Black hole thermodynamics in Horndeski and Generalized Proca theories

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Introduction

- Many models of dark energy and modified gravity are included in Horndeski theories known as the most general scalar-tensor theories with the 2nd-order EOMs
- Horndeski theories have been extended to vector-tensor theories, called Generalized Proca (GP) theories
- The purpose of this work is to provide the general framework to discuss BH thermodynamics in Horndeski and GP theories by applying the Iyer-Wald formulation
- Since theories beyond Horndeski/GP theories contain similar BH solutions, our work could provide a direct hint for BH thermodynamics in these extended theories

Horndeski theories

$$S_{\rm H}=\int d^4x\sqrt{-g}\mathcal{L}_{\rm H}=\int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_{{\rm H}(i)},$$

$$\begin{aligned}
\mathcal{L}_{\rm H(2)} &:= G_2(\phi,X),\\
\mathcal{L}_{\rm H(3)} &:= -G_3(\phi,X)\Box\phi,\\
\mathcal{L}_{\rm H(4)} &:= G_4(\phi,X)R+G_{4X}(\phi,X)\left[(\Box\phi)^2-\left(\phi^{\alpha\beta}\phi_{\alpha\beta}\right)\right],\\
\mathcal{L}_{\rm H(5)} &:= G_5(\phi,X)G_{\mu\nu}\phi^{\mu\nu}\\
&- \frac{1}{6}G_{5X}(\phi,X)\left[(\Box\phi)^3-3\Box\phi\left(\phi^{\alpha\beta}\phi_{\alpha\beta}\right)+2\phi_\alpha{}^\beta\phi_\beta{}^\rho\phi_\rho{}^\alpha\right]
\end{aligned}$$

$$X:=-\frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu,\quad \phi_\mu=\nabla_\mu\phi,\quad \phi_{\mu\nu}=\nabla_\mu\nabla_\nu\phi$$

$$\Longrightarrow\quad \delta S_{\rm H}=\int d^4x\sqrt{-g}\left(E_{(g)\mu\nu}\delta g^{\mu\nu}+E_{(\phi)}\delta\phi+\sum_{i=2}^{5}\nabla_\mu J^\mu_{{\rm H}(i)}\right)$$

Euler-Lagrange (EL) equations: $E_{(g)\mu\nu}=0$ and $E_{(\phi)}=0$

$$
\begin{aligned}
J^{\mu}_{\mathrm{H}(2)} &= -G_{2X}\phi^{\mu}\delta\phi, \\
J^{\mu}_{\mathrm{H}(3)} &= -\frac{1}{2}G_{3}\left(\mathfrak{h}\phi^{\mu} - 2\mathfrak{h}^{\mu\nu}\phi_{\nu} + 2\nabla^{\mu}\delta\phi\right) + \delta\phi G_{3X}\Box\phi\phi^{\mu} + \delta\phi\nabla^{\mu}G_{3}, \\
J^{\mu}_{\mathrm{H}(4)} &= G_{4X}\Box\phi\left(\mathfrak{h}\phi^{\mu} - 2\mathfrak{h}^{\mu\nu}\phi_{\nu}\right) + 2G_{4X}\Box\phi\nabla^{\mu}(\delta\phi) - 2\nabla^{\mu}\left(G_{4X}\Box\phi\right)\delta\phi \\
&\quad + G_{4XX}\left(\phi^{\alpha\beta}\phi_{\alpha\beta} - (\Box\phi)^{2}\right)\phi^{\mu}\delta\phi + G_{4X}\left(2\phi^{\mu\rho}\phi^{\sigma} - \phi^{\rho\sigma}\phi^{\mu}\right)\mathfrak{h}_{\rho\sigma} \\
&\quad - 2G_{4X}\phi^{\mu\nu}\nabla_{\nu}\delta\phi + 2\nabla_{\nu}\left(G_{4X}\phi^{\mu\nu}\right)\delta\phi - \mathfrak{h}^{\mu\nu}\nabla_{\nu}G_{4} + G_{4}\nabla_{\nu}\mathfrak{h}^{\mu\nu} + \mathfrak{h}\nabla^{\mu}G_{4} \\
&\quad - G_{4}\nabla^{\mu}\mathfrak{h} - G_{4X}R\phi^{\mu}\delta\phi, \\
J^{\mu}_{\mathrm{H}(5)} &= \frac{1}{4}G_{5X}\phi^{\alpha\beta}\phi_{\alpha\beta}\left(\mathfrak{h}\phi^{\mu} - 2\mathfrak{h}^{\mu\nu}\phi_{\nu}\right) - \frac{1}{2}\mathfrak{h}_{\rho\sigma}G_{5X}\Box\phi\left(2\phi^{\sigma\mu}\phi^{\rho} - \phi^{\sigma\rho}\phi^{\mu}\right) \\
&\quad - \frac{1}{4}G_{5X}(\Box\phi)^{2}\left(\mathfrak{h}\phi^{\mu} + 2\nabla^{\mu}\delta\phi - 2\mathfrak{h}^{\mu\nu}\phi_{\nu}\right) + \frac{1}{2}\nabla^{\mu}\left[G_{5X}(\Box\phi)^{2}\right]\delta\phi \\
&\quad + \frac{1}{2}G_{5X}\phi^{\alpha\beta}\phi_{\alpha\beta}\nabla^{\mu}\delta\phi - \frac{1}{2}\delta\phi\nabla^{\mu}\left[G_{5X}\left(\phi_{\alpha\beta}\right)^{2}\right] + G_{5X}\Box\phi\phi^{\mu\nu}\nabla_{\nu}\delta\phi \\
&\quad - \delta\phi\nabla_{\nu}\left(G_{5X}\Box\phi\phi^{\mu\nu}\right) - G_{5X}\phi^{\mu\sigma}\phi_{\nu\sigma}\nabla^{\nu}\delta\phi + \delta\phi\nabla^{\nu}\left(G_{5X}\phi^{\mu\sigma}\phi_{\nu\sigma}\right) \\
&\quad + \frac{1}{6}G_{5XX}\left[(\Box\phi)^{3} - 3\Box\phi(\phi_{\alpha\beta})^{2} + 2(\phi_{\alpha\beta})^{3}\right]\phi^{\mu}\delta\phi \\
&\quad + \frac{1}{2}G_{5X}\phi^{\nu}_{\sigma}\left(2\phi^{\sigma\mu}\phi^{\rho} - \phi^{\sigma\rho}\phi^{\mu}\right)\mathfrak{h}_{\rho\nu}
\end{aligned}
$$

where $\mathfrak{h}_{\mu\nu} = \delta g_{\mu\nu}$, $\mathfrak{h}^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}\mathfrak{h}_{\rho\sigma}$, and $\mathfrak{h} = g^{\rho\sigma}\mathfrak{h}_{\rho\sigma}$

Generalized Proca (GP) theories (up to 'quartic' terms)

$$
\begin{aligned}
S_{\rm GP} \;&:=\; \int d^4x\sqrt{-g}\,\mathcal{L}_{\rm GP} = \int d^4x\sqrt{-g}\sum_{i=2}^{4}\mathcal{L}_{{\rm GP}(i)} \\
&\mathcal{L}_{{\rm GP}(2)} = \tilde{G}_2\left(\mathcal{F},\tilde{\mathcal{F}},Y\right) \\
&\mathcal{L}_{{\rm GP}(3)} = -\tilde{G}_3(Y)\nabla^\mu A_\mu \\
&\mathcal{L}_{{\rm GP}(4)} = \tilde{G}_4(Y)R + \tilde{G}_{4Y}(Y)\Big[(\nabla^\mu A_\mu)^2 - \nabla^\rho A^\sigma \nabla_\sigma A_\rho\Big]
\end{aligned}
$$

with $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, $\tilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$,

$Y := -\frac{1}{2}g^{\mu\nu}A_\mu A_\nu$, $\mathcal{F} := -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, $\tilde{\mathcal{F}} := -\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu}$

For $A_\mu \to \nabla_\mu\varphi$ and $Y \to X = -\frac{1}{2}\varphi^\mu\varphi_\mu$,
GP theories $\Longrightarrow$ shift-symmetric Horndeski theories

$$G_2(X) := \tilde{G}_2\left(0,0,X\right), \quad G_3(X) := \tilde{G}_3(X), \quad G_4(X) := \tilde{G}_4(X)$$

$$\Longrightarrow \delta S_{\mathrm{GP}} = \int d^4x\sqrt{-g}\left(E_{(g)\mu\nu}\delta g^{\mu\nu} + E_{(A)\mu}\delta A^{\mu} + \sum_{i=2}^{4}\nabla_{\mu}J^{\mu}_{\mathrm{GP}(i)}\right)$$

EL-equations: $E_{(g)\mu\nu} = 0$ and $E_{(A)\mu} = 0$

$$\begin{aligned}
J^{\rho}_{\mathrm{GP}(2)} &= -\tilde{G}_{2\mathcal{F}}F^{\rho\sigma}\delta A_{\sigma} - \tilde{G}_{2\tilde{\mathcal{F}}}\tilde{F}^{\rho\sigma}\delta A_{\sigma} \\
J^{\rho}_{\mathrm{GP}(3)} &= \tilde{G}_{3}(Y)\left(g^{\rho\mu}A^{\nu} - \frac{1}{2}A^{\rho}g^{\mu\nu}\right)\mathfrak{h}_{\mu\nu} - \tilde{G}_{3}(Y)g^{\rho\nu}\delta A_{\nu} \\
J^{\rho}_{\mathrm{GP}(4)} &= \tilde{G}_{4}(Y)\left[\nabla^{\nu}\left(g^{\rho\sigma}\mathfrak{h}_{\sigma\nu}\right) - \nabla^{\rho}\mathfrak{h}\right] \\
&+ \tilde{G}_{4Y}(Y)A^{\alpha}\left[g^{\nu\rho}\nabla^{\sigma}A_{\alpha} - g^{\sigma\nu}\nabla^{\rho}A_{\alpha}\right]\mathfrak{h}_{\sigma\nu} \\
&+ \tilde{G}_{4Y}(Y)\nabla^{\nu}A_{\nu}\left[2g^{\rho\sigma}\delta A_{\sigma} - 2g^{\rho\sigma}A^{\nu}\mathfrak{h}_{\nu\sigma} + A^{\rho}\mathfrak{h}\right] \\
&- 2\tilde{G}_{4Y}(Y)\left[\left(\nabla^{\sigma}A^{\rho}\right)\delta A_{\sigma} - A^{\nu}\nabla^{(\sigma}A^{\rho)}\mathfrak{h}_{\nu\sigma} + \frac{1}{2}A^{\rho}\nabla^{\nu}A^{\sigma}\mathfrak{h}_{\sigma\nu}\right]
\end{aligned}$$

Noether charge for the diffeomorphism invariance

- Horndeski and GP theories are diffeomorphism invariant
- Under the infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu(x^\mu)$, $\mathfrak{h}^{(\xi)}_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$, $\delta_\xi \phi = \xi^\mu \phi_\mu$, and $\delta_\xi A_\mu = \xi^\sigma \nabla_\sigma A_\mu + A_\sigma \nabla_\mu \xi^\sigma$
- With use of EL equations, the Noether current is obtained

$$J^\mu_{\mathrm{H/GP}(\xi)} - \xi^\mu \mathcal{L}_{\mathrm{H/GP}} = 2\nabla_\nu K^{[\nu\mu]}_{\mathrm{H/GP}(\xi)}$$

- Noether charge for Horndeski theories $K^{[\nu\mu]}_{\mathrm{H}(\xi)} = \sum_{i=2}^{5} K^{[\nu\mu]}_{\mathrm{H}(i)(\xi)}$

$$\begin{aligned}
K^{\mu\nu}_{\mathrm{H}(2)(\xi)} &= 0 \\
K^{\mu\nu}_{\mathrm{H}(3)(\xi)} &= -G_3 \xi^\mu \phi^\nu \\
K^{\mu\nu}_{\mathrm{H}(4)(\xi)} &= 2G_{4X}\left[\Box\phi \xi^\mu \phi^\nu - \xi_\sigma \phi^{\sigma\mu}\phi^\nu\right] + 2\xi^\mu \nabla^\nu G_4 + G_4 \nabla^\mu \xi^\nu \\
K^{\mu\nu}_{\mathrm{H}(5)(\xi)} &= -\frac{1}{2} G_{5X}\left[\left(\Box\phi^2 - \phi^{\alpha\beta}\phi_{\alpha\beta}\right)\xi^\mu\phi^\nu + 2\left(\xi^\rho \phi_{\rho\sigma} - \Box\phi\xi_\sigma\right)\phi^{\sigma\mu}\phi^\nu\right] \\
&\quad + \xi^\mu \nabla_\sigma\left(\phi^{\nu\sigma} G_5\right) - \xi_\sigma \nabla^\mu\left(\phi^{\nu\sigma} G_5\right) - \xi^\mu \nabla^\nu\left(G_5 \Box\phi\right) \\
&\quad + \frac{1}{2} G_5\left(2\xi_\sigma G^{\sigma\mu}\phi^\nu - 2(\nabla_\sigma \xi^\mu)\phi^{\nu\sigma} - \Box\phi \nabla^\mu \xi^\nu\right)
\end{aligned}$$

- Noether charge for GP theories $K^{[\nu\mu]}_{\mathrm{GP}(\xi)} = \sum_{i=2}^{4} K^{[\nu\mu]}_{\mathrm{GP}(i)(\xi)}$

$$
\begin{aligned}
K^{\mu\nu}_{\mathrm{GP}(2)(\xi)} &= \frac{1}{2}\left(\tilde{G}_{2\mathcal{F}} F^{\mu\nu} + \tilde{G}_{2\tilde{\mathcal{F}}} \tilde{F}^{\mu\nu}\right) \\
K^{\mu\nu}_{\mathrm{GP}(3)(\xi)} &= -\tilde{G}_3(Y)\xi^\mu A^\nu \\
K^{\mu\nu}_{\mathrm{GP}(4)(\xi)} &= \tilde{G}_4(Y)\nabla^\mu\xi^\nu + 2\tilde{G}_{4Y}(Y)A^\alpha\left(\nabla^\mu A_\alpha\right)\xi^\nu \\
&- 2\tilde{G}_{4Y}(Y)\left(\nabla^\alpha A_\alpha\right)A^\mu\xi^\nu - \frac{1}{2}\tilde{G}_{4Y}(Y)F^{\mu\nu}A^\alpha\xi_\alpha \\
&+ \tilde{G}_{4Y}(Y)\left(A^\mu\nabla^\nu A^\alpha + A^\mu\nabla^\alpha A^\nu\right)\xi_\alpha
\end{aligned}
$$

- Tensors dual to J^μ and $K^{\mu\nu}_{(\xi)}$

$$
\begin{aligned}
\Theta_{\alpha\beta\gamma} &:= J^\mu \varepsilon_{\mu\alpha\beta\gamma} \\
Q_{(\xi)\alpha\beta} &:= -\epsilon_{\alpha\beta\mu\nu} K^{\mu\nu}_{(\xi)}
\end{aligned}
$$

Black hole thermodynamics

- A static and spherically symmetric BH spacetime

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2\gamma_{ab}d\theta^a d\theta^b$$

- $h(r) > 0$ and $f(r) > 0$ for $r > r_g$, where $h(r_g) = f(r_g) = 0$
- Choosing ξ^μ to the timelike Killing vector, the integration of $\delta Q_{(\xi)\alpha\beta} - \xi^\nu \Theta_{\nu\alpha\beta}$ on the boundaries of the Cauchy surface provides the variation of the Hamiltonian

Wald (93); Iyer and Wald (94)

$$\delta\mathcal{H}_{\mathrm{H/GP}} := \delta\mathcal{H}_\infty - \delta\mathcal{H}_H$$
$$= -\left[\int d\Omega \left(\delta\left(r^2\sqrt{\frac{h}{f}}K^{[tr]}_{\mathrm{H/GP}(\xi)}\right) + r^2\sqrt{\frac{h}{f}}J^{[t}_{\mathrm{H/GP}}\xi^{r]}\right)\right]^{r\to\infty}_{r=r_g}$$

- Variation is taken for integration constants c_j

$$\mathfrak{h}_{tt} = -\sum_j \frac{\partial h}{\partial c_j}\delta c_j, \quad \mathfrak{h}_{rr} = -\frac{1}{f^2}\sum_j \frac{\partial f}{\partial c_j}\delta c_j, \quad \mathfrak{h}_{ab} = 0,$$

$$\delta\phi = \sum_j \frac{\partial \phi}{\partial c_j}\delta c_j, \quad \delta A_0 = \sum_j \frac{\partial A_0}{\partial c_j}\delta c_j, \quad \delta A_1 = \sum_j \frac{\partial A_1}{\partial c_j}\delta c_j$$

- The variation at the horizon idenfied as that of the entropy

$$\delta\mathcal{H}_H = T_{\mathrm{H(H/GP)}}\delta S_{\mathrm{H/GP}}$$

with the Hawking temperature $T_{\mathrm{H(H/GP)}} := \frac{\sqrt{h'(r_g)f'(r_g)}}{4\pi}$

- The variation at the infinity idenfied as that of the thermodynamic mass

$$\delta\mathcal{H}_\infty = \delta M_{\mathrm{H/GP}}$$

- The conservation of the Hamiltonian $\delta\mathcal{H}_{\mathrm{H/GP}} = 0$
$\Longrightarrow$ the 1st law of BH thermodynamics

$$T_{\mathrm{H(H/GP)}}\delta S_{\mathrm{H/GP}} = \delta M_{\mathrm{H/GP}}$$

General Relativity (GR)

$$\mathcal{L}_{\rm GR} = \frac{R-2\Lambda}{16\pi G} \Longrightarrow G_2 = -\frac{1}{8\pi G}\Lambda, \quad G_4 = \frac{1}{16\pi G}, \quad G_3 = G_5 = 0$$

$$\Longrightarrow \quad -\delta\left(r^2\sqrt{\frac{h}{f}}K^{[tr]}_{\rm GR(\xi)}\right) - r^2\sqrt{\frac{h}{f}}J^{[t}_{\rm GR}\xi^{r]} = -r^2\sqrt{\frac{h}{f}}\frac{\delta f}{8\pi G r}$$

Schwarzschild-(A)dS BHs as the unique vacuum solutions

$$f(r) = h(r) = 1 - \frac{\Lambda}{3}r^2 - \frac{r_g}{r}\left(1 - \frac{\Lambda}{3}r_g^2\right)$$

$$\Longrightarrow \quad T_{\rm H(GR)}\delta S_{\rm GR} = \delta M_{\rm GR} = \frac{1}{2G}\left(1 - r_g^2\Lambda\right)\delta r_g$$

$$T_{\rm H(GR)} = \frac{1}{4\pi r_g}(1 - r_g^2\Lambda)$$ Hawking temperature

Bekenstein-Hawking formula $\quad S_{\rm GR} := \frac{\pi r_g^2}{G}\left[= \frac{A_{\rm H}}{4G}\right]$

Thermodynamic mass $\quad M_{\rm GR} = \frac{r_g}{2G}\left(1 - \frac{1}{3}\Lambda r_g^2\right) = M_{\rm ADM}$

Einstein-scalar-Gauss-Bonnet (EsGB) theories

$$\mathcal{L}_{\rm H} = \frac{R}{16\pi G} + \eta X + k(\phi)\left(R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}\right)$$

- hairy BH solutions
- standard Wald's entropy formula

As a check, we check the recovery of the same entropy formula from the equivalent description of EsGB theories

$$G_2 = \eta X + 8k^{(4)}(\phi)X^2\,(3 - \ln X), \;\; G_3 = 4k^{(3)}(\phi)X\,(7 - 3\ln X),$$
$$G_4 = \frac{1}{16\pi G} + 4k^{(2)}(\phi)X(2 - \ln X), \quad G_5 = -4k^{(1)}(\phi)\ln X$$

Expansion of hairy BH solution near $r = r_g$

$$h(r) = h_1(r_g)\,(r - r_g) + h_2(r_g)\,(r - r_g)^2 + \cdots\,,$$
$$f(r) = f_1(r_g)\,(r - r_g) + f_2(r_g)\,(r - r_g)^2 + \cdots\,,$$
$$\phi(r) = \psi_H(r_g) + \psi_1(r_g)\,(r - r_g) + \psi_2(r_g)\,(r - r_g)^2 + \cdots$$

$$\Longrightarrow T_{\mathrm{H(H)}}\delta S_{\mathrm{H}} \quad = \quad \frac{\sqrt{f_1(r_g)h_1(r_g)}}{2G}\left(r_g + 32\pi G k^{(1)}[\psi_H(r_g)]\frac{\partial\psi_H(r_g)}{\partial r_g}\right)\delta r_g$$

with Hawking temperature $T_{\mathrm{H(H)}} = \frac{\sqrt{h_1(r_g)f_1(r_g)}}{4\pi}$

$$\Longrightarrow \delta S_{\mathrm{H}} = \frac{2\pi r_g}{G}\left(1 + \frac{32\pi G k^{(1)}[\psi_H(r_g)]}{r_g}\frac{\partial\psi_H(r_g)}{\partial r_g}\right)\delta r_g$$

For the couplings with $k(0) = 0$

$$S_{\mathrm{H}} = \frac{\pi}{G}\left(r_g^2 + 64\pi G k[\psi_H(r_g)]\right)$$

recovery of the standard formula

BCL solution

Babichev, Charmousis and Lehébel (17)

$$G_2 = \eta X - \frac{\Lambda}{8\pi G}, \quad G_4 = \frac{1}{16\pi G} + \alpha(-X)^{\frac{1}{2}}, \quad G_3 = G_5 = 0$$

The unique hairy BH solution

$$f(r) = h(r) = 1 - \frac{8\pi G\alpha^2}{r^2\eta} - \frac{\Lambda}{3}r^2 - \frac{1}{r}\left(r_g - \frac{8\pi G\alpha^2}{\eta r_g} - \frac{\Lambda r_g^3}{3}\right),$$

$$\phi(r) = \frac{\sqrt{2}\alpha}{\eta}\int \frac{dr}{r^2\sqrt{f(r)}}$$

$$\Longrightarrow T_{\mathrm{H(H)}}\delta S_{\mathrm{H}} = \delta M_{\mathrm{H}} = \frac{\delta r_g}{2G}\left(1 - r_g^2\Lambda + \frac{8\pi G\alpha^2}{\eta r_g^2}\right)$$

Hawking temperature $T_{\mathrm{H(H)}} = \frac{1}{4\pi r_g}\left(1 - r_g^2\Lambda + \frac{8\pi G\alpha^2}{\eta r_g^2}\right)$

$$\Longrightarrow S_{\mathrm{H}} = \frac{\pi r_g^2}{G}, \quad M_{\mathrm{H}} = \frac{r_g}{2G}\left(1 - \frac{8\pi G\alpha^2}{\eta r_g^2} - \frac{\Lambda r_g^2}{3}\right) = M_{\mathrm{ADM}}$$

BHs with $\phi = qt + \psi(r)$

Babichev and Charmousis (13)

$$G_2(X) = -\frac{\Lambda}{8\pi G} - \eta X, \quad G_4(X) = \frac{1}{16\pi G} + \beta X, \quad G_3 = G_5 = 0$$

- Schwarzschild solutions for $\Lambda = \eta = 0$ ($G_2 = 0$)
  - Stealth Schwarzschild solution

$$f = h = 1 - \frac{r_g}{r}, \quad \psi(r) = 2q\sqrt{r_g}\left[\sqrt{r} - \sqrt{r_g}\,\mathrm{arctanh}\left(\sqrt{\frac{r_g}{r}}\right)\right]$$

$$\Longrightarrow S_{\mathrm{H}} = \frac{\pi r_g^2}{G}\left(1 - 8\pi G q^2 \beta\right), \quad M_{\mathrm{H}} = \frac{r_g}{2G}\left(1 - 8\pi G q^2 \beta\right)$$

  - GR Schwarzschild BHs with $\phi = \mathrm{const}$

$$f = h = 1 - \frac{r_g}{r}, \quad \phi = \mathrm{const} \Longrightarrow S_{\mathrm{GR}} = \frac{\pi r_g^2}{G}, \qquad M_{\mathrm{GR}} = \frac{r_g}{2G}$$

  - For the same thermodynamic mass $M_{\mathrm{GR}} = M_{\mathrm{H}}$

$$S_{\mathrm{H}} = \frac{S_{\mathrm{GR}}}{1 - 8\pi G q^2 \beta} \Longrightarrow S_{\mathrm{GR}} > S_{\mathrm{H}} \qquad (\beta < 0)$$

- Schwarzschild-(A)dS solutions for $\eta \neq 0$ and $\Lambda \neq 0$ ($G_2 \neq 0$)
  - Horndeski Schwarzschild-(A)dS solutions

$$f(r) = h(r) = 1 - \frac{\bar{\Lambda}}{3} r^2 - \frac{r_g}{r}\left(1 - \frac{\bar{\Lambda}}{3} r_g^2\right), \quad \bar{\Lambda} := -\frac{\eta}{2\beta},$$

$$\psi(r) = q \int dr \frac{\sqrt{1-h(r)}}{\sqrt{f(r)h(r)}}, \quad q = \sqrt{\frac{\Lambda - \bar{\Lambda}}{8\pi G\eta}}, \quad \bar{\ell} = \sqrt{\frac{3}{|\bar{\Lambda}|}}$$

$$\Longrightarrow S_{\mathrm{H}} = \frac{\pi r_g^2}{G} \frac{\Lambda + \bar{\Lambda}}{2\bar{\Lambda}}, \quad M_{\mathrm{H}} = \frac{r_g}{2G}\left(1 - \frac{\bar{\Lambda} r_g^2}{3}\right) \frac{\Lambda + \bar{\Lambda}}{2\bar{\Lambda}}$$

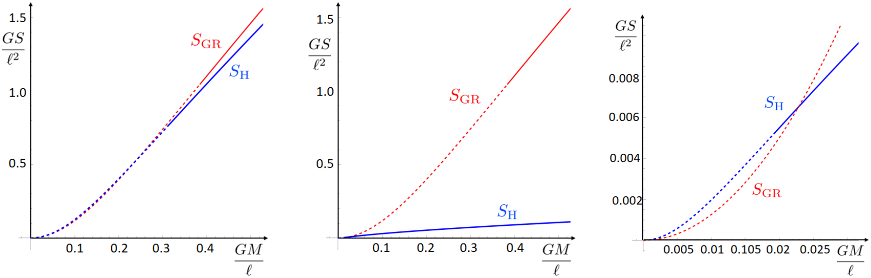
  - GR Schwarzschild-(A)dS solution with $\phi = \mathrm{const}$

$$f(r) = h(r) = 1 - \frac{\Lambda}{3} r^2 - \frac{r_g}{r}\left(1 - \frac{\Lambda}{3} r_g^2\right), \quad \ell = \sqrt{\frac{3}{|\Lambda|}}$$

$$\Longrightarrow S_{\mathrm{GR}} := \frac{\pi r_g^2}{G} \quad M_{\mathrm{GR}} = \frac{r_g}{2G}\left(1 - \frac{1}{3}\Lambda r_g^2\right)$$

(1) $\Lambda \geq \bar{\Lambda} > 0$ Schwarzschild-dS solutions ($\beta < 0$), $S_{\mathrm{GR}} > S_{\mathrm{H}}$

(2) $\bar{\Lambda} \leq \Lambda < 0$ Schwarzschild-AdS solutions ($\beta > 0$)



(a) $\bar{\ell}/\ell = 0.9$ (b) $\bar{\ell}/\ell = 0.1$ (c) enlarged (b)

- Hawking-Page transition for smaller masses
- In most mass regions, $S_{\mathrm{GR}} > S_{\mathrm{H}}$
- For $0 < \bar{\ell}/\ell < 0.48835$, there is a region where $S_{\mathrm{H}} > S_{\mathrm{GR}}$

Stealth Schwarzschild solution in a class of GP theories

$$\tilde{G}_2 = \mathcal{F}, \qquad \tilde{G}_4 = \frac{1}{16\pi G} + \beta Y, \qquad \tilde{G}_3 = 0$$

- Arbitrary $\beta \Longleftrightarrow$ counterparts in Horndeski with $A_\mu \to \partial_\mu \varphi$

$$h = f = 1 - \frac{r_g}{r}, \quad A_0 = q, \quad A_1 = q\frac{\sqrt{r r_g}}{r - r_g}$$

$$\Longrightarrow S_{\mathrm{GP}} = \frac{\pi r_g^2}{G}\left(1 - 8\pi G q^2 \beta\right), \quad M_{\mathrm{GP}} = \frac{r_g}{2G}\left(1 - 8\pi G q^2 \beta\right)$$

same thermodynamic quantities as in Horndeski counterpart

- $\beta = \frac{1}{4}$ Tasinato (16) $\Longrightarrow$ no counterparts in Horndeski theories

$$h = f = 1 - \frac{r_g}{r}, \quad A_0 = q + \frac{Q}{r}, \quad A_1 = \frac{\sqrt{Q^2 + 2qQr + q^2 r r_g}}{r - r_g}$$

$$\Longrightarrow S_{\mathrm{GP}} = \frac{\pi r_g^2}{G}\left(1 - 2\pi G q^2\right), \quad M_{\mathrm{GP}} = \frac{r_g}{2G}\left(1 - 2\pi G q^2\right)$$

Q does not contribute to BH thermodynamics

Schwarzschild-(A)dS solutions in a class of GP theories

$$\tilde{G}_2 = \mathcal{F} + 2m^2 Y - \frac{\Lambda}{8\pi G}, \qquad \tilde{G}_4 = \frac{1}{16\pi G} + \beta Y, \qquad \tilde{G}_3 = 0$$

- Arbitrary $\beta \Longrightarrow$ counterparts in Horndeski with $A_\mu \to \partial_\mu \varphi$

$$h(r) = f(r) = -\frac{\bar{\Lambda}}{3} r^2 + 1 - \frac{r_g}{r}\left(1 - \frac{\bar{\Lambda}}{3} r_g^2\right), \qquad \bar{\Lambda} := -\frac{m^2}{\beta}$$

$$A_0(r) = \frac{\sqrt{\Lambda - \bar{\Lambda}}}{4m\sqrt{\pi G}}, \quad A_1(r) = \frac{\sqrt{\Lambda - \bar{\Lambda}}}{4m\sqrt{\pi G}} \frac{\sqrt{1 - f(r)}}{f(r)}$$

$$\Longrightarrow S_{\mathrm{GP}} = \frac{\pi r_g^2}{G} \frac{\Lambda + \bar{\Lambda}}{2\bar{\Lambda}}, \quad M_{\mathrm{GP}} = \frac{r_g}{2G}\left(1 - \frac{\bar{\Lambda} r_g^2}{3}\right) \frac{\Lambda + \bar{\Lambda}}{2\bar{\Lambda}}$$

same thermodynamic quantities as in Horndeski counterpart

- $\beta = \frac{1}{4}$ Minamitsuji (16) $\Longrightarrow$ no counterparts in Horndeski theories

$$h(r) = f(r) = -\frac{\bar{\Lambda}}{3} r^2 + 1 - \frac{r_g}{r}\left(1 - \frac{\bar{\Lambda}}{3} r_g^2\right), \quad \bar{\Lambda} := -4m^2$$

$$A_0(r) = \frac{Q}{r} + \frac{1}{4m\sqrt{\pi}} \sqrt{\frac{\Lambda - \bar{\Lambda}}{G}}, \quad Y(r) = \frac{\Lambda - \bar{\Lambda}}{32\pi G m^2}$$

Q does not contribute to BH thermodynamics

Stealth Schwarzschild-(A)dS solutions

Cisterna, Hassaine, Oliva, and Rinaldi (16)

$$\tilde{G}_2 = g_2(Y), \qquad \tilde{G}_3 = g_3(Y), \qquad \tilde{G}_4 = \frac{1}{16\pi G}$$

BHs with the constant norm of the vector field $Y = Y_0$

$$\begin{aligned}
h(r) &= f(r) = \frac{8\pi G g_2(Y_0)}{3} r^2 + 1 - \frac{r_g}{r}\left(1 + \frac{8\pi G r_g^2 g_2(Y_0)}{3}\right), \\
A_0(r)^2 &= \frac{1}{9}\left(48\pi G Y_0 g_2(Y_0) + \frac{g_{2Y}(Y_0)^2}{g_{3Y}(Y_0)^2}\right) r^2 + 2Y_0 \\
&\quad + \frac{1}{r}\left(-\frac{2r_g Y_0}{3}\left(3 + 8\pi G r_g^2 g_2(Y_0)\right) - \frac{2P g_{2Y}(Y_0)}{3 g_{3Y}(Y_0)}\right) + \frac{P^2}{r^4}, \\
A_1(r) &= \frac{1}{r^2 f(r)}\left(P - \frac{r^3}{3}\frac{g_{2Y}(Y_0)}{g_{3Y}(Y_0)}\right)
\end{aligned}$$

$$\Longrightarrow S_{\rm GP} = \frac{\pi r_g^2}{G}, \quad M_{\rm GP} = \frac{r_g}{2G}\left(1 + \frac{8\pi G}{3} r_g^2 g_2(Y_0)\right) = M_{\rm ADM}$$

- P does not contribute to BH thermodynamics
- $S_{\rm GR} > S_{\rm GP}$ for a timelike vector field $(Y_0 > 0)$

RN-(A)dS solutions in a class of GP theories

$$\tilde{G}_2 = \mathcal{F} - \frac{\Lambda}{8\pi G}, \qquad \tilde{G}_4 = \frac{1}{16\pi G}, \qquad \tilde{G}_3 = g_3(Y)$$

BHs with the constant norm $Y = Y_0 = \mathrm{const}$

$$h(r) = f(r) = -\frac{\Lambda r^2}{3} + 1 - \frac{r_g}{r}\left(\frac{4\pi G Q^2}{r_g^2} + 1 - \frac{r_g^2 \Lambda}{3}\right) + \frac{4\pi G Q^2}{r^2},$$

$$A_0(r) = q + \frac{Q}{r}, \quad A_1(r) = \frac{1}{f(r)}\sqrt{A_0(r)^2 - 2Y_0 f(r)},$$

$$\frac{\partial g_3}{\partial Y}(Y_0) = 0$$

the same first law for RN-(A)dS BH in Einstein-Maxwell theory

$$\Longrightarrow S_{\mathrm{GP}} = \frac{\pi r_g^2}{G}, \quad M_{\mathrm{GP}} = \frac{r_g}{2G}\left(1 + \frac{4\pi G Q^2}{r_g^2} - \frac{\Lambda r_g^2}{3}\right) = M_{\mathrm{ADM}}$$

$$T_{\mathrm{H(H)}}\delta S_{\mathrm{GP}} = \delta M_{\mathrm{GP}} - \Phi_H \delta Q, \quad \Phi_H := -4\pi\left(A_0(\infty) - A_0(r_g)\right)$$

Summary

- We provided general framework for thermodynamics of static and spherically symmetric BHs in Horndeski and GP theories
- We applied it to several exact BH solutions including stealth BHs

Future issues

- Extension to theories beyond Horndeski and GP
- BH thermodynamics associated with different propagation speeds and effective horizons

Thank you !