4}{(\tilde r^3 + \tilde\sigma^3)^2} - \frac{3\left[4\tilde r^9 + 2\tilde\sigma^3 \tilde r^6(8\tilde r - 1) + \tilde\sigma^6 r^3(\tilde r - 7) - \tilde\sigma^9\right]}{4(\tilde r^3 + \tilde\sigma^3)^4} \right\}$$

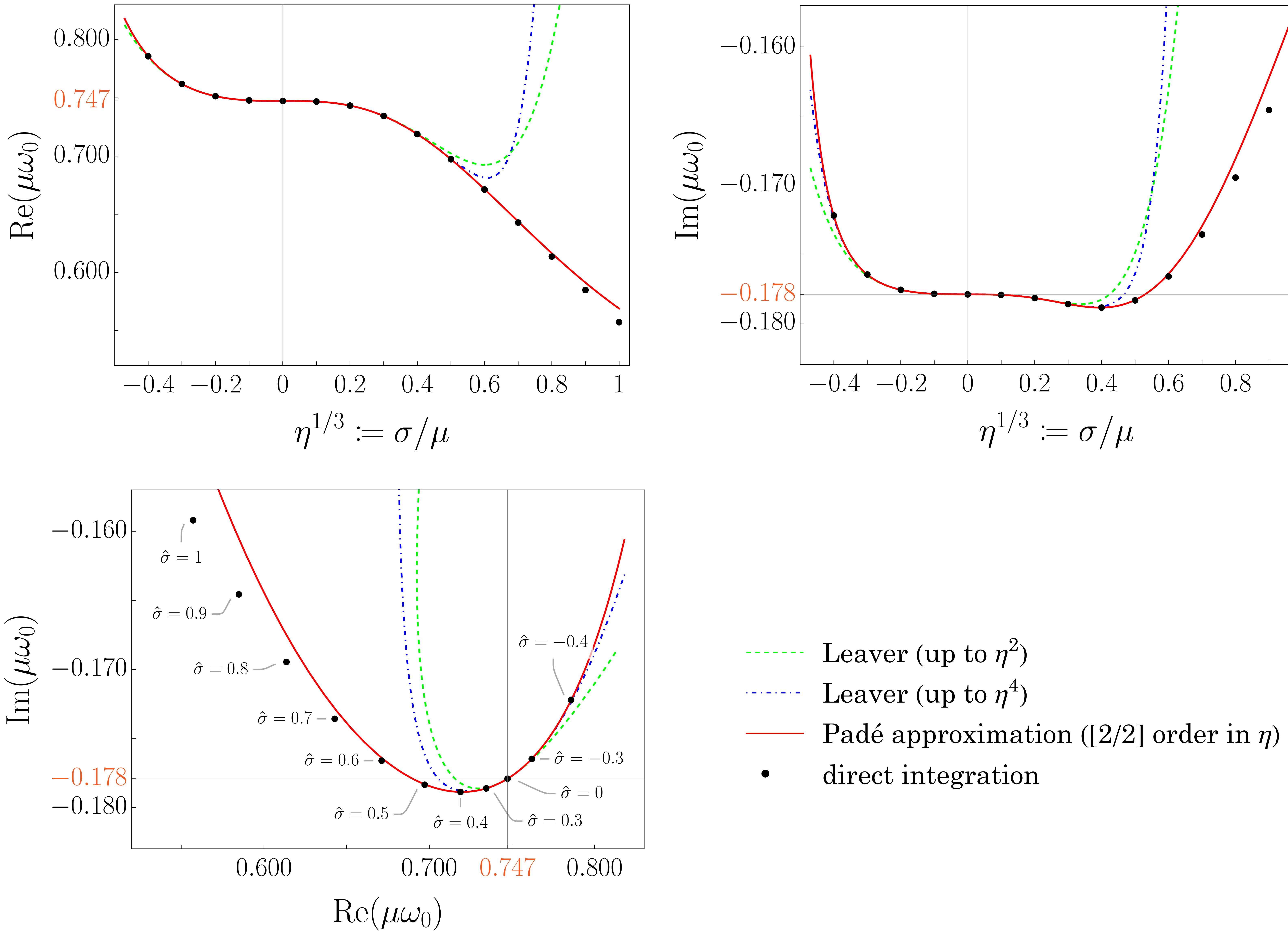
$\tilde r \equiv r/\mu$ $\tilde\sigma \equiv \sigma/\mu$

The only free parameter is σ/μ

- The graviton horizon satisfies

$$r_g^4 - \mu(r_g^3 + \sigma^3) = 0$$

QNM in Hayward case



- The fundamental QNM frequencies are fully numerics, using direct integration — black dot

- Other methods e.g. Leaver method + Padé approx. are a useful cross-check.

- Higher overtones give a possible probe of geometry around the horizon (Konoplya 23)

Dynamics of even sector

- There are two propagating DoFs (1 scalar + 1 tensor) $\delta g_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$

$$\delta g_{\tau\tau} = A(r)\, H_0(\tau, r) \qquad \delta g_{\tau r} = \sqrt{1-A}\, H_1(\tau, r) \qquad \delta\Phi = \Phi - \bar{\Phi}(\tau)$$

$$\delta g_{rr} = H_2(\tau, r) \qquad \delta g_{\tau a} = \delta g_{ra} = 0 \qquad \delta g_{ab} = r^2 K(\tau, r)\gamma_{ab}$$

Gauge transformation

The background $\simeq$ stealth Schwarchild solution (De Felice et al. 23)

- Scordatura term resolves strong coupling problem when $c_s^2 \to 0$ Motohashi and Mukohyama 19, +++

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\left[\frac{M_{\rm Pl}^2}{2}R + F_0(X) + \frac{\alpha_{\rm L}(\Box\Phi)^2}{\Lambda^4}\right] \Rightarrow$$

Dispersion relation of $\delta\Phi$: $\omega^2 \simeq \dfrac{\alpha_{\rm L} k^4}{\Lambda^2}$

- Preliminary result: dynamics of $\delta\Phi$ at linear order decouples from others

Mukohyama, Takahashi, Tomikawa, VY, in prep.

Rotating background

Mukohyama, Oshita, Takahashi, Wang, VY, in prep.

- General EFT can be applied to any background metric
- Our task is to realize the coordinate system where $\bar{\Phi}$ is spatially uniform
- Stationary and axisymmetric background:

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \bar{g}_{tt}(r,\theta)\mathrm{d}t^2 + \bar{g}_{rr}(r,\theta)\mathrm{d}r^2 + \bar{g}_{\theta\theta}(r,\theta)\mathrm{d}\theta^2 + \bar{g}_{\phi\phi}(r,\theta)\mathrm{d}\phi^2 + 2\bar{g}_{t\phi}(r,\theta)\mathrm{d}t\mathrm{d}\phi\,.$$

- For simplicity, we consider $\bar{X} = \bar{g}^{\mu\nu}\partial_\mu\bar{\Phi}\partial_\nu\bar{\Phi} = \mathrm{const}$ and $\bar{\Phi}(\tau) \propto \tau$

$$\bar{g}^{\tau\tau} = -1 \quad \Rightarrow \quad \boxed{\bar{g}^{\mu\nu}\frac{\partial\tau}{\partial x^\mu}\frac{\partial\tau}{\partial x^\nu} = -1}$$

Hamilton-Jacobi eq for a timelike test particle

Solve these eqs. to obtain coordinate transformations where EFT is constructed

(The form of EFT is the same)

Extension to vector-tensor gravity

- Formulate the EFT of BH perturbations in vector-tensor

Aoki, Gorji, Mukohyama, Takahashi and VY 23

$(A_\mu, g_{\mu\nu})$: 3+2 dofs

$\boldsymbol{n}_\mu \propto \delta^\tau_\mu + g_M A_\mu \equiv \boldsymbol{\delta}^\tau_\mu \qquad \boldsymbol{n}_\mu \boldsymbol{n}^\mu = -1$

preferred time direction not preferred time slicing

$\boldsymbol{h}_{\mu\nu} \equiv g_{\mu\nu} + \boldsymbol{n}_\mu \boldsymbol{n}_\nu$ Projection tensor ≠ induced metric

Sym. Breaking pattern

Unitary gauge

4d diffeo. $\Rightarrow$ 3d diffeo. +

$\boldsymbol{\delta}^\tau_\mu = \delta^\tau_\mu + g_M A_\mu$

$\left.\begin{array}{l} A_\mu \to A_\mu + \partial_\mu \chi \\ \tau \to \tau - g_M \chi \end{array}\right]$

- General EFT in unitary gauge: $\boldsymbol{g}^{\tau\tau} = g^{\tau\tau} + 2 g_M A^\tau + g_M^2 A_\mu A^\mu \qquad \boldsymbol{K}_{\mu\nu} \equiv \boldsymbol{h}^\alpha_{\ (\mu|} \nabla_\alpha \boldsymbol{n}_{|\nu)}$

$$S = \int d^4x \sqrt{-g}\ \mathscr{L}(\boldsymbol{g}^{\tau\tau}, {}^{(3)}\boldsymbol{R}_{\mu\nu}, \boldsymbol{K}_{\mu\nu}, \boldsymbol{E}_{\mu\nu}, \boldsymbol{B}_{\mu\nu}, \pounds_{\boldsymbol{n}}, \boldsymbol{D}_\mu)$$

The orthogonal spatial Ricci curvature

orthogonal covariant derivative

Time coord. is not a good EFT building block! $\Rightarrow$ Explicit time-dependence is not allowed

NEED two sets of consistency relations

- EFT of vector-tensor theories on cosmological background

Aoki et. al. 22

Conclusions/Future directions

Conclusions

- Formulated the EFT on a generic background with $\bar{\Phi}(\tau)$
- Studied odd-sector with the EFT and shift- and Z_2- symmetries
- QNM spectrum in odd sector

Future directions

- QNM spectrum of even-parity perturbations
- Dynamics of perturbations in EFT of vector-tensor
- EFT with matter: Neutron star

Backup

Introduction: Motivation

Is it possible to formulate an EFT of perturbations on an inhomogeneous background with timelike scalar profile?

e.g. Schwarzschild—dS, Schwarzschild—FRW

A. Single EFT that works on both cosmological and black hole regimes: DE + BH
B. Accommodate the scordatura mechanism avoiding strong coupling problem around stealth solutions e.g. DHOST or Horndeski

   The problem typically arises when $c_s \to 0$ since $E_{\rm NL} \sim c_s$

- Similar work on the EFT with timelike scalar profile (Khoury et al. 22)
- The EFT of perturbations on a black hole background with spacelike scalar profile (Franciolini et al. 18, Hui et al. 21)

General construction of EFTs

Vector-tensor EFT : timelike vector field $\boldsymbol{\delta}^{\tau}_{\mu} = \delta^{\tau}_{\mu} + g_M A_\mu$ spontaneously breaks time diffeo.

$(A_\mu, g_{\mu\nu})$: 3+2 dofs

4d diffeo.

$(A_\mu, g_{\mu\nu})$: 3+2 dofs

U(1) Stückelberg →

4d diffeo. + $\left[\begin{array}{l} A_\mu \to A_\mu + \partial_\mu \chi \\ \tilde{\tau} \to \tilde{\tau} - g_M \chi \end{array} \right.$

$v_\mu = \partial_\mu \tilde{\tau} + g_M A_\mu$

$(A_\mu, \tilde{\tau}, g_{\mu\nu})$: 2+1+2 dofs

$\tilde{\tau} = \tau$ →

Unitary gauge

3d diffeo. + $\left. \begin{array}{l} A_\mu \to A_\mu + \partial_\mu \chi \\ \tau \to \tau - g_M \chi \end{array} \right]$

EFT symmetries in unitary gauge

$(A_\mu, \tilde{\tau}, g_{\mu\nu})$: 2+0+3 dofs

$\boldsymbol{n}_\mu \propto \delta^{\tau}_{\mu} + g_M A_\mu \equiv \boldsymbol{\delta}^{\tau}_{\mu}$ $\qquad \boldsymbol{n}_\mu \boldsymbol{n}^\mu = -1$

Not hypersurface orthogonal

- EFT action in unitary gauge

$$\boldsymbol{g}^{\tau\tau} \equiv g^{\mu\nu} \boldsymbol{\delta}^{\tau}_{\mu} \boldsymbol{\delta}^{\tau}_{\nu} = g^{\tau\tau} + 2 g_M A^\tau + g_M^2 A_\mu A^\mu$$

$$\boldsymbol{h}_{\mu\nu} \equiv g_{\mu\nu} + \boldsymbol{n}_\mu \boldsymbol{n}_\nu$$

Projection tensor ≠ induced metric

$$\boldsymbol{K}_{\mu\nu} \equiv \boldsymbol{h}^{\alpha}_{(\mu|} \nabla_\alpha \boldsymbol{n}_{|\nu)}$$

$$S = \int d^4x \sqrt{-g}\ \mathscr{L}(\boldsymbol{g}^{\tau\tau}, {}^{(3)}\boldsymbol{R}_{\mu\nu}, \boldsymbol{K}_{\mu\nu}, \boldsymbol{E}_{\mu\nu}, \boldsymbol{B}_{\mu\nu}, \pounds_{\boldsymbol{n}}, \boldsymbol{D}_\mu)$$

The orthogonal spatial Ricci curvature

orthogonal covariant derivative

Time coord. is not a good EFT building block! $\Rightarrow$ Explicit time-dependence is not allowed

For $g_M = 0$ all the geometrical quantities coincide with usual time slicing quantities

Consistency relations

Vector-tensor EFT : Expand the EFT action around a generic background

$$S=\int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}f(y)\boldsymbol{R}-\Lambda(y)-c(y)\boldsymbol{g}^{\tau\tau}-d(y)\boldsymbol{K}+\frac{1}{2}m_2^4(y)(\delta\boldsymbol{g}^{\tau\tau})^2+\frac{1}{2}M_1^3(y)\delta\boldsymbol{g}^{\tau\tau}\delta\boldsymbol{K}+\cdots\right]$$

$$\delta\boldsymbol{g}^{\tau\tau}\equiv\boldsymbol{g}^{\tau\tau}-\bar{\boldsymbol{g}}^{\tau\tau}(\tau,x_i)\qquad \delta\boldsymbol{K}^\mu_\nu=\boldsymbol{K}^\mu_\nu-\bar{\boldsymbol{K}}^\mu_\nu(\tau,x_i)\qquad \boldsymbol{g}^{\tau\tau}=g^{\tau\tau}+2g_MA^\tau+g_M^2A_\mu A^\mu$$

- Two sets of consistency relations ensures the 3d diffeo. invariance and the residual U(1) of the EFT

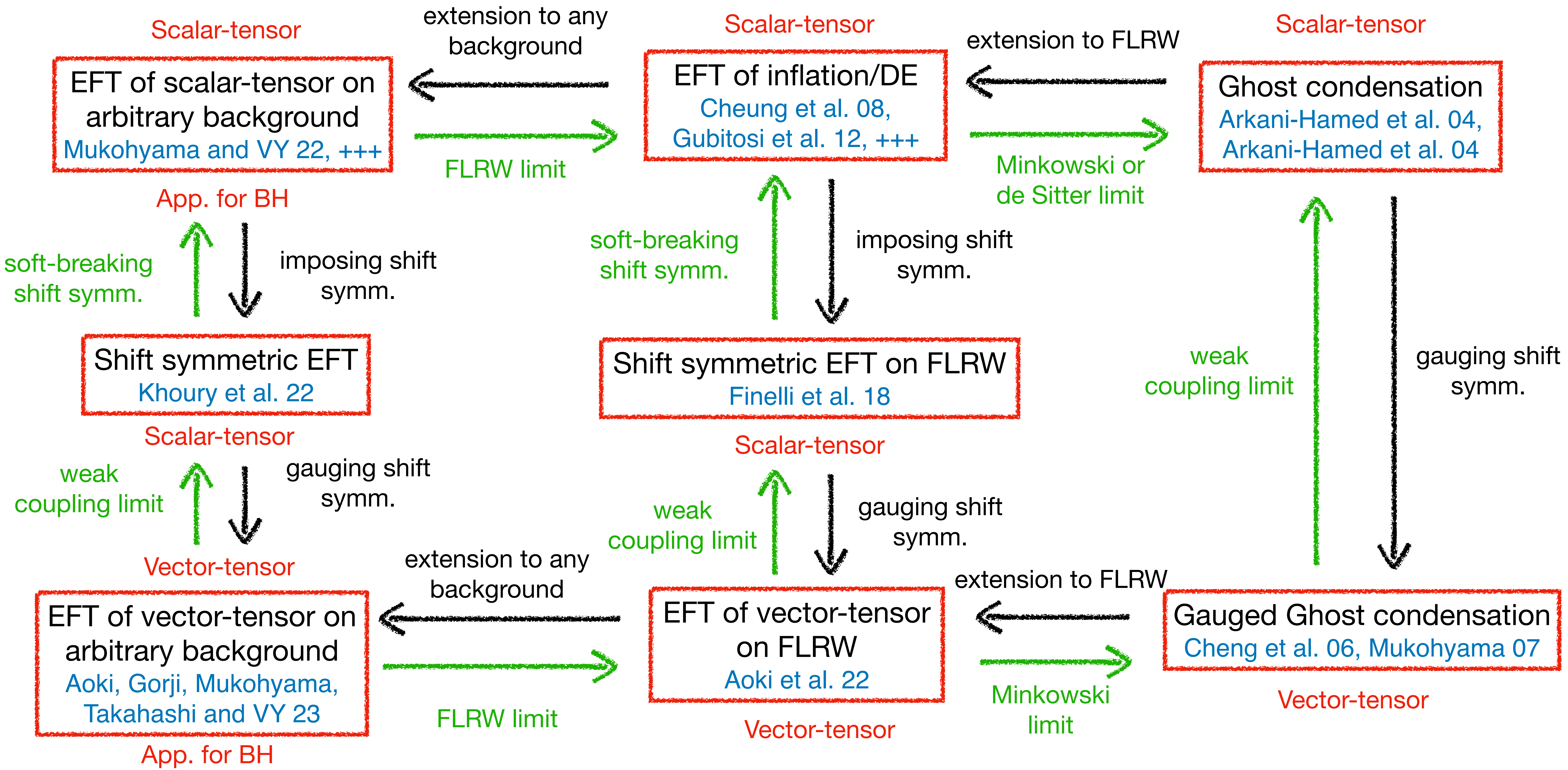
Chain rule in x^i-directions
$$\partial_i\Lambda+\bar{\boldsymbol{g}}^{\tau\tau}\partial_ic-\frac{M_\star^2}{2}\left(\bar{\boldsymbol{K}}^2-\bar{\boldsymbol{K}}^\mu{}_\nu\bar{\boldsymbol{K}}^\nu{}_\mu+{}^{(3)}\bar{\boldsymbol{R}}\right)\partial_if+\bar{\boldsymbol{K}}\partial_id=0$$

⋮

Chain rule in τ-directions
$$\dot{\Lambda}+\bar{\boldsymbol{g}}^{\tau\tau}\dot{c}-\frac{M_\star^2}{2}\left(\bar{\boldsymbol{K}}^2-\bar{\boldsymbol{K}}^\mu{}_\nu\bar{\boldsymbol{K}}^\nu{}_\mu+{}^{(3)}\bar{\boldsymbol{R}}\right)\dot{f}+\bar{\boldsymbol{K}}\dot{d}=0$$

⋮

The Web of EFTs



Dynamics in odd-parity sector

- Background metric: $ds^2 = -d\tau^2 + [1 - A(r)]d\rho^2 + r^2 d\Omega^2$ $\Leftrightarrow$ $ds^2 = -A(r)dt^2 + \dfrac{dr^2}{B(r)} + r^2 d\Omega^2$

  Static and spherically symmetric

- Background scalar: $\bar{\Phi}(\tau) = \mu^2 \tau \quad \Rightarrow \quad \bar{X} = -\mu^4$ with $\Phi \to \Phi + const.$ and $\Phi \to -\Phi$

- Metric perturbations in odd sector (no scalar): $\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ $\qquad \delta g_{\tau\tau} = \delta g_{\tau\rho} = \delta g_{\rho\rho} = 0$

$$\delta g_{\tau a} = \sum_{\ell,m} r^2 h_{0,\ell m}(\tau,\rho) E_a{}^b \bar{\nabla}_b Y_{\ell m}(\theta,\phi) \qquad \delta g_{\rho a} = \sum_{\ell,m} r^2 h_{1,\ell m}(\tau,\rho) E_a{}^b \bar{\nabla}_b Y_{\ell m}(\theta,\phi)$$

Covariant on S_2

$a \equiv \{\theta, \phi\}$

$$\delta g_{ab} = \sum_{\ell,m} r^2 h_{2,\ell m}(\tau,\rho) E_{(a}{}^c \bar{\nabla}_{|c|} \bar{\nabla}_{b)} Y_{\ell m}(\theta,\phi)$$

Epsilon tensor on S_2

- Gauge transformation: $x^\mu \to x^\mu + \epsilon^\mu \ \Rightarrow \ h_2 = 0$

- Only one physical degree of freedom — (h_0, h_1) aren't physical

Parameters p's

- Quadratic Lagrangian from EFT with $A(r) = B(r)$ and Z_2 and shift symmetries:

$$\mathcal{L}_2 = p_1 h_0^2 + p_2 h_1^2 + p_3[(\dot{h}_1 - \partial_\rho h_0)^2 + 2p_4 h_1 \partial_\rho h_0]$$

$$p_1 \equiv \frac{1}{2}(j^2-2)r^2\sqrt{1-A}\,(M_\star^2 + M_3^2) \qquad p_2 \equiv -(j^2-2)\frac{r^2 M_\star^2}{2\sqrt{1-A}} + (p_3 p_4)^{\cdot}$$

$$p_3 \equiv \frac{(M_\star^2 f + M_3^2)r^4}{2\sqrt{1-A}} \qquad p_4 \equiv \sqrt{\frac{B}{A(1-A)}}\left(\frac{A'}{2} + \frac{1-A}{r}\right)\frac{\alpha + M_3^2}{M_\star^2 + M_3^2}$$

- Integrate out (h_0, h_1) : $\quad \mathcal{L}_2 = p_1 h_0^2 + \tilde{p}_2 h_1^2 + p_3[-\chi^2 + 2\chi(\dot{h}_1 - \partial_\rho h_0 - p_4 h_1)]$

$$\tilde{p}_2 \equiv p_2 - (p_3 p_4)^{\cdot} - p_3 p_4^2$$

$$h_0 = -\frac{\partial_\rho(p_3\chi)}{p_1} \qquad h_1 = \frac{(p_3\chi)^{\cdot} + p_3 p_4 \chi}{\tilde{p}_2}$$

Parameters s's

$$\mathcal{L}_2 = s_1\dot{\chi}^2 - s_2(\partial_\rho\chi)^2 - s_3\chi^2$$

$$s_1 = \frac{j^2-2}{2\sqrt{1-A}}\frac{(M_\star^2+M_3^2)^2r^6}{(j^2-2)M_\star^2+(M_\star^2+M_3^2)r^2p_4^2} \qquad s_2 = \frac{(M_\star^2+M_3^2)r^6}{2(1-A)^{3/2}}$$

$$s_3 = j^2\frac{(M_\star^2+M_3^2)r^4}{2\sqrt{1-A}} + \mathcal{O}(j^0)$$

- The absence of ghost and gradient instabilities require:

$$s_1 > 0\,, \quad c_\rho^2 > 0\,, \quad c_\theta^2 > 0$$

$$M_\star^2 + M_3^2 > 0\,, \quad M_\star^2 > 0$$

$\mathscr{L}_2$ in $(t, r)-$ and $(\tilde{t}, r)-$coordinates

- We use $dt = \frac{1}{A}d\tau - \frac{1-A}{A}d\rho$ and $dr = -\sqrt{\frac{B(1-A)}{A}}d\tau + \sqrt{\frac{B(1-A)}{A}}d\rho$

$$\mathscr{L}_2 = \tilde{a}_1(\partial_t\chi)^2 - a_2(\partial_r\chi)^2 + 2a_3(\partial_t\chi)(\partial_r\chi) - a_4\chi^2$$

$$\tilde{a}_1 = \frac{s_1 - (1-A)^2 s_2}{\sqrt{A^3B(1-A)}} \qquad a_2 = \sqrt{\frac{B(1-A)}{A}}(s_2 - s_1) \qquad a_3 = \frac{(1-A)s_2 - s_1}{A} \qquad a_4 = \sqrt{\frac{B}{A(1-A)}}s_3$$

- Remove the crossed term by $\tilde{t} = t + \int \frac{a_3}{a_2} dr$ and $r \to r$

All the parameters are explicitly independent of t

$$\mathscr{L}_2 = a_1(\partial_{\tilde{t}}\chi)^2 - a_2(\partial_r\chi)^2 - a_4\chi^2 \quad \text{with} \quad a_1 = \tilde{a}_1 + \frac{a_3^2}{a_2}$$

Beyond Horndeski theories

- The most general scalar-tensor theories without the Ostrogradski ghost : (Beyond) Horndeski Theories

$$L_2 = G_2(\Phi, X) \qquad L_3 = G_3(\Phi, X)\Box\Phi$$

Modifies speed of GW on $\Phi(t)$

$$L_4 = G_4(\Phi, X)R - 2G_{4X}(\Phi, X)[(\Box\Phi)^2 - \Phi_{\mu\nu}\Phi^{\mu\nu}] - F_4(X, \Phi)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\Phi_{\mu}\Phi_{\mu'}\Phi_{\nu\nu'}\Phi_{\rho\rho'}$$

$$L_5 = G_5(\Phi, X)G_{\mu\nu}\Phi^{\mu\nu} + \frac{1}{3}G_{5X}(\Phi, X)[(\Box\Phi)^3 - 3(\Box\Phi)\Phi_{\mu\nu}\Phi^{\mu\nu} + 2\Phi_{\mu\nu}\Phi^{\sigma\mu}\Phi^{\nu}{}_{\sigma}]$$

$$-F_5(\Phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\Phi_{\mu}\Phi_{\mu'}\Phi_{\nu\nu'}\Phi_{\rho\rho'}\Phi_{\sigma\sigma'}$$

Horndeski 74, Deffayet et al. 11, Zumalacárregui and García-Bellido 14, Gleyzes et al. 14

$$\Phi_{\mu} \equiv \nabla_{\mu}\Phi$$

- DHOST Theories: Lagrangian contains second-order derivatives of a scalar field

Langlois and Noui 15, Langlois 17

Motivated example

- Timelike scalar solution on Schwarzschild background   Mukohyama 05

Stealth solution i.e. non-trivial scalar profile on a metric of GR

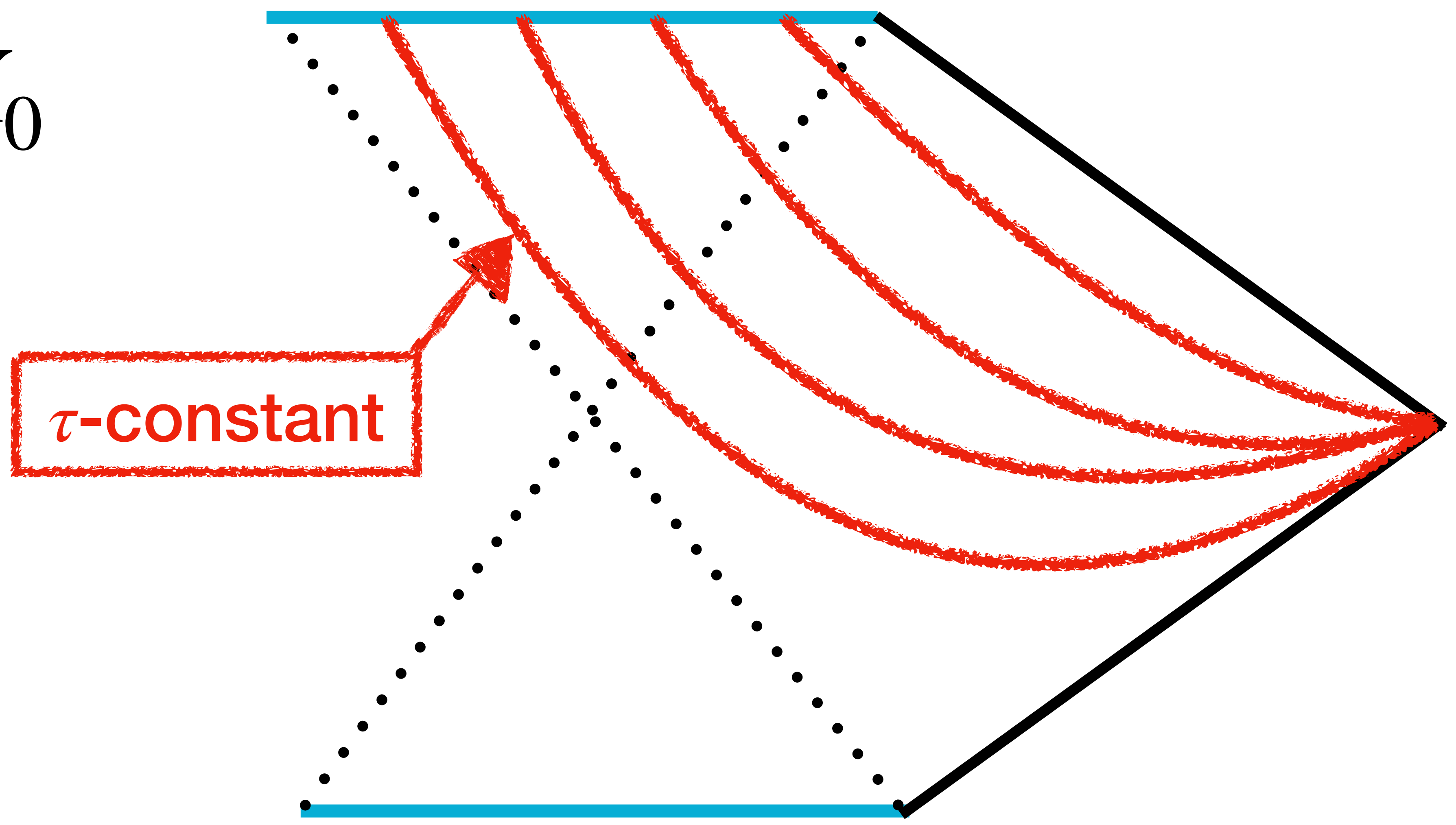
$$S = \int d^4x \sqrt{-g}\; P(X) \qquad T^{\Phi}_{\mu\nu} = P(X_0) g_{\mu\nu} \qquad P(X_0) = P'(X_0) = 0$$

The solution is $\Phi(\tau) = \sqrt{-X_0}\;\tau$ with constant X_0

$$ds^2 = -\, d\tau^2 + \frac{r_s}{r} d\rho^2 + r^2 d\Omega^2$$

$$r(\tau, \rho) = r_s^{1/3} \left[\frac{3}{2}(\rho - \tau) \right]^{2/3}$$

Lemaitre coordinates



$u^\mu = -\, \partial^\mu \Phi$: timelike everywhere

Formulation of the EFT

- Consistency relations among EFT parameters:

$$\partial_i\Lambda + \bar{g}^{\tau\tau}\partial_i c - \frac{1}{2}M_\star^2\,{}^{(3)}\!\bar{R}\partial_i f + \frac{1}{3}\bar{K}(M_\star^2\bar{K}\partial_i f + 3\partial_i\beta) - \frac{1}{2}\bar{\sigma}^\mu_\nu(M_\star^2\bar{\sigma}^\nu_\mu\partial_i f - 2\partial_i\alpha^\nu_\mu) + \bar{r}^\mu_\nu\partial_i\gamma^\nu_\mu \simeq 0$$

$$\partial_i c + m_2^4\partial_i\bar{g}^{\tau\tau} + \frac{1}{2}(M_1^3 + \frac{1}{3}h^\mu_\nu\lambda^\nu_{1\mu})\partial_i\bar{K} + \frac{1}{2}\lambda^\nu_{1\mu}\partial_i\bar{\sigma}^\mu_\nu + (\mu_1^2 + \frac{1}{3}h^\mu_\nu\lambda^\nu_{2\mu})\partial_i\,{}^{(3)}\!\bar{R} + \frac{1}{2}\lambda^\nu_{2\mu}\partial_i\bar{r}^\mu_\nu \simeq 0$$

$$\vdots$$

- EFT corresponding to Horndeski theories:

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_\star^2}{2}f(y)R - \Lambda(y) - c(y)g^{\tau\tau} - \beta(y)K - \alpha^\mu_\nu(y)\sigma^\nu_\mu - \gamma^\mu_\nu(y)r^\nu_\mu + \frac{1}{2}m_2^2(y)(\delta g^{\tau\tau})^2$$

$$+\frac{1}{2}M_1^3(y)\delta g^{\tau\tau}\delta K + \frac{1}{2}\mu_1^2(y)\delta g^{\tau\tau}\delta^{(3)}R - \frac{1}{2}m_4^2(y)(\delta K^2 - \delta K^\mu_\nu\delta K^\nu_\mu) + \cdots\Big]$$

Dictionary w/ quadratic HOST theories
see S. Mukohyama, K. Takahashi and VY 22

Decoupling limit action for π

Neglect the mixing with gravity $\Rightarrow$ no need to solve for N and N_i

$$S = \int d^4x\sqrt{-g}\left[\frac{M_\star^2}{2}f(y)R - \Lambda(y) - c(y)g^{\tau\tau} + \frac{1}{2}m_2^4(y)(\delta g^{\tau\tau})^2 + \frac{1}{3!}m_3^4(y)(\delta g^{\tau\tau})^3\right]$$

Use $g^{\tau\tau} \to g^{\tau\tau} + 2g^{\tau\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$

$$S_\pi = \int d\tau d^3\tilde{x}\sqrt{-g}\ c_s c\left[\left(\dot{\pi}^2 - h^{ij}\tilde{\partial}_i\pi\tilde{\partial}_j\pi\right) + c_s^2\left(\frac{1}{c_s^2} - 1\right)\dot{\pi}(\dot{\pi}^2 - h^{ij}\tilde{\partial}_i\pi\tilde{\partial}_j\pi) + \frac{8}{3}\frac{m_3^4 c_s^2}{c}\dot{\pi}^3\right]$$

where $x_i = c_s\tilde{x}_i$ and $\dfrac{1}{c_s^2} \equiv 1 + \dfrac{2m_2^4}{c}$

$$\frac{\mathscr{L}_2}{\mathscr{L}_3} \sim 1 \quad \Rightarrow \quad E_{\text{Cubic}} \sim \frac{(c_s M_\star^2 E^2)^{1/4}}{\sqrt{1 - c_s^2}}$$

Strong coupling scale above which $\mathscr{L}_3$ becomes important compared to $\mathscr{L}_2$

Dictionary e.g. Quartic Horndeski

$$L_4 = G_4(\Phi, X)R - 2G_{4X}(\Phi, X)(\Box\Phi^2 - \nabla_\nu \nabla_\mu \Phi \nabla^\nu \nabla^\mu \Phi)$$

This can be rewritten as $L_4 = G_4{}^{(3)}R + (2XG_{4X} - G_4)\mathcal{K}_2 - 2\sqrt{-X}G_{4\Phi}K$ $\quad \mathcal{K}_2 \equiv K^2 - K_{\mu\nu}K^{\mu\nu} = 2K^2/3 - \sigma_{\mu\nu}\sigma^{\mu\nu}$

Determine $\bar{F}$ and its derivatives e.g. $\bar{F} = \bar{G}_4{}^{(3)}\bar{R} + (2\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X} - \bar{G}_4)\bar{\mathcal{K}}_2 - 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi}\,, \quad \bar{F}_{^{(3)}R} = \bar{G}_4$

Use the matching we obtain

$$M_\star^2 f = 2\bar{G}_4 \quad \Lambda = \bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}({}^{(3)}\bar{R} + 3\bar{\mathcal{K}}_2) + 2(-\bar{g}^{\tau\tau})^2\dot{\bar{\Phi}}^4\bar{G}_{4XX}\bar{\mathcal{K}}_2 - \sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi} + 2(-\bar{g}^{\tau\tau})^{3/2}\dot{\bar{\Phi}}^3\bar{K}\bar{G}_{4\Phi X}$$

$$c = -\dot{\bar{\Phi}}^2\bar{G}_{4X}{}^{(3)}\bar{R} - \mathcal{G}_4\bar{\mathcal{K}}_2 - \frac{1}{\sqrt{-\bar{g}^{\tau\tau}}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi} + 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}^3\bar{K}\bar{G}_{4\Phi X} \quad \beta = -\frac{8}{3}\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{K}\bar{G}_{4X} + 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}\bar{G}_{4\Phi}\,, \quad \alpha^\mu_\nu = 4\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}\bar{\sigma}^\mu_\nu$$

$$m_2^4 = \dot{\bar{\Phi}}^4\bar{G}_{4XX}{}^{(3)}\bar{R} + (3\dot{\bar{\Phi}}^4\bar{G}_{4XX} + 2\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^6\bar{G}_{4XXX})\bar{\mathcal{K}}_2 + \frac{1}{2(-\bar{g}^{\tau\tau})^{3/2}}\dot{\bar{\Phi}}\bar{K}\bar{G}_{4\Phi} + \frac{2}{\sqrt{-\bar{g}^{\tau\tau}}}\dot{\bar{\Phi}}^3\bar{K}\bar{G}_{4\Phi X} - 2\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}^5\bar{K}\bar{G}_{4\Phi XX}$$

$$M_1^3 = \frac{8}{3}\bar{K}\mathcal{G}_4 + \frac{2}{\sqrt{-\bar{g}^{\tau\tau}}}\dot{\bar{\Phi}}\bar{G}_{4\Phi} - 4\sqrt{-\bar{g}^{\tau\tau}}\dot{\bar{\Phi}}^3\bar{G}_{4\Phi X}\,, \quad M_2^2 = 4\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}\,, \quad M_3^2 = -4\bar{g}^{\tau\tau}\dot{\bar{\Phi}}^2\bar{G}_{4X}\,, \quad \mu_1^2 = 2\dot{\bar{\Phi}}^2\bar{G}_{4X}\,, \quad \lambda^\mu_{1\ \nu} = -4\mathcal{G}_4\bar{\sigma}^\mu_\nu$$

Applicable for any solution of $\bar{\Phi}(\tau)$

Dipole perturbations $(\ell = 1)$

- h_2 trivially vanishes by definition in terms of spherical harmonics
- Setting $\partial_\rho \Xi = h_1 \Rightarrow h_1 = 0$ is an incomplete gauge fixing since one is still free to choose Ξ up to an arbitrary function of τ
- Setting $h_1 = 0$ in $\mathscr{L}_2$ leads to a loss of indep. Eqns of motion
- Eqns of motion for h_0 and h_1:

$$\partial_\rho[p_3(\dot{h}_1 - \partial_\rho h_0)] + \partial_\rho(p_3 p_4 h_1) = 0 \qquad [p_3(\dot{h}_1 - \partial_\rho h_0)]^{\cdot} - (p_3 p_4)^{\cdot} h_1 - p_4 p_3 \partial_\rho h_0 = 0$$

- Setting $h_1 = 0$ gives $\quad \partial_\rho(p_3 \partial_\rho h_0) = 0 \quad$ and $\quad \dot{C}_1(\tau) + p_4 C_1(\tau) = 0$

where $\quad p_3 \partial_\rho h_0 = C_1(\tau)$

Dipole perturbations $(\ell = 1)$

- From $\dot{C}_1(\tau) + p_4 C_1(\tau) = 0 \Rightarrow C_1(\tau) = (const.)e^{-\int p_4 d\tau}$

- Avoid infinite c_ρ^2 $\Rightarrow rp_4$ to be finite as $r \to \infty \Rightarrow p_4 = p_4(r)$ with $r = r(\rho - \tau)$

- But this is incompatible with $\partial_\rho(p_3\partial_\rho h_0) = 0$ only if $C_1(\tau) = 0$

$$p_3\partial_\rho h_0 = C_1(\tau)$$

- For $C_1(\tau) = 0$ we have $h_0 = h_0(\tau)$ since $p_3 \neq 0$

Gauge parameter

- Setting $h_1 = 0$ gives $\Xi = \Xi(\tau)$ which can be chosen such that $h_0 = 0$

$$h_0 \to h_0 - \dot{\Xi}\,, \quad h_1 \to h_1 - \partial_\rho \Xi$$

No slowly rotating black hole solution

Dipole perturbations $(\ell = 1)$

- We set $p_4 = 0 \Rightarrow C_1 = const.$ and

$$h_0 = 2C_1 \int d\rho \frac{\partial_\rho r}{(M_\star^2 + M_3^2) r^4} \sqrt{\frac{A}{B}}$$

Slowly rotating BH exists

- For $A = B$ and $M_3^2 = const.$:

$$h_0 = -\frac{J}{4\pi (M_\star^2 + M_3^2) r^3}$$

Angular momentum of slowly rotating BH

$$J \equiv \frac{8\pi C_1}{3}$$

Dipole perturbations correspond to the angular momentum of slowly rotating BH

e.g. Expanding Kerr metric up to linear order in J gives $\# J dt d\phi$

- For $A \neq B$ and/or $M_3^2 \neq const.$: a rotating BH doesn't belong to Kerr family, even at the linear level