Modular Average and Weyl Anomaly in Two-Dimensional Schwarzian Theory

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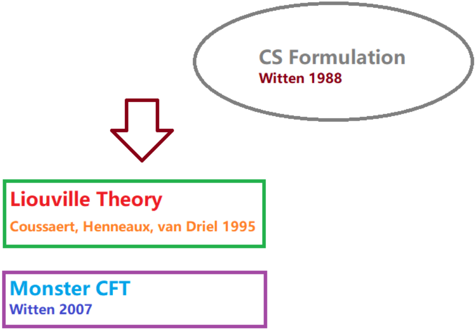
History


CS Formulation
Witten 1988
Liouville Theory
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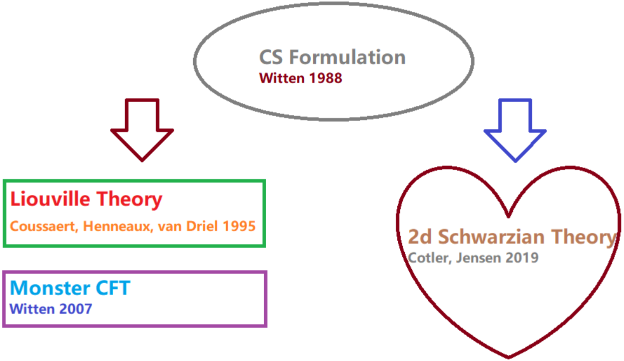
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2d Schwarzian Theory
Cotler, Jensen 2019

Gauge Formulation of Einstein Gravity Theory

- SL(2)×SL(2) Chern-Simons formulation

$$\frac{k}{2\pi}\int d^3x\ \mathrm{Tr}\left(A_t F_{r\theta} - \frac{1}{2}(A_r\partial_t A_\theta - A_\theta\partial_t A_r)\right)$$
$$-\frac{k}{2\pi}\int d^3x\ \mathrm{Tr}\left(\bar{A}_t \bar{F}_{r\theta} - \frac{1}{2}(\bar{A}_r\partial_t \bar{A}_\theta - \bar{A}_\theta\partial_t \bar{A}_r)\right)$$
$$-\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}\left(\frac{E_t^+}{E_\theta^+}A_\theta^2\right) + \frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}\left(\frac{E_t^-}{E_\theta^-}\bar{A}_\theta^2\right)$$

- $k \equiv l/(4G_3)$, $1/l^2 \equiv -\Lambda$, where Λ is a cosmological constant
- bulk metric $g_{\mu\nu} = 2\mathrm{Tr}(e_\mu e_\nu)$, where $A - \bar{A} \sim e$, $A + \bar{A} \sim \omega$
- boundary conditions for the torus:

$$(E_\theta^+ A_t - E_t^+ A_\theta)|_{r\to\infty} = 0;\ (E_\theta^- \bar{A}_t - E_t^- \bar{A}_\theta)|_{r\to\infty} = 0 \qquad (1)$$

Boundary Theory for Torus

- two-dimensional Schwarzian theory [Cotler, Jensen 2019]

$$\begin{aligned} & S_{\mathrm{Gb}} \\ = \; & \frac{k}{2\pi}\int dtd\theta\,\left(\frac{3}{2}\frac{(D_-\partial_\theta\mathcal{F})(\partial_\theta^2\mathcal{F})}{(\partial_\theta\mathcal{F})^2}-\frac{D_-\partial_\theta^2\mathcal{F}}{\partial_\theta\mathcal{F}}\right) \\ & -\frac{k}{2\pi}\int dtd\theta\,\left(\frac{3}{2}\frac{(D_+\partial_\theta\bar{\mathcal{F}})(\partial_\theta^2\bar{\mathcal{F}})}{(\partial_\theta\bar{F})^2}-\frac{D_+\partial_\theta^2\bar{\mathcal{F}}}{\partial_\theta\bar{\mathcal{F}}}\right), \end{aligned} \quad (2)$$

where

$$\mathcal{F}\equiv\frac{F}{E_\theta^+}, \qquad \bar{\mathcal{F}}\equiv\frac{\bar{F}}{E_\theta^-}; \quad (3)$$

$$D_+=\frac{1}{2}\partial_t-\frac{1}{2}\frac{E_t^-}{E_\theta^-}\partial_\theta, \qquad D_-=\frac{1}{2}\partial_t-\frac{1}{2}\frac{E_t^+}{E_\theta^+}\partial_\theta \quad (4)$$

- measure $\int d\mathcal{F}d\bar{\mathcal{F}}\,\left(1/(\partial_\theta\mathcal{F}\partial_\theta\bar{\mathcal{F}})\right)$

Liouville Theory

- consider a general Weyl transformation $E^{\pm} \to \exp(\sigma) E^{\pm}$ on a torus
- boundary conditions become:

$$\begin{aligned} (E^{+}_{\theta} A_t - E^{+}_{t} A_{\theta} + E^{+}_{t} A^2_{\theta} J_2)|_{r\to\infty} &= 0; \\ (E^{-}_{\theta} \bar{A}_t - E^{-}_{t} \bar{A}_{\theta} + E^{-}_{t} \bar{A}^2_{\theta} J_2)|_{r\to\infty} &= 0, \end{aligned} \tag{5}$$

where $A^2_{\theta} = \bar{A}^2_{\theta} = -\partial_t \sigma$

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- additional boundary term appears, and it is necessary for obtaining the Liouville theory not the same as in the claim of Ref. [Cotler, Jensen 2019]

$$-\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}(A_\theta A_\theta + \bar{A}_\theta\bar{A}_\theta) + \frac{k}{8\pi}\int dtd\theta\ (A^2_\theta A^2_\theta + \bar{A}^2_\theta\bar{A}^2_\theta) \tag{6}$$

Application

- implement the averaging of a modular group to compute the Rényi-2 mutual information for disjoint two-intervals
- non-perturbative effect kills the phase transition

Modular Averaging

- partition function of 2d Schwarzian theory on a torus is [Cotler, Jensen 2019]

$$Z_T(\tau) = |q|^{-\frac{c}{12}} \frac{1}{\prod_{n=2}^{\infty} |1-q^n|^2}, \quad q \equiv e^{2\pi i \tau}, \quad c = \frac{3}{2G_3} + 13, \quad (7)$$

  where τ is a complex structure

- summation is over the $SL(2, \mathbb{Z})$ group (or modular group) in the boundary theory, which is equivalent to the path integration for all asymptotic AdS_3 boundary [Maloney, Witten 2010]

$$Z_M(\tau) = \sum_{c_1, d_1;\ (c_1, d_1)=1} Z_T\left(\frac{a_1\tau + b_1}{c_1\tau + d_1}\right), \quad (8)$$

  where $a_1 d_1 - b_1 c_1 = 1, \ a_1, b_1, c_1, d_1 \in \mathbb{Z}$

Rényi-2 Mutual Information

- Rényi entropy of order q is

$$R^{(q)} \equiv \frac{\ln \mathrm{Tr}\rho^q}{1-q} = \frac{\ln Z^{(q)} - q \ln Z^{(1)}}{1-q}, \tag{9}$$

where ρ is a reduced density matrix, and $Z^{(q)}$ is a q-sheet partition function

- Rényi mutual information of order q is

$$I^{(q)}_{[u_1,v_1],[u_2,v_2]} = R^{(q)}_{[u_1,v_1]} + R^{(q)}_{[u_2,v_2]} - R^{(q)}_{[u_1,v_1]\cup[u_2,v_2]} \tag{10}$$

- Rényi-2 mutual information is [Headrick 2010]

$$I^{(2)} = \ln Z_T(\tau) + c \ln\left(\frac{\theta_2(\tau)}{2\eta(\tau)}\right) \tag{11}$$

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- modular averaging is equivalent to replacing Z_T with Z_M

Rényi-2 Mutual Information

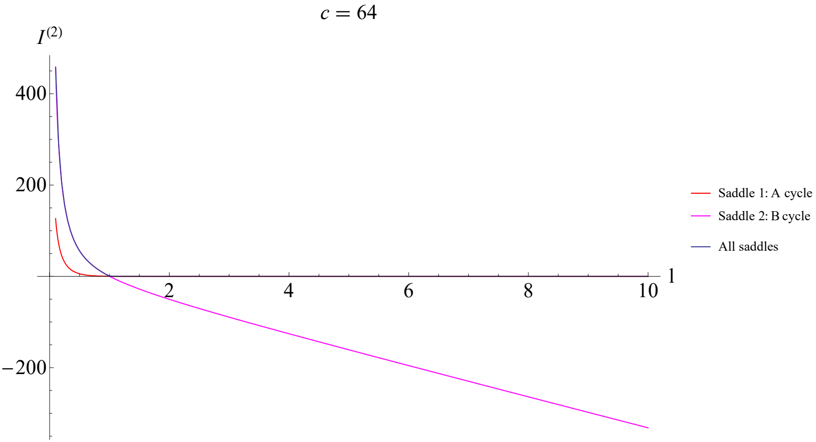

Figure: We show the Rényi-2 mutual information from summing all saddle points, A-cycle, and B-cycle with $c = 64$. Not all saddle points decay to zero when $l \equiv -i\tau \to \infty$ (like B-cycle), where τ is a complex structure.

Logarithmic Partition Function

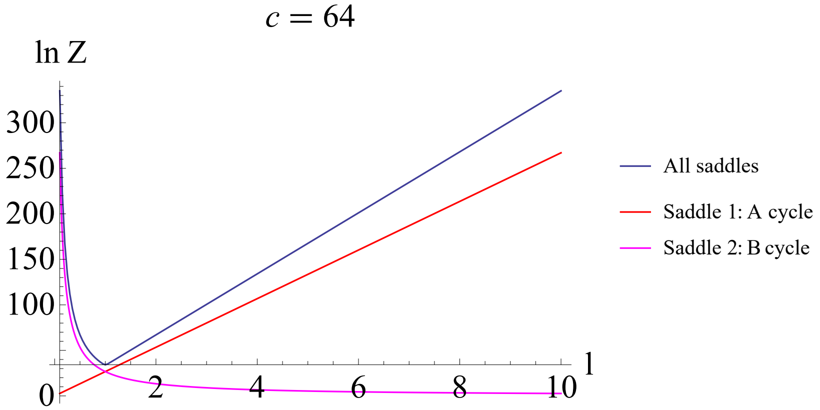

Figure: The logarithmic partition function is continuous for l.

First-Order Derivative

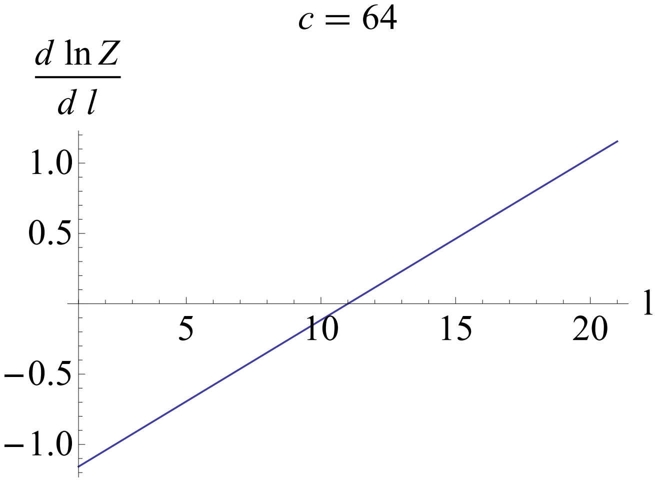

Figure: The first-order derivative of the logarithmic partition function is a continuous function for l.

Discussion and Conclusion

- agrees with the Liouville Theory for a torus manifold

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- implement the averaging of a modular group to compute the Rényi-2 mutual information for disjoint two-intervals
- non-perturbative effect kills the phase transition
- boundary theory=resummation of perturbative gravity

Thank you!

Cylinder→Sphere

- respects the result from the Cylinder manifold to Sphere Manifold

$$e^a \to \exp\big(\sigma(t)\big)e^a = \mathrm{sech}(\psi)e^a \tag{12}$$

but additional finite term appears

$$\frac{k}{2\pi}\int_{-\infty}^{\infty} d\psi \int_0^{2\pi} d\theta\ \partial_\psi^2\sigma = -2k \tag{13}$$