A conserved charge in general relativity

Sinya AOKI
Center for Gravitational Physics and Quantum Information,
Yukawa Institute for Theoretical Physics, Kyoto University

YITP long-term workshop
"Gravity and Cosmology 2024"
January 29 - March 1, 2024
Yukawa Institute for Theoretical Physics, Kyoto University

This talk is based on

SA, Y. Hidaka, K. Kawana and K. Shimada, "Geometric conservation in curved spacetime and entropy", arXiv:2312.09712[hep-th]

Related references

SA, T. Onogi and S. Yokoyama, "Charge conservation, Entropy Current, and Gravitation", Int. J. Mod. Phys. A36 (2021)2150201.

SA and K. Kawana, "Entropy and its conservation in expanding Universe", International Journal of Modern Physics A38 (2023) 2350072 [arXiv:2210.03323 [hep-th]].

SA, T. Onogi and T. Yamaoka, "Energies and a gravitational charge for massive particles in general relativity", [arXiv:2305.09849 [gr-qc]].

I. Introduction

Motivation

Questions Is there a covariantly conserved quantity in general relativity ?

If exists, what is its physical meaning ?

Energy ?

Some conclusions from our previous studies

1. There exists no covariant definition of conserved energy in general relativity, due to Noether's 2nd theorem.

Einstein's pseudo-tensor (non-covariant), quasi-local energy (absence of local energy density)

2. A (matter) energy covaraintly defined in general relativity is not conserved in general.

I will not discuss this anymore in this talk, due to the limitation of time.

For more details, please take a look at

SA and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", Int. J. Mod. Phys. A36 (2022) 2250129,

Results in this talk

1. There exists a covariant and geometric conservation for a general class of energy momentum tensor in curved spacetime.

2. The geometric conserved charge becomes "entropy" for a perfect fluids.

content

II. Set up

(which may not be found in textbooks)

1. Decomposition of energy momentum tensor

Energy Momentum Tensor (EMT)

In this talk, we consider the EMT of the Hawking-Ellis type I, which is given by

$$T_{\mu\nu} = \varepsilon u_\mu u_\nu + P_{\mu\nu}, \quad P_{\mu\nu} u^\nu = u^\mu P_{\mu\nu} = 0$$

ε: energy density $\quad u^\mu$: a time-like unit vector $\quad P_{\mu\nu}$: pressure tensor

This EMT covers standard classical matters in 3+1 dimensions.

Hawking&Ellis 1973, Martin-Moruno&Visser 2018

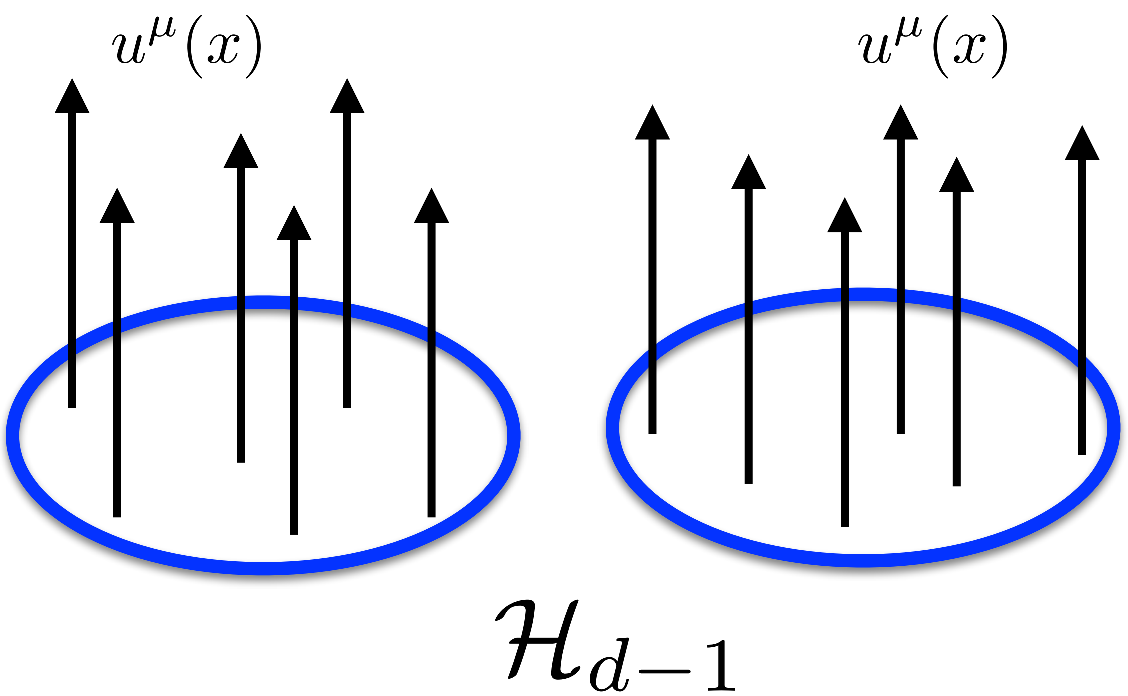
Conservation law $\quad \nabla_\mu T^{\mu\nu} = 0$

Other conserved currents such as electric charge or baryon number may exist:

$$\nabla_\mu N_i^\mu = 0 \qquad i = 1, 2 \cdots, f$$

2. Initial hyper-surface

First we pick up an initial space-like hyper-surface $\mathscr{H}_{d-1}$.

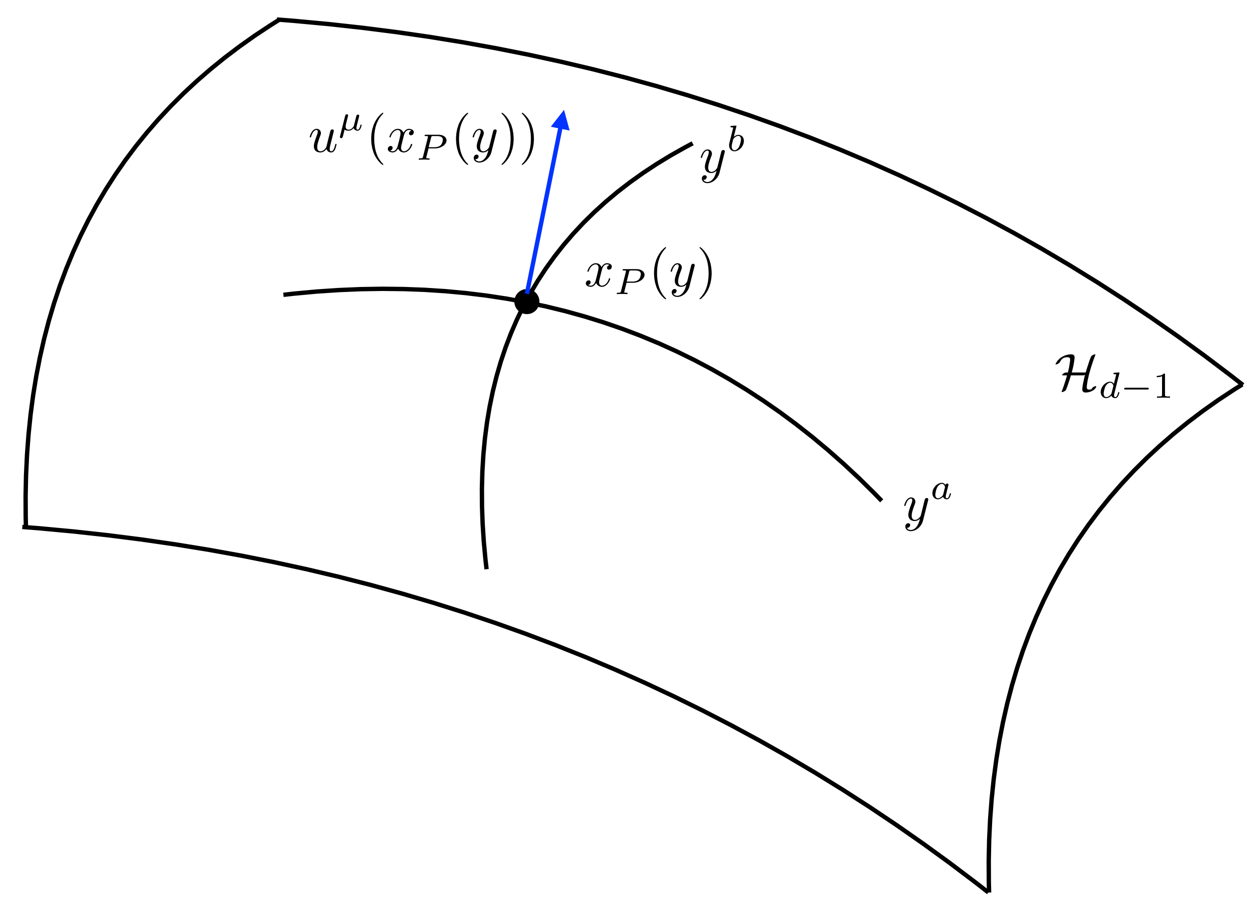


EMT is non-zero on $\mathscr{H}_{d-1}$:

$$u^\mu(x) \neq 0 \text{ at } {}^\forall x \in \mathcal{H}_{d-1}$$

$\mathscr{H}_{d-1}$ may not be connected.

Coordinate y^a $(a = 1,2,\cdots, d-1)$ on $\mathscr{H}_{d-1}$.



$$\mathcal{H}_{d-1} = \{x_P^\mu(y) \mid y \in H_{d-1}\}$$

H_{d-1} is a $d-1$ dimensional subspace of $\mathbb{R}^{d-1}$, which may not be necessarily connected.

3. Time-like curves and a foliation of hyper-surfaces

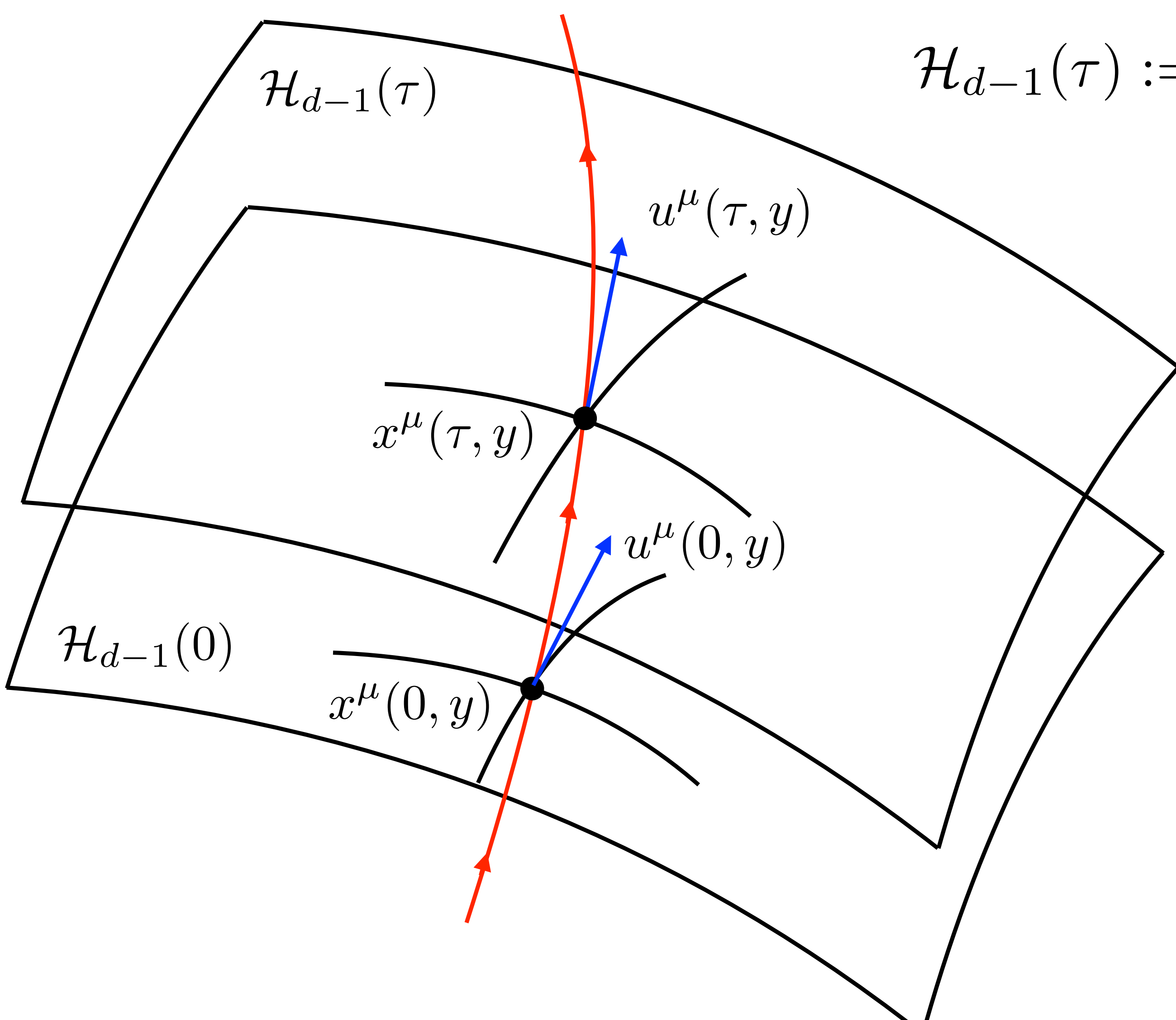
We define a time-like curve $x^\mu(\tau, y)$ starting from an arbitrary point $x_P(y)$ on $\mathscr{H}_{d-1}$:

$$\frac{dx^\mu(\tau,y)}{d\tau} = u^\mu(x^\mu(\tau,y)), \quad x^\mu(0,y) = x_P^\mu(y)$$

$$\Rightarrow \quad x^\mu(\tau,y) = x_P^\mu(y) + \int_0^\tau d\eta\, u^\mu(x(\eta,y))$$

τ can be negative.

We can construct a foliation of hyper-surfaces $\mathscr{H}_{d-1}(\tau)$ using time-like curves.



$$\mathcal{H}_{d-1}(\tau) := \left\{ x^\mu(\tau,y) \mid {}^\exists\tau, {}^\forall y \in H_{d-1} \right\}$$

$$\mathcal{H}_{d-1}(0) = \mathcal{H}_{d-1}$$

Assumption: $\nabla_\mu T^{\mu\nu} = 0$ implies $u^\mu(\tau, x)$ never end or emerge.

Hereafter, we use simplified notations such as $u^\mu(\tau,y) := u^\mu(x(\tau,y))$.

4. “3+1” decomposition

We consider a new coordinate (3+1 decomposition) as $y^A = (y^0 = \tau, y^a)$:

$$\tilde{g}_{AB} dy^A dy^B = -\lambda^2 d\tau^2 + h_{ab}(dy^a + N^a d\tau)(dy^b + N^b d\tau)$$

$$\tilde{g}_{AB} = \begin{pmatrix} -1, & N_b \\ N_a, & h_{ab} \end{pmatrix} \qquad \tilde{g}^{AB} = \frac{1}{\lambda^2}\begin{pmatrix} -1, & N^b \\ N^a, & \lambda^2 B^{ab} \end{pmatrix}$$

$$\lambda^2 := h_{ab} N^a N^b + 1 \qquad N_a := g_{\mu\nu} u^\mu e^\nu_a$$

$$h_{ab} := g_{\mu\nu} e^\mu_a e^\nu_b \qquad e^\nu_a := \frac{\partial x^\mu}{\partial y^a}$$

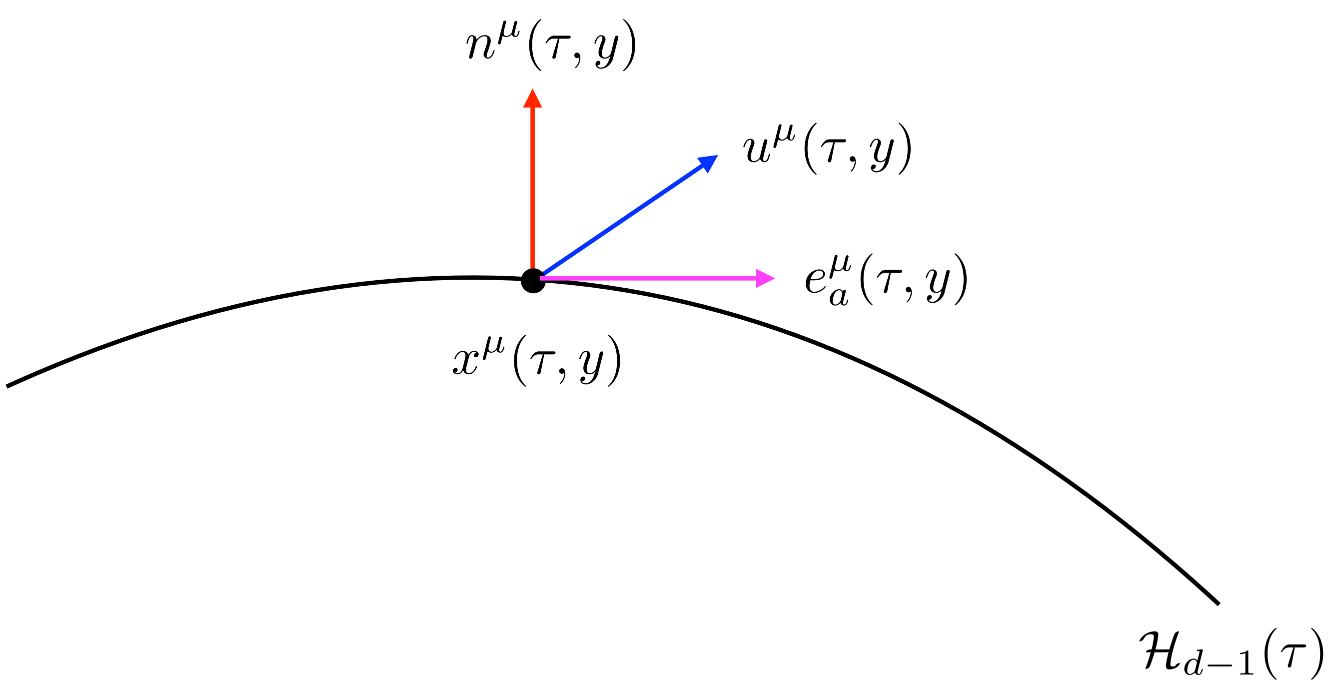
$$B^{ab} := g^{\mu\nu} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu} = h^{ab} - \frac{N^a N^b}{\lambda^2}$$

unit normal to $\mathcal{H}_{d-1}(\tau)$: $\quad \tilde{n}_A = \frac{\partial x^\mu}{\partial y^A} n_\mu = -\lambda\, \delta^0_A \quad (\lambda > 0)$

On the other hand $\quad \tilde{u}^A = \frac{\partial y^A}{\partial x^\mu} u^\mu = \delta^A_0$

Therefore

$$u \cdot n = \tilde{u} \cdot \tilde{n} = -\lambda \quad (\lambda > 0)$$



5. Evaluation of $K := \nabla_\mu u^\mu$

For a latter use, we calculate $K = \nabla_\mu u^\mu$ as

$$K = g^{\mu\nu}\nabla_\mu u_\nu = \frac{1}{2}g^{\mu\nu}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) = \frac{1}{2}g^{\mu\nu}\mathcal{L}_u(g_{\mu\nu}) = \frac{1}{\sqrt{-g}}\mathcal{L}_u(\sqrt{-g})$$

$\mathcal{L}_u$: Lie derivative along with u^μ $\qquad g := \det g_{\mu\nu}$

Since $\tilde{u}^A = \delta^A_0$ is a constant vector in y^A coordinate, we have

$$K = \frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial y^0} = \frac{1}{(u\cdot n)\sqrt{h}}\frac{\partial}{\partial\tau}\left\{(u\cdot n)\sqrt{h}\right\} = \frac{\partial}{\partial\tau}\log\left\{-(u\cdot n)\sqrt{h}\right\}$$

Here we use $-\tilde{g} := -\det\tilde{g}_{AB} = \lambda^2 h = (u\cdot n)^2 h$ with $h := \det h_{ab}$.

$$\tilde{g}_{AB} = \begin{pmatrix} -1, & N_b \\ N_a, & h_{ab} \end{pmatrix}$$

III. Conserved current and conserved charge

(Our proposal)

1. Construction of conserved current

Conserved current

refinement of the proposal in Aoki, Onogi & Yokoyama 2021

We construct the conserved current from the EMT using $u^\mu(x)$ as

$$J^\mu(x) := T^\mu{}_\nu(x)\zeta(x)u^\nu(x) = -\varepsilon(x)\zeta(x)u^\mu(x)$$

This definition is coordinate independent.

We determine $\zeta(x)$ in order to satisfy the conservation law $\nabla_\mu J^\mu = 0$ as

$$\nabla_\mu J^\mu(x) = -u^\mu\partial_\mu(\zeta\varepsilon) - \zeta\varepsilon K = -\frac{\partial}{\partial\tau}(\zeta\varepsilon) - \zeta\varepsilon K = 0$$

$$\Rightarrow \quad \zeta(\tau,y)\varepsilon(\tau,y) = \zeta(0,y)\varepsilon(0,y)\exp\left[-\int_0^\tau d\eta\, K(\eta,y)\right] = \zeta(0,y)\varepsilon(0,y)\frac{(n\cdot u)\sqrt{h}(0,y)}{(n\cdot u)\sqrt{h}(\tau,y)}$$

where we use $\quad K = \dfrac{\partial}{\partial\tau}\log\left\{-(u\cdot n)\sqrt{h}\right\}$

The conserved current is determined as

$$J^\mu(\tau,y) = -\zeta(0,y)\varepsilon(0,y)n(0,y)\cdot u(0,y)\sqrt{h(0,y)}\frac{u^\mu(\tau,y)}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

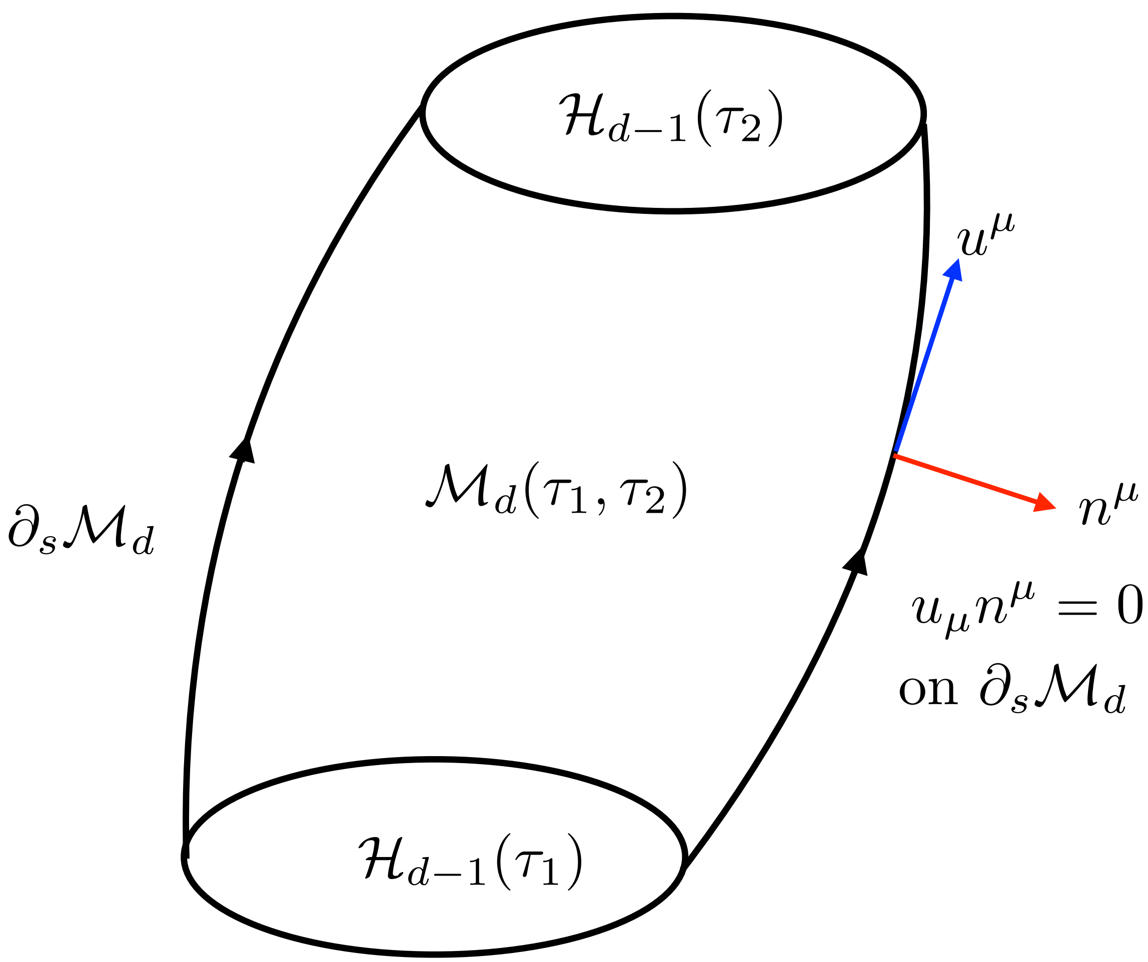
2. Conservation law and conserved charge

We consider a foliation of $\mathscr{H}_{d-1}(\tau)$ as $\mathcal{M}_d(\tau_1, \tau_2) := \{\mathcal{H}_{d-1}(\tau) \mid \tau_1 \leq \tau \leq \tau_2\}$

Integral of conservation law on $\mathscr{M}_d(\tau_1, \tau_2)$

$$0 = \int_{\mathcal{M}_d(\tau_1,\tau_2)} d^d x \sqrt{-g} \nabla_\mu J^\mu = Q(\mathcal{H}_{d-1}(\tau_2)) - Q(\mathcal{H}_{d-1}(\tau_1)) + \int_{\partial_s \mathcal{M}_d} d\Sigma_\mu J^\mu$$

where we define $$Q(\mathcal{H}_{d-1}(\tau)) := \int_{\mathcal{H}_{d-1}(\tau)} d\Sigma_\mu J^\mu$$



By construction, the current is zero on $\partial_s \mathscr{M}_d$ as

$$d\Sigma_\mu J^\mu \propto d\Sigma_\mu u^\mu \propto n_\mu u^\mu = 0$$

$$Q(\mathcal{H}_{d-1}({}^\forall \tau_2)) = Q(\mathcal{H}_{d-1}({}^\forall \tau_1)) := Q$$

conserved charge

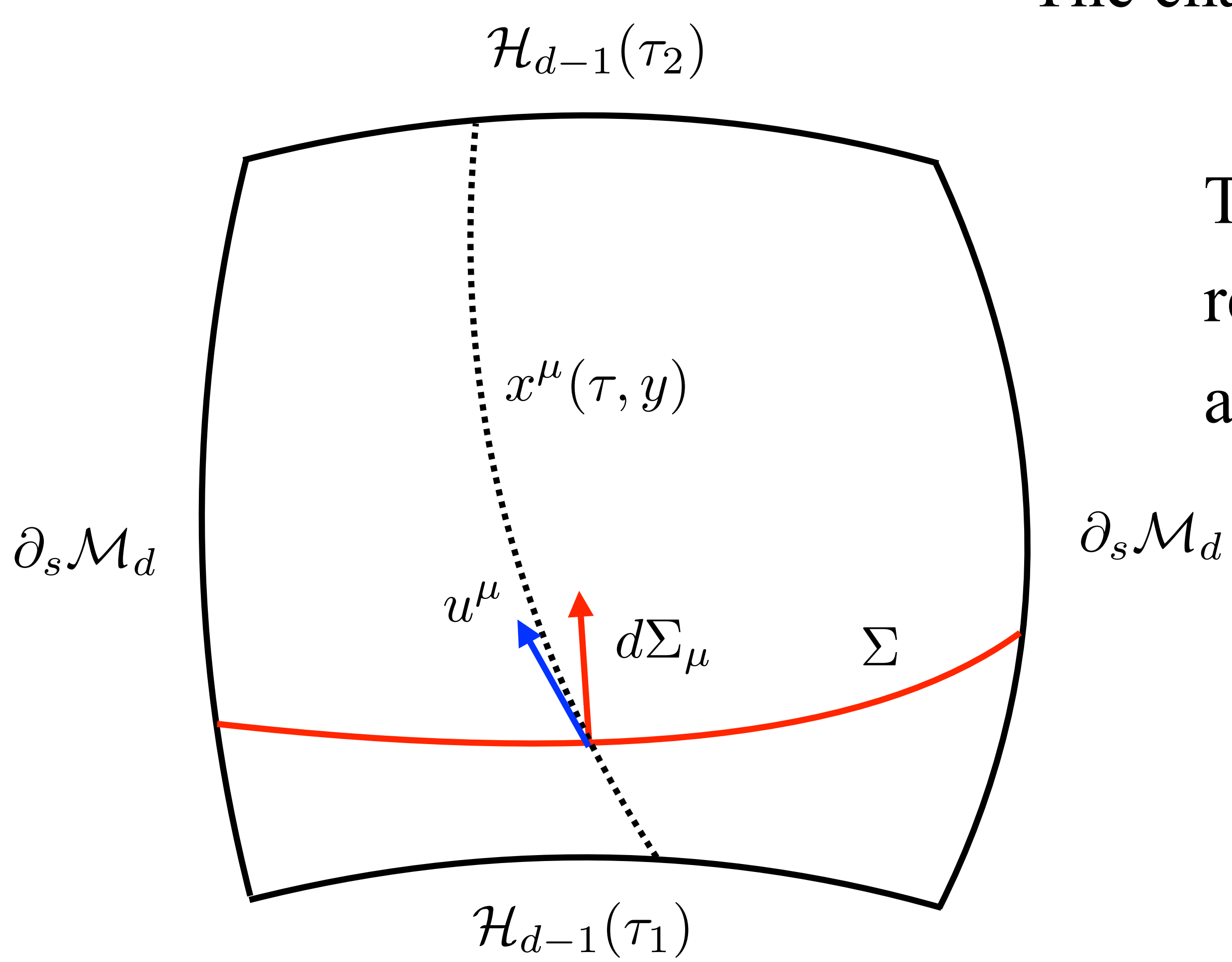
3. Explicit form of the conserved charge

$$J^\mu(\tau, y) = -\zeta(0, y)\varepsilon(0, y)n(0, y) \cdot u(0, y)\sqrt{h(0, y)}\frac{u^\mu(\tau, y)}{n(\tau, y) \cdot u(\tau, y)\sqrt{h(\tau, y)}}$$

$$Q(\mathcal{H}_{d-1}(\tau)) := \int_{\mathcal{H}_{d-1}(\tau)} d\Sigma_\mu \, J^\mu \qquad d\Sigma_\mu = -d^{d-1}y\sqrt{h}n_\mu$$

$$Q(\mathcal{H}_{d-1}(\tau)) = -\int_{H_{d-1}} d^{d-1}y\sqrt{h(\tau, y)}\, n_\mu(\tau, y)J^\mu(\tau, y) = \int_{H_{d-1}} d^{d-1}y\, \zeta(0, y)\varepsilon(0, y)\, n_\mu(0, y)u^\mu(0, y)\sqrt{h(0, y)}$$

The charge is indeed τ independent, and thus conserved.



The charge Q takes the same value even if we replace $\mathscr{H}_{d-1}(\tau)$ with an arbitrary hyper-surface Σ as in the left figure.

$$Q(\mathcal{H}_{d-1}(\tau)) = Q(\Sigma) = Q$$

IV. Geometric conservation and entropy

(trivial conservation but non-trivial interpretation)

1. (Special) initial condition for ζ

$$J^\mu(\tau,y) = -\zeta(0,y)\varepsilon(0,y)n(0,y)\cdot u(0,y)\sqrt{h(0,y)}\frac{u^\mu(\tau,y)}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

The conserved current depends on an initial value $\zeta(0,y)$.

We take the following (special) choice : $\zeta(0,y)\varepsilon(0,y)n(0,y)\cdot u(0,y)\sqrt{h(0,y)} = 1$

(We take the y-coordinate Cartesian or similar.)

geometric conserved current $J^\mu(\tau,y) = -\dfrac{u^\mu(\tau,y)}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$

geometric conserved charge
("gravitational charge") $Q = \displaystyle\int_{H_{d-1}} d^{d-1}y$

These are coordinate independent, but depend on a choice of an initial hyper-surface $\mathscr{H}_{d-1}$.

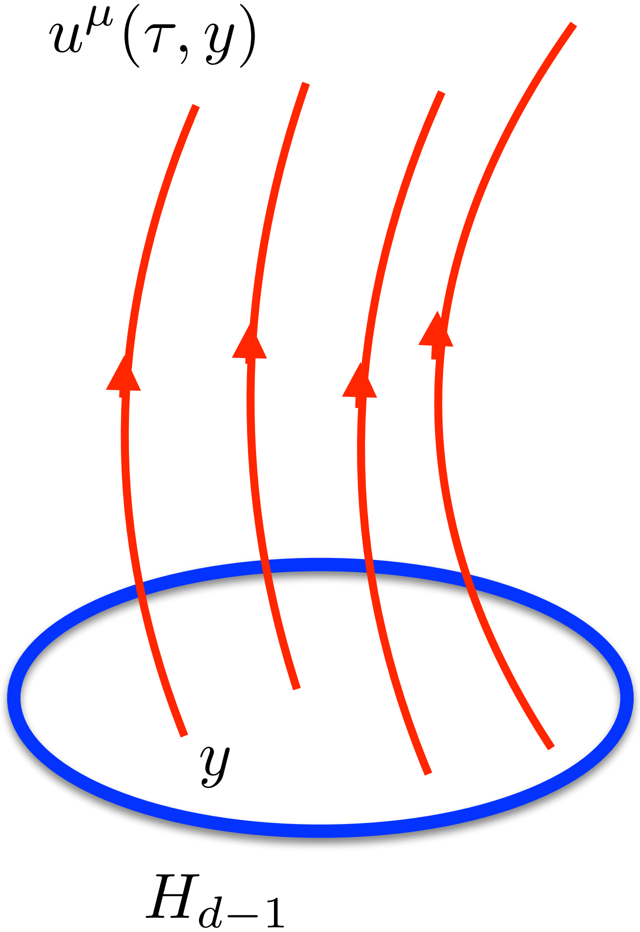
Q is invariant under the volume preserving diffeo. of y^a .

The conserved current looks "trivial". As we will show, however,
$J^\mu =$ "entropy current" in the case of perfect fluids.

2. Geometric conserved current

An essence of the geometric conservation law is an existence of time-like curves $u^\mu(\tau, y)$ which never end or emerge as

$$u^\mu(\tau,y) \neq 0 \text{ at } {}^\forall\tau \text{ for } y \in H_{d-1} \qquad u^\mu(\tau,y) = 0 \text{ at } {}^\forall\tau \text{ for } y \notin H_{d-1}$$



$$\tilde{g}^{AB} = \frac{1}{\lambda^2}\begin{pmatrix} -1, & N^b \\ N^a, & \lambda^2 B^{ab} \end{pmatrix} \qquad B^{ab} = g^{\mu\nu}\frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu}$$

A cofactor $\dfrac{\det B^{ab}}{\det \tilde{g}^{AB}} = \tilde{g}_{\tau\tau} = -1$ implies

$$b := \sqrt{\det B^{ab}} = \frac{1}{\sqrt{-\tilde{g}}} = -\frac{1}{(u\cdot n)\sqrt{h}}$$

Geometric current $J^\mu = bu^\mu$

Other representation (without derivation)

$$J^\mu = -\frac{1}{(d-1)!}\frac{1}{\sqrt{-\tilde{g}}}e^{\mu\alpha_1\cdots\alpha_{d-1}}\tilde{e}_{0a_1\cdots a_{d-1}}\partial_{\alpha_1}y^{a_1}\cdots\partial_{\alpha_{d-1}}y^{a_{d-1}}$$

Therefore $\nabla_\mu J^\mu = 0$ and $J^\mu \partial_\mu y^a = 0$. Furthermore we see $J^\mu \propto \dfrac{1}{\sqrt{-g}}$

3. Effective field theory for perfect fluids

We extend an argument in Dubovsky, Hui, Nicolis & Son 2012 to a curved spacetime.

Dynamical variable u^μ : fluid 4-velocity

$y^a(x)$: co-moving coordinate of fluids, $\psi(x)$: phase of a conserved quantity.

Symmetry

Poincare symmetry, volume preserving diffeo: $y^a \to f^a(y)$ with $\det(\partial_b f^a) = 1$

phase transformation: $\psi(x) \to \psi(x) + c$

low energy effective theory

derivative expansion

$$S = \int d^d x \sqrt{-g} F(b, z) \qquad b = \sqrt{\det B^{ab}}, \; z := u^\mu \partial_\mu \psi = \frac{J^\mu}{b} \partial_\mu \psi \qquad B^{ab} = g^{\mu\nu} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu}$$

$$J^\mu = b u^\mu$$

Conserved Noether current for $\psi(x) \to \psi(x) + c$: $N_1^\mu(x) := \dfrac{\delta S}{\delta \partial_\mu \psi} = n_1(x) u^\mu(x)$

charge density $n_1 = F_z := \partial_z F$

4. Entropy of perfect fluids

EMT $$T^{\mu\nu}(x) := \frac{2}{\sqrt{-\tilde{g}}}\frac{\partial S}{\partial g_{\mu\nu}(x)} = g^{\mu\nu}F + \frac{2F_z}{b}\frac{\partial(J^\mu\partial_\mu\psi)}{\partial g_{\mu\nu}} - 2(F_z z - F_b b)\frac{\partial b}{b\partial g_{\mu\nu}}$$

$$\frac{\partial(J^\mu\partial_\mu\psi)}{\partial g_{\mu\nu}} = -\frac{bz}{2}g^{\mu\nu} \qquad \frac{\partial b}{\partial g_{\mu\nu}} = -\frac{b}{2}g^{\mu\alpha}g^{\nu\beta}\underline{B_{ab}\partial_\alpha y^a\partial_\beta y^b} = -\frac{b}{2}(g^{\mu\nu} + u^\mu u^\nu)$$

$$= g_{\alpha\beta} + u_\alpha u_\beta$$

$$T^\mu{}_\nu = (F - F_b b)\delta^\mu_\nu + (F_z z - F_b b)u^\mu u_\nu$$

P $\qquad$ $\varepsilon + P$

$$P = F - F_b b$$
$$\varepsilon = F_z z - F$$

+ charge density

$$n_1 = F_z$$

$$d\varepsilon = z dn_1 - F_b db$$

compared with thermodynamic relation $d\varepsilon = Tds + \mu_1 dn_1$

$$s = b,\ T = -F_b,\ \mu_1 = z$$

μ_1 : chemical potential for n_1

b is an entropy density !

Other thermodynamics equation $\varepsilon + P - \mu_1 n_1 = -F_b b = Ts$ automatically follows.

V. Conclusion

Conclusion

1. Geometric conservation always holds in a curved spacetime.

conserved current $$J^{\mu}(\tau, y) = \frac{-u^{\mu}(\tau, y)}{n(\tau, y) \cdot u(\tau, y) \sqrt{h(\tau, y)}}$$

gravitational charge $$Q = \int_{H_{d-1}} d^{d-1} y$$

2. The geometric conserved charge is entropy for perfect fluids.

Interpretation

1. A source of gravity is "entropy", as the electric charge is the source of EM interaction.

   c.f. "Gravity is entropic force". T. Jacobson 1995, E.P. Verlinde 2011.

2. Through Einstein's equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$, the geometric conservation holds in spacetime . What is its meaning ?

Future studies

1. What is a physical interpretation of the geometric conservation for dissipative fluids ?

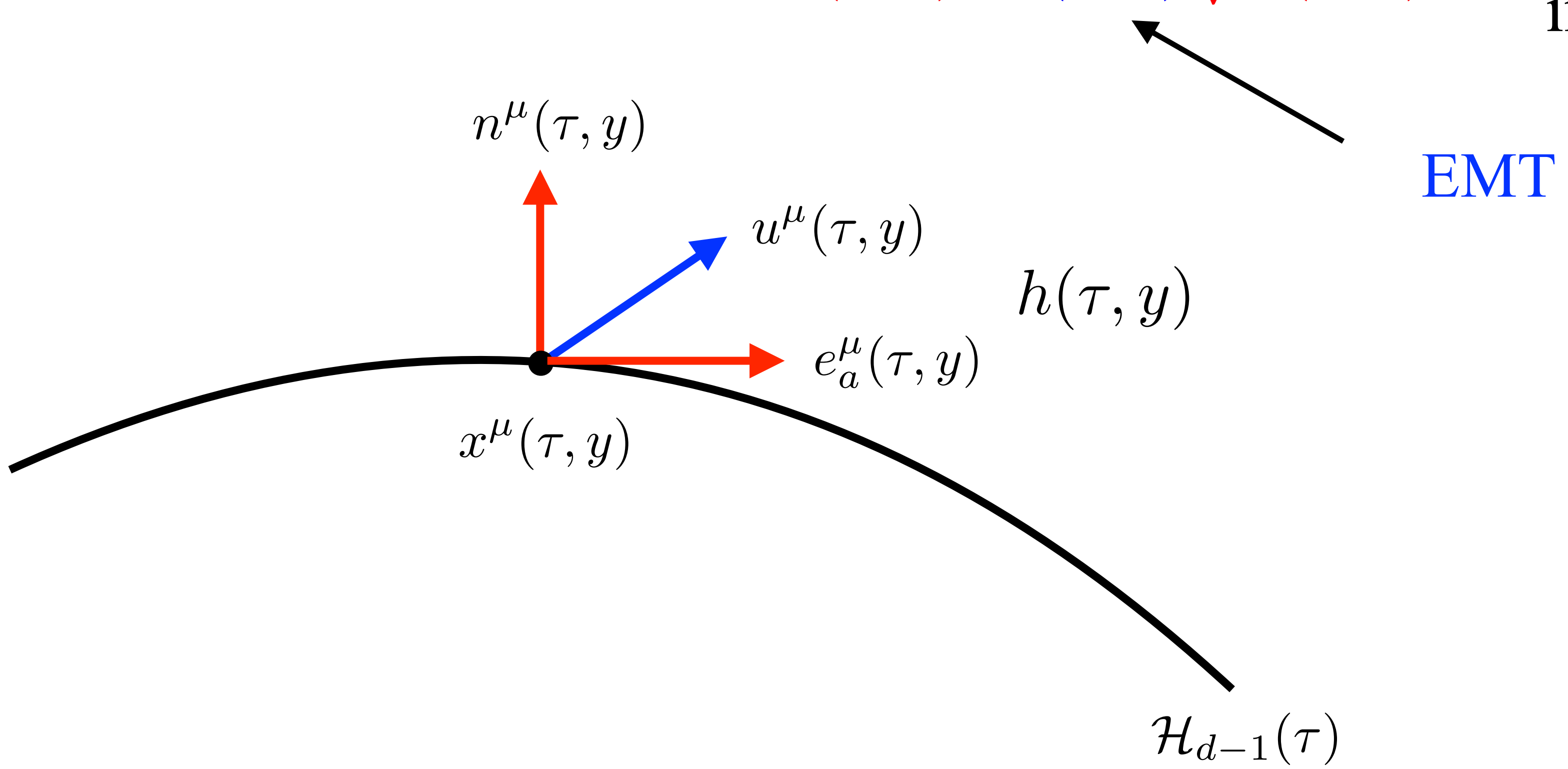
In gravitational systems, instead of energy, is entropy conserved ?

2. Applications of the geometric conservation.

A magic (universal) formula for "entropy" density

$$s(x(\tau,y)) = \frac{-1}{n(\tau,y)\cdot u(\tau,y)\sqrt{h(\tau,y)}}$$

Please calculate entropy density in your favorite spacetime.



Thank you for your attention.

Backup

B1. More general "3+1" decomposition

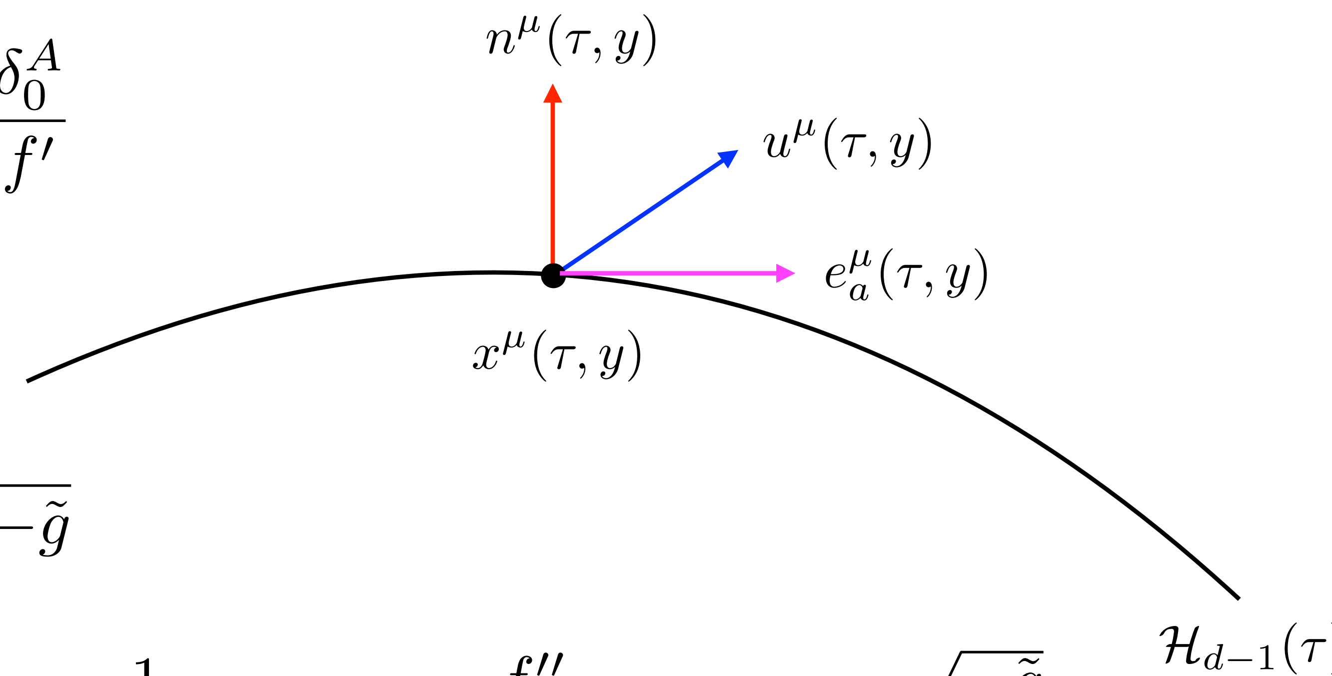
ADM decomposition $y^A = (y^0, y^a),\ \tau = f(y^0), f'(y^0) > 0$

$$ds^2 = -(f'\lambda)^2 (dy^0)^2 + h_{ab}(dy^a + f' N^a dy^0)(dy^b + f' N^b dy^0)$$

$$\tilde{g}_{AB} = \begin{pmatrix} -(f')^2, & f' N_b \\ f' N_a, & h_{ab} \end{pmatrix} \qquad \tilde{g}^{AB} = \frac{1}{(f'\lambda)^2} \begin{pmatrix} -1, & f' N^b \\ f' N^a, & (f'\lambda)^2 B^{ab} \end{pmatrix}$$

$$-\tilde{g} = (f'\lambda)^2 h \qquad \tilde{n}_A = -f'\lambda\, \delta^0_A \qquad \tilde{u}_A = \frac{\delta^A_0}{f'}$$

$$u \cdot n = \tilde{u} \cdot \tilde{n} = -\lambda$$



Calculation of $K = \nabla_\mu u^\mu$ $\quad K = \frac{1}{\sqrt{-\tilde{g}}} \mathcal{L}_u \sqrt{-\tilde{g}}$

$$\mathcal{L}_u \sqrt{-\tilde{g}} = \tilde{u}^A \partial_A \sqrt{-\tilde{g}} + \frac{d}{dt} \det \frac{\partial y^A}{\partial (y')^B}\bigg|_{t=1} = \frac{1}{f'} \partial_0 \sqrt{-g} - \frac{f''}{(f')^2} \sqrt{-\tilde{g}} = \partial_0 \frac{\sqrt{-\tilde{g}}}{f'}$$

$$y' = y - tu \qquad \frac{d}{dt} \det \left(\frac{\partial y^A}{\partial (y')^B} \right) = 1 - \frac{f''}{(f')^2}$$

$$K = \frac{1}{(u \cdot n)\sqrt{h} f'} \frac{\partial (u \cdot n)\sqrt{h}}{\partial y^0} = \partial_\tau \log[-(u \cdot n)\sqrt{h}]$$

Same result as before