1st Mar. 2024. GC2024
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PHOTON REGION IN STATIONARY CYLINDRICALLY SYMMETRIC SPACETIMES

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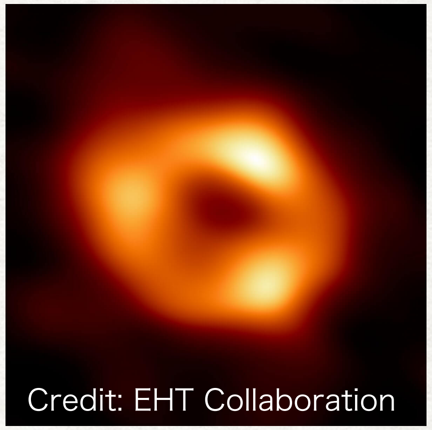
PHOTON SPHERE

- Collection of unstable circular photon orbits in static spherically symmetric spacetimes

- It plays an important role in gravitational physics

  e.g. [K. S. Virbhadra & G. F. R. Ellis (2000)] [V. Cardoso+ (2016)]



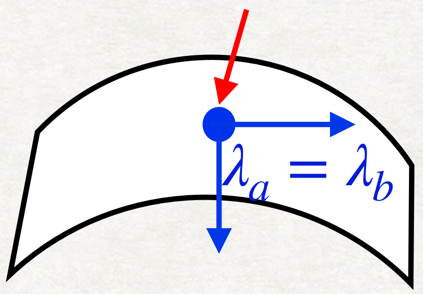
Generalize → **photon surface** …etc.

[C. M. Claudel + (2001)]

- Totally umbilical submanifold of Lorentzian manifold

  [C. M. Claudel + (2001)], [V. Perlick (2005)]

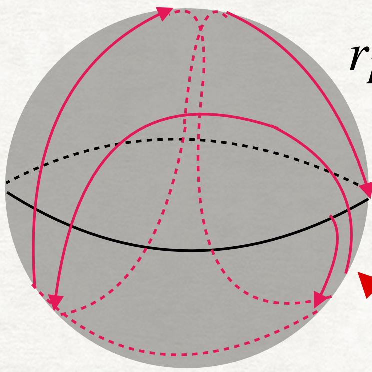
  - Second fundamental form: $h(X, Y) = H\hat{g}(X, Y), \ \ \forall X, Y \in TS$



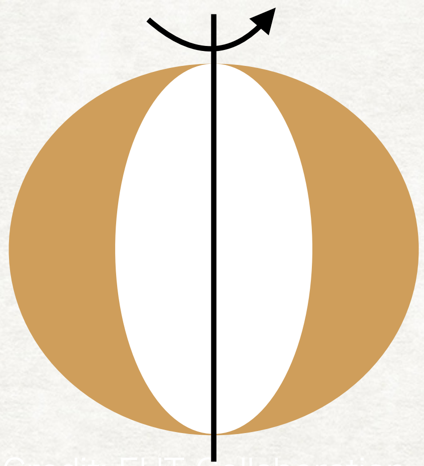
- Uniqueness theorem of static photon surfaces

  [C. Cederbaum (2014)], [S. Yazadjiev (2015)], [H. Yoshino (2017)], [Y. Tomikawa + (2017)]…etc.

PHOTON REGION (PR)



- Set of spherical photon orbits (SPOs)
  [E. Teo (2003)]



- Edge of black hole (BH) shadow
  (if SPOs are unstable)
  [A. Grenzebach + (2014), (2015)]

- Smooth submanifold of tangent bundle (of non-extremal Kerr)
  [C. Cederbaum & S. Jahns (2019)]

Generalize → **Fundamental photon region**

[K. V. Kobialko & D. V. Gal'tsov (2020)]

in any stationary axisymmetric spacetimes

- Uniqueness of photon region? -> No

∵. No appropriate definition in a stationary spacetime

CYLINDRICALLY SYMMETRIC (CS) SPACETIMES

- Two spacelike Kiiling vector fields
  - For the definition of cylindrical symmetry, see [Carot +(1999)]
- Have been used to explain some astrophysical phenomena
  - Cosmic string, jets, ···etc
- Have been used to understand some fundamental issues
  - Definition of the energy, hoop conjecture ···etc
- Geodesic structure
  - Circular null geodesics in Levi-Civita spacetime, Lewis spacetime,···etc.

    e.g. [K. A. Bronnikov + (2020)]
  - Photon cylinder(totally umbilical hypersurface) in Melvin universe

    [G. W. Gibbons & C. M. Warnick (2016)]

Our goal: To understand PR with different topology

First step: we define PR in stationary CS spacetimes

NOTATIONS

$(c = G = 1)$

- (M, g): 4-dimensional Lorentzian manifold with

$$g = -N(\rho)^2 dt^2 + g_{\phi\phi}(\rho)(d\phi - \omega(\rho)dt)^2 + g_{zz}(\rho)dz^2 + g_{\rho\rho}(\rho)d\rho^2, \quad z \in (-\infty . \infty), \phi \in [0,2\pi)$$

  - $N = \sqrt{-g_{tt} + \omega^2 g_{\phi\phi}}$: lapse function, $\omega = -g_{t\phi}/g_{\phi\phi}$: angular velocity of ZAMO
  - Note that this spacetime has the reflection symmetry $\phi \to -\phi$ and $z \to -z$

Def.) Timelike hypersurface S of M is called **invariant** if the Killing vector fields ∂_t , ∂_ϕ and ∂_z are tangent vector fields to S

- Σ_t: 3-dimensional spacelike hypersurface,

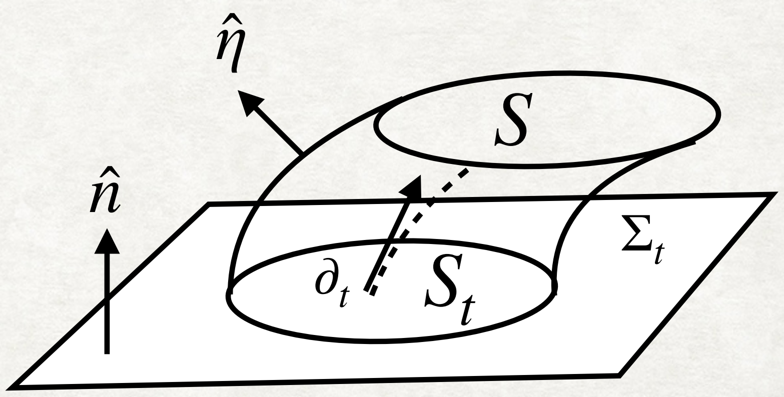
  S: Invariant Timelike hypersurface

  $S_t := S \cap \Sigma_t$,

  $\hat{n} = N^{-1}[\partial_t + \omega\partial_\phi]$: unit normal vector field to Σ_t,

  $\hat{\eta} = g_{\rho\rho}^{-1/2}\partial_\rho$: unit normal vector field to S

  $\nabla, \hat{\nabla}$: Levi-Civita connections on M and S, resp.



SECOND FUNDAMENTAL FORM

Def.) $h(X,Y) := g(\nabla_X Y, \hat{\eta}) = -g(\nabla_x \hat{\eta}, Y),\ \forall X, Y \in TS$

$$\longrightarrow \nabla_X Y = \underbrace{\hat{\nabla}_X Y}_{\text{Tangent to S}} + \underbrace{h(X,Y)\hat{\eta}}_{\text{Normal to S}}$$

Tangent to S   Normal to S

cf. [B. O'Neill (1983)]

PARTIALLY UMBILICAL HYPERSURFACE

- Let $V \subset TS$ (TS is a tangent bundle over S) and $\hat{g}$ the induced metric on S.

Def.) A point $p \in S$ is called a V-umbilic point of S if for some $H_p \in \mathbb{R}$,

$$h(X_p, Y_p) = H_p \hat{g}(X_p, Y_p),\ \forall X_p, Y_p \in V_p.$$

[K. V. Kobialko & D. V. Gal'tsov (2020)]

- We will call $\tilde{S} = \{p \in S \mid V\text{-umbilic points of } S\}$ the partially umbilical hypersurface of S
  - If V=TS, then S is a totally umbilical hypersurface

V-UMBILIC EQUATIONS

($\cdot$: derivative with respect to affine parameter)

- Let γ be a null geodesic on $\tilde{S}$ and $\dot{\gamma}$ its tangent vector

$\because \dot{\gamma}$ is a null vector

$$\longrightarrow \nabla_{\dot{\gamma}}\dot{\gamma} = \underbrace{\hat{\nabla}_{\dot{\gamma}}\dot{\gamma}}_{=0} + h(\dot{\gamma},\dot{\gamma})\hat{\eta} = H_{\tilde{S}}\,\hat{g}(\dot{\gamma},\dot{\gamma}) = 0 \quad \Longleftrightarrow \quad \sigma(\dot{\gamma},\dot{\gamma}) := h(\dot{\gamma},\dot{\gamma}) - \frac{H}{3}\hat{g}(\dot{\gamma},\dot{\gamma}) = 0$$

Def. of V-umbilic point

Trace-free part of h

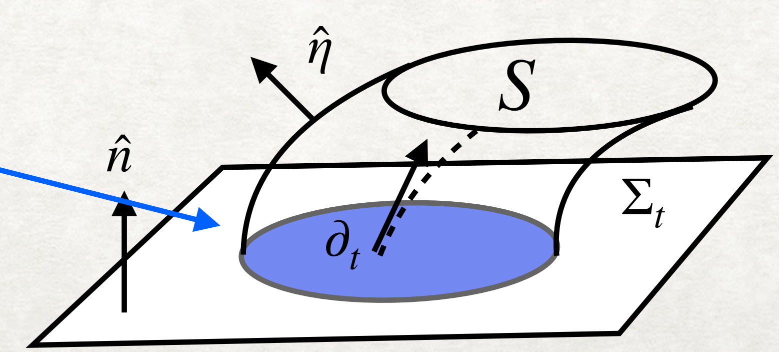
- In the orthogonal basis $\left\{\hat{n},\ \hat{\phi} = g_{\phi\phi}^{-1/2}\partial_\phi,\ \hat{z} = g_{zz}^{-1/2}\partial_z\right\}$,

$$(\lambda_\phi - \lambda_z)\xi^2 + 2v\xi + (\lambda_z - \lambda_n) = 0, \quad \xi^2 := \frac{N^2L_E^2}{(1-\omega L_E)^2 g_{\phi\phi}} \leq 1$$

$\because\ 0 = g(\dot{\gamma},\dot{\gamma}) \iff g_{\rho\rho}\dot{\rho}^2 + g_{zz}\dot{z}^2 = N^2\dot{t}^2 - g_{\phi\phi}(\dot{\phi} - \omega\dot{t})^2 \geq 0$

$\lambda_\phi = h(\hat{\phi},\hat{\phi}),\ \lambda_z := h(\hat{z},\hat{z}),\ \lambda_n := -\dfrac{\hat{\eta}(N)}{N}$, and $v := \sqrt{g_{\phi\phi}}\dfrac{\hat{\eta}(\omega)}{2N}$

Principal curvatures of $S_t = S \cap \Sigma_t$



where $E := -g(\dot{\gamma},\partial_t)$, $L_z := g(\dot{\gamma},\partial_\phi)$ and $L_E := L_z/E$

$$(\lambda_\phi - \lambda_z)\xi^2 + 2v\xi + (\lambda_z - \lambda_n) = 0, \quad \xi^2 := \frac{N^2 L_E^2}{(1-\omega L_E)^2 g_{\phi\phi}} \leq 1$$

$$\Longrightarrow \quad L_E = \frac{\sqrt{g_{\phi\phi}}\,\xi_{sol}}{N \pm \sqrt{g_{\phi\phi}}\,\omega\xi_{sol}}, \quad P_E^2 = g_{zz}\frac{(1-\omega L_E)^2(1-\xi_{sol})^2}{N^2}$$

where $P_z := g(\dot{\gamma}, \partial_z)$ and $P_E := P_z/E$

- Null geodesics with $L_E = 0$ $(\xi = 0)$ $\Longrightarrow \lambda_z = \lambda_n$
- Circular null geodesics: $\xi = \pm 1 \Longrightarrow P_E = 0, \quad \lambda_\phi - \lambda_n \pm 2v = 0$
- In the static case

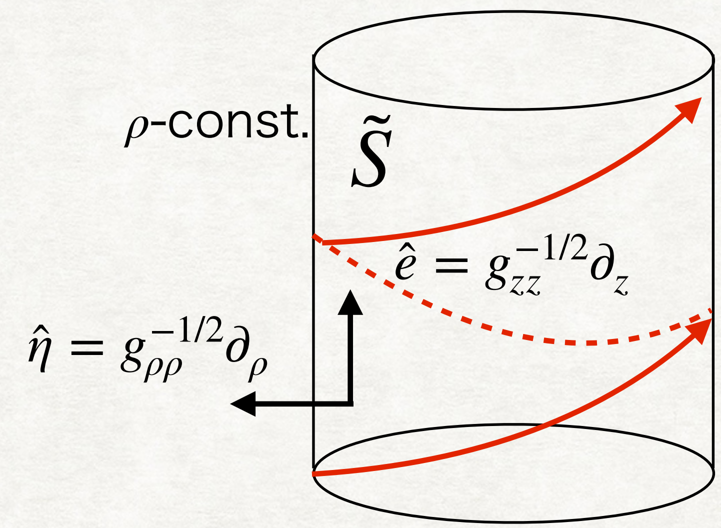
$$\omega = 0 \Longrightarrow (\lambda_z - \lambda_\phi)(g_{tt}L_E)^2/g_{\phi\phi} + (\lambda_z - \lambda_t) = 0$$

  - $\lambda_z \neq \lambda_\phi \Longrightarrow L_E^2 = \dfrac{g_{\phi\phi}(\lambda_t - \lambda_z)}{g_{tt}(\lambda_\phi - \lambda_z)}, \quad P_E^2 = \dfrac{g_{zz}(\lambda_t - \lambda_\phi)}{g_{tt}(\lambda_z - \lambda_\phi)}$
  - $\lambda_z = \lambda_\phi \Longrightarrow \lambda_z = \lambda_\phi = \lambda_t$

S is a totally umbilical hypersurface $\Longleftrightarrow$ photon surface

- A null geodesic that starts tangential to $\tilde{S}$ remains within $\tilde{S}$

$$\because \quad \nabla_{\dot\gamma}\dot\gamma = \hat\nabla_{\dot\gamma}\dot\gamma + h(\dot\gamma,\dot\gamma)\hat\eta = 0$$



- We call a null geodesic with (L_E, P_E) a **helical photon orbit**, where

$$L_E = \frac{\sqrt{g_{\phi\phi}}\xi_{sol}}{N \pm \sqrt{g_{\phi\phi}}\omega\xi_{sol}}, \quad P_E^2 = g_{zz}\frac{(1-\omega L_E)^2(1-\xi_{sol})^2}{N^2}$$

↑ corresponds to a spherical photon orbit in Kerr spacetime

- We define the **photon region** in a stationary cylindrically symmetric spacetime as the **set of helical photon orbits**

- Metric: [J. P. S Lemos & V. T. Zanchin (1995)] $z \in (-\infty . \infty), \phi \in [0,2\pi)$

$$g = -A(\rho)dt^2 - \frac{16J}{3\alpha\rho}dtd\phi + \left(\rho^2 + \frac{4M(1-\Omega)}{\alpha^3\rho}\right)d\phi^2 + \alpha^2\rho^2 dz^2 + \frac{d\rho^2}{B(\rho)},$$

  - $A(\rho) = \alpha^2\rho^2 - \dfrac{2M(1+\Omega)}{\alpha\rho},$
  - $B(\rho) = \alpha^2\rho^2 - \dfrac{2M(3\Omega-1)}{\alpha\rho}.$

M, J: mass and angular momentum per unit length of the axis, resp.,

$\alpha^2 = -\Lambda/3 > 0, \ \ \Omega := \sqrt{1-(8J^2\alpha^2)/(9M^2)}$

- Horizon: $\{\rho \,|\, B(\rho) = 0\}$ (if $0 < \alpha^2 J^2 M^{-2} < 1$)

- Conditions for existence of a photon region

$\iff$ $(\lambda_z - \lambda_\phi)\xi^2 + 2v\xi + (\lambda_z - \lambda_{\hat{n}}) = 0$ has one or two solutions $\xi_{sol} \in [-1.1]$

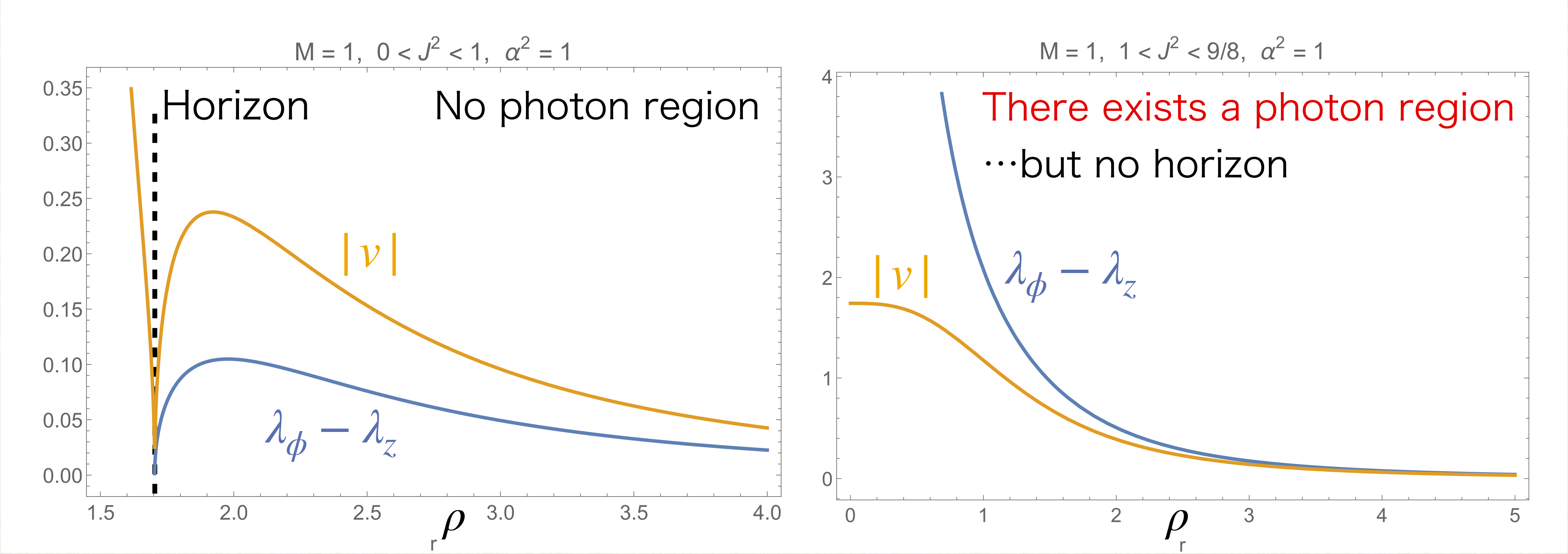
$\iff$ $v^2 \geq (\lambda_\phi - \lambda_z)(\lambda_z - \lambda_{\hat{n}})$ and $\begin{pmatrix} \lambda_n - \lambda_\phi \leq 2|v| \leq 2(\lambda_\phi - \lambda_z) \text{ if } \lambda_\phi > \lambda_z \\ \lambda_n - \lambda_\phi \geq 2|v| \leq 2(\lambda_\phi - \lambda_z) \text{ if } \lambda_\phi < \lambda_z \end{pmatrix.$

- Result

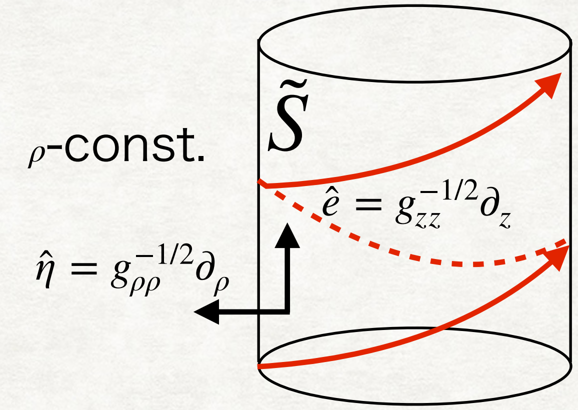
$$\lambda_z - \lambda_n = \frac{3M(1+\Omega)}{\alpha g_{33}\sqrt{B(\rho)}}, \quad \lambda_\phi - \lambda_z = \frac{6M(1-\Omega)}{\alpha^3 \rho^2 g_{33}}\sqrt{B(\rho)} > 0, \quad v = \frac{4J}{\alpha \rho g_{33}}$$

$$\Rightarrow \quad v^2 - (\lambda_\phi - \lambda_z)(\lambda_z - \lambda_{\hat{n}}) = 0$$

We need to check $|\xi_{sol}| = \left|-v/(\lambda_\phi - \lambda_z)\right| < 1 \iff |v| < \lambda_\phi - \lambda_z$



SUMMARY



- Partially umbilical hypersurface of $S \subset M$:

  $\tilde{S} = \{p \in S \mid V\text{-umbilic points of } S\}$

- Helical photon orbit is a null geodesic that stays on $\tilde{S}$

- PR in stationary CS spacetimes
  $\Longleftrightarrow$ the set of helical photon orbits

FUTURE WORKS

- Relaxing the definition of an invariant hypersurface
  ⟶ There are other partially umbilical hypersurfaces

- Does the partially umbilical hypersurface have a physical meaning?

  e.g) the centrifugal force vanishes on a photon sphere

  [W. Hasse & V. Perlick (2002)]

- Stability in terms of geometrical quantity?