Note on the TRG calculation for $N_f = 2$ Schwinger model

Hayato Kanno[a]

[a] *RIKEN BNL Research Center,*
Brookhaven National Laboratory, Upton, NY 11973, USA

Abstract

This is the note for the TRG calculation and to construct the initial tensor for $N_f = 2$ Schwinger model. This note is written for this research [1].

last update : 2025.09.25

Contents

1 Introduction

We will focus on the Schwinger model, which is 2-dimensional QED ($U(1)$ gauge theory with fermionic matter). This theory has the similar IR phase structure to 4-dimensional QCD, because the global symmetry and its confinement are very similar. So this is a good toy model to understand the IR phase structure of 4-dimensional QCD.

In this note, we focus on the Schwinger model which has massive N_f Dirac fermions as the matter, and $N_f \geq 2$. We are interested in the parameter space of the fermion mass. The massless case, this theory becomes the CFT ($SU(N_f)_1$ WZW model) in the IR limit. Massive limit is also well-known, because this limit corresponds to the 2-dimensional Maxwell theory ($U(1)$ gauge theory) and it has no propagating degrees of freedom. However, finite fermion mass case is difficult to solve. Small mass case is understood by the bosonized theory with the mass perturbation, but there is no way to understand the finite mass case.

The Schwinger model with finite mass is already studied well, by the Monte Carlo simulation. However, Massive Schwinger model with θ parameter is hard to calculate by the Monte Carlo simulation. We will use the Tensor RenormalizationGroup method (in short, TRG) to understand the finite θ parameter case. We Will start from the lattice action with a staggered fermion and the U(1) gauge field.

2 Review for the Schwinger model

In this note, we study $N_f = 2$ Schwinger model with θ parameter.

$$\begin{aligned} S &= \int d^2x \left\{ \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \bar{\psi}i\not{D}\psi + m\bar{\psi}\psi \right\} \\ &= \int d^2x \left\{ \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \bar{\psi}i\gamma^\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi \right\} \end{aligned} \tag{2.1}$$

2.1 Bosonization and mass perturbation

$$\begin{aligned} S = &\int d^2x \left\{ \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{i}{2\pi}\left(-i\mathrm{tr}[\log U] - \theta\right)\epsilon^{\mu\nu}F_{\mu\nu} + \mathrm{tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] + \mathrm{tr}[mU + U^\dagger m^\dagger] \right\} \\ &+ \int \mathrm{tr}[(UdU^\dagger)^3] \end{aligned}$$

$$\begin{aligned} U =& \exp\left(i\pi(x)\right) \in U(N_f) \\ \pi(x) =& \eta' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \pi^0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \pi^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \pi^- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \eta' =& -i\mathrm{tr}[\log U] = -i\log\det U \end{aligned} \tag{2.2}$$

2.2 Known facts for the θ dependence

3 The formulation on the lattice

In this section, we create the lattice action and the initial tensor for the Schwinger model. We use the staggered fermion for the lattice fermion.

3.1 Lattice action

We use the 2-dim Euclidean action.

$$\begin{aligned} S &= \int d^2x \left\{ \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \bar{\psi} i \not{D} \psi + m\bar{\psi}\psi \right\} \\ &= \int d^2x \left\{ \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \bar{\psi} i \gamma^\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi \right\} \end{aligned} \tag{3.1}$$

We discretize the fermions by the staggered action, so (3.1) is discretized as below.

$$\begin{aligned} S = \sum_{n,\mu} \Bigg[&- \frac{1}{g^2}\cos(A_p(n)) - \frac{i\theta}{2\pi}\tilde{A}_p(n) \\ &+ \frac{1}{2}\Big[\eta_\mu(n) \big\{ \bar{\chi}(n) U_\mu(n) \chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu}) U_\mu^\dagger(n) \chi(n) \big\} + m\bar{\chi}(n)\chi(n) \Big] \Bigg] \end{aligned} \tag{3.2}$$

Where,

$$\begin{aligned} A_p(n) =& A_1(n) + A_2(n+\hat{1}) - A_1(n+\hat{2}) - A_2(n), \\ \tilde{A}_p(n) \equiv& A_p(n) \qquad\qquad (\mathrm{mod}\ 2\pi), \end{aligned} \tag{3.3}$$

$$\begin{aligned} -\pi \le& A_\mu(n) < \pi, \\ -4\pi \le& A_p(n) < 4\pi, \\ -\pi \le& \tilde{A}_p(n) < \pi. \end{aligned} \tag{3.4}$$

$U_\mu(n) = \exp(iA_\mu(n)) \in U(1)$, $\chi(n)$ and $\bar{\chi}(n)$ are Grassmannian valued fields and they are one component complex valued fields. θ is a 2π periodic parameter, and $0 \le \theta < 2\pi$. $\eta_\mu(n)$ is a staggered phase;

$$\begin{aligned} \eta_1(n_x, n_y) =& +1, \\ \eta_2(n_x, n_y) =& -1 \qquad (\text{where } n_x \text{ is an odd number}), \\ \eta_2(n_x, n_y) =& +1 \qquad (\text{where } n_x \text{ is an even number}). \end{aligned} \tag{3.5}$$

$\chi(n)$ is the staggered fermion. This theory is 2dim theory, so this one component complex valued field corresponds to two Dirac fermions in the continuous limit. Because of the staggered phase η_μ, this lattice action has two-site shift symmety. So 2×2 sites correspond to the local degrees of freedom in the continuous theory. In continuous theory, 2dim Dirac fields has two component complex valued field. Then, this lattice fermion which includes four complex degrees of freedom each site correspond to two

Dirac fermions in continuous limit. Therefore, this lattice theory goes to $N_f = 2$ Schwinger model in the continuous limit. This is the starting point to us.

In the following subsections, we derive the continuous action (3.1) from the lattice action (3.2). We will see the fermion part in section 3.2, and the gauge part in section 3.3.

3.2 Fermion part

The lattice action is below.

$$S = \sum_{n,\mu} \frac{1}{2}\Big[\eta_\mu(n)\big\{\bar{\chi}(n)\chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu})\chi(n)\big\} + m\bar{\chi}(n)\chi(n)\Big] \tag{3.6}$$

First, we rewrite the staggered phase η_μ explicitly. We introduce the "large lattice coordinate", written as N, and rewrite the action (3.6). The definition is,

$$\begin{aligned} N =& \frac{n}{2} \qquad\qquad\qquad\qquad\qquad\qquad (\text{for } n = (2x, 2y)) \\ N \ni& \{n, n+\hat{1}, n+\hat{2}, n+\hat{1}+\hat{2}\}. \end{aligned}$$

We also define the fermions λ on the large lattice,

$$\lambda_{11}(N) = \chi(n), \lambda_{12}(N) = \chi(n+\hat{1}), \lambda_{21}(N) = \chi(n+\hat{2}), \lambda_{22}(N) = \chi(n+\hat{1}+\hat{2}) \tag{3.7}$$

$$\begin{array}{ccccc} & (\hat{2}) & & & \\ & | & & | & \\ - & \lambda_{21} & - & \lambda_{22} & - \\ & | & & | & \\ - & \lambda_{11} & - & \lambda_{12} & -(\hat{1}) \\ & | & & | & \end{array} \tag{3.8}$$

Then, the action (3.6) becomes,

$$\begin{aligned} S =\sum_N \frac{1}{2}\Big[& \bar{\lambda}_{11}(N)\lambda_{12}(N) - \bar{\lambda}_{12}(N)\lambda_{11}(N) + \bar{\lambda}_{11}(N)\lambda_{21}(N) - \bar{\lambda}_{21}(N)\lambda_{11}(N) \\ &- \big(\bar{\lambda}_{12}(N)\lambda_{22}(N) - \bar{\lambda}_{22}(N)\lambda_{12}(N)\big) + \bar{\lambda}_{21}(N)\lambda_{22}(N) - \bar{\lambda}_{22}(N)\lambda_{21}(N) \\ &+ \bar{\lambda}_{12}(N)\lambda_{11}(N+\hat{1}) - \bar{\lambda}_{11}(N+\hat{1})\lambda_{12}(N) + \bar{\lambda}_{21}(N)\lambda_{11}(N+\hat{2}) - \bar{\lambda}_{11}(N+\hat{2})\lambda_{21}(N) \\ &+ \bar{\lambda}_{22}(N)\lambda_{21}(N+\hat{1}) - \bar{\lambda}_{21}(N+\hat{1})\lambda_{22}(N) - \big(\bar{\lambda}_{22}(N)\lambda_{12}(N+\hat{2}) - \bar{\lambda}_{12}(N+\hat{2})\lambda_{22}(N)\big) \\ &+ m\big(\bar{\lambda}_{11}(N)\lambda_{11}(N) + \bar{\lambda}_{12}(N)\lambda_{12}(N) + \bar{\lambda}_{21}(N)\lambda_{21}(N) + \bar{\lambda}_{22}(N)\lambda_{22}(N)\big)\Big]. \end{aligned} \tag{3.9}$$

To understand them as Dirac fermions, we need to introduce one more representation of the fermions $\psi(N)$. The definition of the ψ is,

$$\begin{aligned} \lambda_{11} &= \psi_{11} + \psi_{22}, & \lambda_{12} &= \psi_{12} + \psi_{21}, \\ \lambda_{21} &= i\psi_{12} - i\psi_{21}, & \lambda_{22} &= -i\psi_{11} + i\psi_{22}. \end{aligned} \tag{3.10}$$

$$\begin{aligned} \bar{\lambda}_{11} &= \bar{\psi}_{11} + \bar{\psi}_{22}, & \bar{\lambda}_{12} &= \bar{\psi}_{12} + \bar{\psi}_{21}, \\ \bar{\lambda}_{21} &= -i\bar{\psi}_{12} + i\bar{\psi}_{21}, & \bar{\lambda}_{22} &= i\bar{\psi}_{11} - i\bar{\psi}_{22}. \end{aligned} \tag{3.11}$$

This definition leads γ matrices. We need to use the gamma matrices to rewrite fermions here. We use the chiral representation for the 2-dimensional γ matirx. We use all plus metric for the Euclidean spacetime, $\eta_{\mu\nu} = \mathrm{diag}(1, 1)$. Then, the γ matrices are,

$$\gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.12}$$

The definition of this γ matrices are determined by the definition of the ψ fields in (3.10) and (3.11). The general translation between λ fields and ψ fields are,[2]

$$\begin{aligned} \psi_{ij}(N) &= \frac{1}{2}\left(\chi(n)\mathbb{I}_{ij} + \chi(n+\hat{1})\left(\gamma_1\right)_{ij} + \chi(n+\hat{2})\left(\gamma_2\right)_{ij} + \chi(n+\hat{1}+\hat{2})\left(\gamma_1\gamma_2\right)_{ij}\right), \\ \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} \lambda_{11} + i\lambda_{22} & \lambda_{12} - i\lambda_{21} \\ \lambda_{12} + i\lambda_{21} & \lambda_{11} - i\lambda_{22} \end{pmatrix}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \bar{\psi}_{ji}(N) &= \frac{1}{2}\left(\bar{\chi}(n)\mathbb{I}_{ij} + \bar{\chi}(n+\hat{1})\left(\gamma_1\right)^{\dagger}_{ij} + \bar{\chi}(n+\hat{2})\left(\gamma_2\right)^{\dagger}_{ij} + \bar{\chi}(n+\hat{1}+\hat{2})\left(\gamma_1\gamma_2\right)^{\dagger}_{ij}\right), \\ \begin{pmatrix} \bar{\psi}_{11} & \bar{\psi}_{21} \\ \bar{\psi}_{12} & \bar{\psi}_{22} \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} \bar{\lambda}_{11} - i\bar{\lambda}_{22} & \bar{\lambda}_{12} - i\bar{\lambda}_{21} \\ \bar{\lambda}_{12} + i\bar{\lambda}_{21} & \bar{\lambda}_{11} + i\bar{\lambda}_{22} \end{pmatrix}, \end{aligned} \tag{3.14}$$

where $\mathbb{I}$ denotes 2×2 unit matrix.

Then we rewite the action again, as follows.

$$
\begin{aligned}
S =\sum_N \Bigg[& \bar{\psi_{11}}(N)\frac{1}{2}\big(\psi_{21}(N+\hat{1})-\psi_{21}(N-\hat{1})\big)+\bar{\psi_{21}}(N)\frac{1}{2}\big(\psi_{11}(N+\hat{1})-\psi_{11}(N-\hat{1})\big)\\
&+\bar{\psi_{12}}(N)\frac{1}{2}\big(\psi_{22}(N+\hat{1})-\psi_{22}(N-\hat{1})\big)+\bar{\psi_{22}}(N)\frac{1}{2}\big(\psi_{12}(N+\hat{1})-\psi_{12}(N-\hat{1})\big)\\
&-i\bar{\psi_{11}}(N)\frac{1}{2}\big(\psi_{21}(N+\hat{2})-\psi_{21}(N-\hat{2})\big)+i\bar{\psi_{21}}(N)\frac{1}{2}\big(\psi_{11}(N+\hat{2})-\psi_{11}(N-\hat{2})\big)\\
&-i\bar{\psi_{12}}(N)\frac{1}{2}\big(\psi_{22}(N+\hat{2})-\psi_{22}(N-\hat{2})\big)+i\bar{\psi_{22}}(N)\frac{1}{2}\big(\psi_{12}(N+\hat{2})-\psi_{12}(N-\hat{2})\big)\\
&-\bar{\psi_{22}}(N)\frac{1}{2}\big(\psi_{21}(N+\hat{1})+\psi_{21}(N-\hat{1})-2\psi_{21}(N)\big)\\
&+\bar{\psi_{12}}(N)\frac{1}{2}\big(\psi_{11}(N+\hat{1})+\psi_{11}(N-\hat{1})-2\psi_{11}(N)\big)\\
&+\bar{\psi_{21}}(N)\frac{1}{2}\big(\psi_{22}(N+\hat{1})+\psi_{22}(N-\hat{1})-2\psi_{22}(N)\big)\\
&-\bar{\psi_{11}}(N)\frac{1}{2}\big(\psi_{12}(N+\hat{1})+\psi_{12}(N-\hat{1})-2\psi_{12}(N)\big)\\
&+i\bar{\psi_{22}}(N)\frac{1}{2}\big(\psi_{21}(N+\hat{2})+\psi_{21}(N-\hat{2})-2\psi_{21}(N)\big)\\
&-i\bar{\psi_{12}}(N)\frac{1}{2}\big(\psi_{11}(N+\hat{2})+\psi_{11}(N-\hat{2})-2\psi_{11}(N)\big)\\
&+i\bar{\psi_{21}}(N)\frac{1}{2}\big(\psi_{22}(N+\hat{2})+\psi_{22}(N-\hat{2})-2\psi_{22}(N)\big)\\
&-i\bar{\psi_{11}}(N)\frac{1}{2}\big(\psi_{12}(N+\hat{2})+\psi_{12}(N-\hat{2})-2\psi_{12}(N)\big)\\
&+m\big(\bar{\psi_{11}}(N)\psi_{11}(N)+\bar{\psi_{12}}(N)\psi_{12}(N)+\bar{\psi_{21}}(N)\psi_{21}(N)+\bar{\psi_{22}}(N)\psi_{22}(N)\big)\Bigg]. \quad (3.15)
\end{aligned}
$$

We can get the continious action from (3.15), by the following relation

$$
\partial_\mu\psi(x)=\frac{1}{2}\big(\psi(N+\hat{\mu})-\psi(N-\hat{\mu})\big),\quad (\partial_\mu)^2\psi(x)=\frac{1}{2}\big(\psi(N+\hat{\mu})+\psi(N-\hat{\mu})-2\psi(N)\big). \quad (3.16)
$$

From this relation, we can write the continuum action for this lattice action as,

$$
\begin{aligned}
S = \int d^2x \Big[& \bar{\psi}_{11}\partial_1\psi_{21} + \bar{\psi}_{21}\partial_1\psi_{11} + \bar{\psi}_{12}\partial_1\psi_{22} + \bar{\psi}_{22}\partial_1\psi_{12} \\
& - i\bar{\psi}_{11}\partial_2\psi_{21} + i\bar{\psi}_{21}\partial_2\psi_{11} - i\bar{\psi}_{12}\partial_2\psi_{22} + i\bar{\psi}_{22}\partial_2\psi_{12} \\
& - \bar{\psi}_{22}\partial_1^2\psi_{21} + \bar{\psi}_{12}\partial_1^2\psi_{11} + \bar{\psi}_{21}\partial_1^2\psi_{22} - \bar{\psi}_{11}\partial_1^2\psi_{12} \\
& + i\bar{\psi}_{22}\partial_2^2\psi_{21} - i\bar{\psi}_{12}\partial_2^2\psi_{11} + i\bar{\psi}_{21}\partial_2^2\psi_{22} - i\bar{\psi}_{11}\partial_2^2\psi_{12} \\
& + m\big(\bar{\psi}_{11}\psi_{11} + \bar{\psi}_{12}\psi_{12} + \bar{\psi}_{21}\psi_{21} + \bar{\psi}_{22}\psi_{22}\big)\Big] \\
= \int d^2x \Big[& \sum_{abi} \left\{ \bar{\psi}_{ai} \left(\gamma_1\right)_{ab} \partial_1\psi_{bi} + \bar{\psi}_{ai} \left(\gamma_2\right)_{ab} \partial_2\psi_{bi} \right\} \\
& + \sum_{abij} \left\{ \bar{\psi}_{ai} \left(\gamma_3\right)_{ab} \left(\sigma_1\sigma_3\right)_{ij} \partial_1^2\psi_{bj} + \bar{\psi}_{ai} \left(\gamma_3\right)_{ab} \left(-\sigma_2\sigma_3\right)_{ij} \partial_2^2\psi_{bj} \right\} \\
& + \sum_{ai} m\big(\bar{\psi}_{ai}\psi_{ai}\big)\Big] \\
= \int d^2x \Big[& \sum_{abi} \bar{\psi}_{ai} \left(\gamma_\mu\right)_{ab} \partial_\mu\psi_{ai} + \sum_{abij} \bar{\psi}_{ai} \left(\gamma_3\right)_{ab} \left(\sigma_\mu^T\sigma_3\right)_{ij} \partial_\mu^2\psi_{bj} + \sum_{ai} m\big(\bar{\psi}_{ai}\psi_{ai}\big)\Big] \, .
\end{aligned}
\tag{3.17}
$$

The indices of ψ_{ai} and $\bar{\psi}_{ai}$ are understood as, a : spinor index and i : flavor index. We can neglect higher derivative terms, because they have higher power of the lattice constant a, and they have only subleading contribution in the lattice constant $a \to 0$ limit. Finally, we get the continuous action

$$
S = \int d^2x \sum_{i=1}^{2} \left[\bar{\Psi}_i(x)\gamma^\mu\partial_\mu\Psi_i(x) + m\bar{\Psi}_i(x)\Psi_i(x) + \text{(higher derivative terms)} \right] \tag{3.18}
$$

with Dirac fermion Ψ as

$$
\Psi_i = \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix} \, , \qquad \bar{\Psi}_i = \left(\bar{\psi}_{1i}, \bar{\psi}_{2i}\right) \, . \tag{3.19}
$$

Be aware that the spinor index of ψ and $\bar{\psi}$ are denote opposite chirality in the continuum theory. We take the chiral representation for the definition of the γ matrices (3.12). In this representation, the spinor index denotes chirality of the fermion because the chirality is the eigenvalue of γ_3. As we see the lattice action (3.17), the kinetic terms are couplong between $\bar{\psi}_{1i}$ and ψ_{2i}, and this term does not mix the chirality. The mass terms are coupling between $\bar{\psi}_{1i}$ and ψ_{1i}, and this term mixes the chirality. The conventional name of chirality $+$ and $-$ fields are right and left respectively. We can write the fermion field as,

$$\Psi_i = \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix} = \begin{pmatrix} \psi^i_R \\ \psi^i_L \end{pmatrix} , \qquad \bar{\Psi}_i = \left(\bar{\psi_{1i}}, \bar{\psi_{2i}}\right) = \left(\bar{\psi}^i_L, \bar{\psi}^i_R\right) , \tag{3.20}$$

in this notation. The chirality twist of $\bar{\psi}$ comes from its definition $\bar{\psi} = \psi^\dagger \gamma_0$ in the continuum theory.

3.2.1 θ term on the fermion

In the Schwinger model, the θ parameter is not only the coefficient of F, but also the phase of the complex mass of fermions. More precisely, this phase of the mass corresponds to the coefficient of the $\bar{\psi}\gamma_3\psi$. In this point of view, we should focus on the phase of the mass even for the free fermion theory.

However, $\bar{\psi}\gamma_3\psi$ term cannot be the nearest neighbor hopping in our set up. In staggered fermion, $\bar{\psi}\gamma_3\psi$ is,

$$\begin{aligned}\sum_{i=1,2} \bar{\psi}_i(N)\gamma_3\psi_i(N) =& \bar{\psi}_{11}\psi_{11} - \bar{\psi}_{12}\psi_{12} + \bar{\psi}_{21}\psi_{21} - \bar{\psi}_{22}\psi_{22} \\ =& \frac{1}{4}\Big\{ \left(\bar{\lambda}_{11} - i\bar{\lambda}_{22}\right)\left(\lambda_{11} + i\lambda_{22}\right) - \left(\bar{\lambda}_{12} + i\bar{\lambda}_{21}\right)\left(\lambda_{12} - i\lambda_{21}\right) \\ &+ \left(\bar{\lambda}_{12} - i\bar{\lambda}_{21}\right)\left(\lambda_{12} + i\lambda_{21}\right) - \left(\bar{\lambda}_{11} + i\bar{\lambda}_{22}\right)\left(\lambda_{11} - i\lambda_{22}\right) \Big\} \\ =& \frac{1}{2}\Big\{ i\bar{\lambda}_{11}\lambda_{22} - i\bar{\lambda}_{22}\lambda_{11} + i\bar{\lambda}_{12}\lambda_{21} - i\bar{\lambda}_{21}\lambda_{12} \Big\} \\ =& \frac{i}{2}\Big\{ \bar{\chi}(N)\chi(N+\hat{1}+\hat{2}) - \bar{\chi}(N+\hat{1}+\hat{2})\chi(N) \\ &+ \bar{\chi}(N+\hat{1})\chi(N+\hat{2}) - \bar{\chi}(N+\hat{2})\chi(N+\hat{1}) \Big\}. \end{aligned} \tag{3.21}$$

We can also write it by the small lattice coordinate $n = (n_1, n_2)$ as,

$$\begin{aligned}\sum_N \sum_{i=1,2} \bar{\psi}_i(N)\gamma_3\psi_i(N) =& \sum_n \frac{i}{2}\left(\frac{1 + (-1)^{n_1} + (-1)^{n_2} + (-1)^{n_1+n_2}}{4}\right)\Big\{ \bar{\chi}(n)\chi(n+\hat{1}+\hat{2}) \\ &- \bar{\chi}(n+\hat{1}+\hat{2})\chi(n) + \bar{\chi}(n+\hat{1})\chi(n+\hat{2}) - \bar{\chi}(n+\hat{2})\chi(n+\hat{1}) \Big\} . \end{aligned} \tag{3.22}$$

This next to nearest neighbor type hopping has its origin in the construction of the staggered fermion. As we saw in from equations (3.10) to (3.14), the definitions of ψ_{ij} and $\bar{\psi}_{ij}$ fields are related to the γ matrices. Roughly speaking, γ_1 and γ_1 corresponds to $\hat{1}$ and $\hat{2}$ direction hopping within the unit lattice, respectively. From this definition, γ_3 corresponds $\hat{1}+\hat{2}$ direction hopping within the unit lattice, inevitably.

For this reason, we do not treat θ parameter as $\bar{\psi}\psi + \bar{\psi}\gamma_3\psi$ type mass term but treat the coefficient of the field strength F_{12}.

3.2.2 $U(1)_A$ symmetry on the lattice

The continuous theory has $U(1)_A$ symmetry for classical action, and it is broken by ABJ-type anomaly. $U(1)_A$ transformation for the fermion is,

$$\begin{aligned} \psi &\to \mathrm{e}^{i\gamma_3\alpha}\psi \\ \bar{\psi} &\to \bar{\psi}\mathrm{e}^{i\gamma_3\alpha} \ . \end{aligned} \tag{3.23}$$

The lattice action does not have $U(1)_A$ symmetry, but the staggered fermion has its discrete part. The staggered fermion has $U(1)$ symmetry in addition to the vector-type $U(1)$ symmetry. The additional $U(1)$ symmetry is related to $\eta_3(n) = (-1)^{n_1+n_2}$ but it is not same as $U(1)_A$ in the continuous theory. Let us call it $U(1)_{\eta_3}$, and its transformation is defined as

$$\begin{aligned} \chi(n) &\to \mathrm{e}^{i\eta_3(n)\alpha}\chi(n) \\ \bar{\chi}(n) &\to \bar{\chi}(n)\mathrm{e}^{i\eta_3(n)\alpha} \ , \end{aligned} \tag{3.24}$$

where χ and $\bar{\chi}$ are staggered fermions and n is site index fer small lattice. The sign $\eta_3(n)$ for a unit lattice can be written as the following way.

$$\begin{array}{ccccc} & | & & | & \\ - & (-1) & - & (+1) & - \\ & | & & | & \\ - & (+1) & - & (-1) & - \\ & | & & | & \end{array} \tag{3.25}$$

The mass term breaks $U(1)_{\eta_3}$ symmetry beause it is the on-site coupling $\bar{\chi}\chi$. To consider the continuous fermion, we read $\chi(n)$ in language of λ_{ij} and ψ_{ij} (3.10).

Before considering $U(1)_A$, let us understand $U(1)_{\eta_3}$ symmetry in the continuum theory. To read the continuum fields from the lattice fermion χ, we should consider λ_{ij} and ψ_{ij}. The $U(1)_{\eta_3}$ transformation for them are,

$$\begin{aligned} \lambda_{11} &= \psi_{11} + \psi_{22} &\to\quad & \mathrm{e}^{i\alpha}\lambda_{11} = \mathrm{e}^{i\alpha}\psi_{11} + \mathrm{e}^{i\alpha}\psi_{22} \ , \\ \lambda_{12} &= \psi_{12} + \psi_{21} &\to\quad & \mathrm{e}^{-i\alpha}\lambda_{12} = \mathrm{e}^{-i\alpha}\psi_{12} + \mathrm{e}^{-i\alpha}\psi_{21} \ , \\ \lambda_{21} &= i\psi_{12} - i\psi_{21} &\to\quad & \mathrm{e}^{-i\alpha}\lambda_{21} = i\mathrm{e}^{-i\alpha}\psi_{12} - i\mathrm{e}^{-i\alpha}\psi_{21} \ , \\ \lambda_{22} &= -i\psi_{11} + i\psi_{22} &\to\quad & \mathrm{e}^{i\alpha}\lambda_{22} = -i\mathrm{e}^{i\alpha}\psi_{11} + i\mathrm{e}^{i\alpha}\psi_{22} \ . \end{aligned}$$

This transformation can be written as transformations for ψ_{ij} as,

$$\begin{aligned} \psi_{11} &\to \mathrm{e}^{i\alpha}\psi_{11} & \psi_{12} &\to \mathrm{e}^{-i\alpha}\psi_{12} \\ \psi_{21} &\to \mathrm{e}^{-i\alpha}\psi_{21} & \psi_{22} &\to \mathrm{e}^{i\alpha}\psi_{22} \ , \end{aligned} \tag{3.26}$$

and for $\bar{\psi}_{ij}$ as,

$$\begin{aligned} \bar{\psi}_{11} &\to \mathrm{e}^{i\alpha}\bar{\psi}_{11} & \bar{\psi}_{12} &\to \mathrm{e}^{-i\alpha}\bar{\psi}_{12} \\ \bar{\psi}_{21} &\to \mathrm{e}^{-i\alpha}\bar{\psi}_{21} & \bar{\psi}_{22} &\to \mathrm{e}^{i\alpha}\bar{\psi}_{22} \ . \end{aligned} \tag{3.27}$$

For the continuum theory, this $U(1)_{\eta_3}$ transformation can be written by the field $\Psi_i(x)$ in (3.20) as,

$$\Psi_i \to \mathrm{e}^{i\gamma_3(\sigma_3)_{ij}\alpha}\Psi_j \qquad\qquad \bar{\Psi}_i \to \bar{\Psi}_j \mathrm{e}^{i\gamma_3(\sigma_3)_{ji}\alpha} \ , \tag{3.28}$$

where σ_3 is a Pauli matrix acts on the flavor index.[1] This transformation is twist for both of the flavor and chirality. Be aware that this transformation (3.28) in the continuum theory is a kind of $U(1)_A$ transformation, but it is anomaly free unlike the standard $U(1)_A$ transformation. If we focus on one of the two flavor, this transformation is exactly same as $U(1)_A$ and anomalous. However, it has an opposite sign comes from the flavor matrix σ_3, so the $U(1)_A$ anomalies are compensated each other.

It is important to consider the pure $U(1)_A$ transformation because it rotates the θ parameter through its anomaly. This transformation is also important to the mass shift in the Hamiltonian formalism.

One exception in the transformation (3.24) is $\alpha = \pi/2$ case. The transformation in (3.26) and (3.27) are not the pure $U(1)_A$ transformation, but this transformation acts the same way for fermion bilinear term without flavor twist. For fermion bilinear terms, the transformation can be written as,

$$\begin{aligned} \bar{\Psi}_i\tilde{\gamma}\Psi_i \ &\to \bar{\Psi}_j \left(i\gamma_3(\sigma_3)_{ji}\right)\tilde{\gamma}\left(i\gamma_3(\sigma_3)_{ij}\right)\Psi_j \\ &= -\bar{\Psi}_i\gamma_3\tilde{\gamma}\gamma_3\Psi_i \\ &= \bar{\Psi}_i \mathrm{e}^{i\gamma_3\frac{\pi}{2}}\tilde{\gamma}\mathrm{e}^{i\gamma_3\frac{\pi}{2}}\Psi_i \ , \end{aligned} \tag{3.29}$$

where $\tilde{\gamma}$ can be any of $1, \gamma_1, \gamma_2, \gamma_3$, derivative and their combinations. Therefore, this $\mathbb{Z}_2$ part of $U(1)_{\eta_3}$ transformation can be considered as a part of the pure $U(1)_A$ transformation. In other words, this action has $\mathbb{Z}_2$ part of the $U(1)_A$ symmetry. However, this $\mathbb{Z}_2$ transformation acts nothing on the θ parameter, unlike [3]. One explanation is that the transformation (3.28) is anomaly free. Another explanation is that this $\mathbb{Z}_2$ transformation shifts $\theta \to \theta + \pi$ for one flavor, but for two flavors, $\theta \to \theta + 2\pi = \theta$. From this reason, we cannot consider the mass shift like [3] in this action.

[1]There is an ambiguity for the choice of this Pauli matrix, that is comes from the definition of the γ matrices. The Pauli matrix σ_3 in equation (3.28) depends on the definition of the γ matrices. If we choose the chiral representation for γ matrices as (3.12), the flavor Pauli matrix in (3.28) can be σ_3. In general, the flavor matrix in (3.28)is defined as the same Pauli matrix as $-i\gamma_1\gamma_2$.

Let us try to consider the pure $U(1)_A$ transformation, even though it is not the exact lattice symmetry. For example, the transformation for λ_{11} can be written as,

$$\begin{aligned}\lambda_{11} = \psi_{11} + \psi_{22} \quad \rightarrow \quad \mathrm{e}^{i\alpha}\psi_{11} + \mathrm{e}^{-i\alpha}\psi_{22} &= \cos(\alpha)(\psi_{11} + \psi_{22}) + i\sin(\alpha)(\psi_{11} - \psi_{22}) \\ &= \lambda_{11}\cos(\alpha) - \lambda_{22}\sin(\alpha) \ .\end{aligned}$$

This type of transformation is highly non-locally for lattice fermion $\chi(n)$, even if we consider $\alpha = \pi/2$ case.[2] This non-locality is similar to that of we considered in section 3.2.1, but this transformation for the kinetic term can be more complicated. For example, $\bar{\lambda}_{11}(N)\lambda_{12}(N-\hat{1})$ term changes to $\bar{\lambda}_{22}(N)\lambda_{21}(N-\hat{1})$, but this term is equal to $\bar{\chi}(n+\hat{1}+\hat{2})\chi(n-\hat{1}-\hat{1}+2)$, under the condition (or the definition of n) of $\lambda_{11}(N) = \chi(n)$. This term includes ∂_1^3 term.[3] If we consider $\alpha \neq \pi/2$ case, the transformation is more complicated. However, at least this kind of $\mathbb{Z}_2$ transformation is realized by (3.29) effectively and we know that it does not affect to the θ term.

[2]Naively, this transformation for $\alpha = \pi/2$ is very similar to the $\hat{1}+\hat{2}$ shift, such as $n \rightarrow n + \hat{1} + \hat{2}$. The definition of the shift transformation for μ direction is,[4, 5]

$$\begin{aligned}&\chi(n) \rightarrow (-1)^{\sum_{\nu>\mu}}\chi(n+\hat{\mu}) \ , \\ &\chi(n) \rightarrow (-1)^{n_2}\chi(n+\hat{1}) \ , \qquad \chi(n) \rightarrow \chi(n+\hat{2}) \ .\end{aligned} \tag{3.30}$$

The factor $(-1)^{\sum_{\nu>\mu}}$ is important to realize the staggered phase. This shift is not commute for the orderig of the shift because of the factor of $(-1)^{\sum_{\nu>\mu}}$. Hereafter, we consider shift for $\hat{1}$ direction first. For example, the shift leads this condition,

$$\lambda_{11} \rightarrow \lambda_{22}, \qquad \lambda_{22} \rightarrow -\lambda_{11}, \qquad \lambda_{12} \rightarrow \lambda_{21}, \qquad \lambda_{21} \rightarrow -\lambda_{12},$$

and for ψ_{ij} fields,

$$\psi_{11} \rightarrow -i\psi_{11}, \qquad \psi_{21} \rightarrow -i\psi_{21}, \qquad \psi_{12} \rightarrow i\psi_{12}, \qquad \psi_{22} \rightarrow i\psi_{22}.$$

This transformation looks like the flavor σ_3 transformation. However, this shift is not exactly the same to the flavor σ_3 transformation because the condition (3.30) should be satisfied within the 2×2 unit lattice. The flavor structure of the shift is the same to the flavor σ_3 transformation, but the shift breaks the unit lattice structure.

[3]If we consider the improvement by ∂^3 terms, it may be possible to realize the $mathbbZ_2$ part of the pure $U(1)_A$ symmetry. The imporovement terms are,

$$\bar{\chi}(n)\chi(n+3\cdot\hat{1}) - \bar{\chi}(n)\chi(n+3\cdot\hat{2}) + \bar{\chi}(n+\hat{1})\chi(n+\hat{1}+3\cdot\hat{2}) + \bar{\chi}(n+\hat{2})\chi(n+\hat{2}+3\cdot\hat{1}) + \text{h.c.} \ ,$$

up to factors (under the condition of $\lambda_{11}(N) = \chi(n)$). The minus sign in the second term comes from the staggered phase. However, it seems difficult to realize this improvement in TRG. I have not understood how this exact $\mathbb{Z}_2$ symmetry is important. I leave some problems with this improvement for future works.

Anyway, this $\mathbb{Z}_2$ symmetry is not much important in this case because it is effectively realized by $U(1)_{\eta_3}$ symmetry.

3.2.3 Preparation for the improvement

To improve the lattice action, we start from the action written by ψ_{ij}, and rewrite λ_{ij}. The ψ_{ij} action is,

$$
\begin{aligned}
S = \sum_N \Bigg[& \bar{\psi_{11}}(N)\frac{1}{2}\big(\psi_{21}(N+\hat{1})-\psi_{21}(N-\hat{1})\big)+\bar{\psi_{21}}(N)\frac{1}{2}\big(\psi_{11}(N+\hat{1})-\psi_{11}(N-\hat{1})\big) \\
& +\bar{\psi_{12}}(N)\frac{1}{2}\big(\psi_{22}(N+\hat{1})-\psi_{22}(N-\hat{1})\big)+\bar{\psi_{22}}(N)\frac{1}{2}\big(\psi_{12}(N+\hat{1})-\psi_{12}(N-\hat{1})\big) \\
& -i\bar{\psi_{11}}(N)\frac{1}{2}\big(\psi_{21}(N+\hat{2})-\psi_{21}(N-\hat{2})\big)+i\bar{\psi_{21}}(N)\frac{1}{2}\big(\psi_{11}(N+\hat{2})-\psi_{11}(N-\hat{2})\big) \\
& -i\bar{\psi_{12}}(N)\frac{1}{2}\big(\psi_{22}(N+\hat{2})-\psi_{22}(N-\hat{2})\big)+i\bar{\psi_{22}}(N)\frac{1}{2}\big(\psi_{12}(N+\hat{2})-\psi_{12}(N-\hat{2})\big) \\
& +m\big(\bar{\psi_{11}}(N)\psi_{11}(N)+\bar{\psi_{12}}(N)\psi_{12}(N)+\bar{\psi_{21}}(N)\psi_{21}(N)+\bar{\psi_{22}}(N)\psi_{22}(N)\big)\Bigg] .
\end{aligned}
\tag{3.31}
$$

We start from this action. We substitute (3.13) and (3.14) for it (by mathematica), the action can be written as,

$$
\begin{aligned}
S = \sum_N \Bigg[& -\bar{\lambda_{12}}(N+\hat{1})\lambda_{11}(N)-\bar{\lambda_{21}}(N+\hat{2})\lambda_{11}(N)-\bar{\lambda_{11}}(N+\hat{1})\lambda_{12}(N)+\bar{\lambda_{22}}(N+\hat{2})\lambda_{12}(N) \\
& -\bar{\lambda_{11}}(N+\hat{2})\lambda_{21}(N)-\bar{\lambda_{22}}(N+\hat{1})\lambda_{21}(N)+\bar{\lambda_{12}}(N+\hat{2})\lambda_{22}(N)-\bar{\lambda_{21}}(N+\hat{1})\lambda_{22}(N) \\
& -\bar{\lambda_{22}}(N)\lambda_{12}(N+\hat{2})+\bar{\lambda_{22}}(N)\lambda_{21}(N+\hat{1})+\bar{\lambda_{11}}(N)\lambda_{12}(N+\hat{1})+\bar{\lambda_{11}}(N)\lambda_{21}(N+\hat{2}) \\
& +\bar{\lambda_{21}}(N)\lambda_{11}(N+\hat{2})+\bar{\lambda_{21}}(N)\lambda_{22}(N+\hat{1})+\bar{\lambda_{12}}(N)\lambda_{11}(N+\hat{1})-\bar{\lambda_{12}}(N)\lambda_{22}(N+\hat{2}) \\
& +m\big(\bar{\lambda_{11}}(N)\lambda_{11}(N)+\bar{\lambda_{12}}(N)\lambda_{12}(N)+\bar{\lambda_{21}}(N)\lambda_{21}(N)+\bar{\lambda_{22}}(N)\lambda_{22}(N)\big)\Bigg] \\
= \sum_N \Bigg[& \bar{\lambda_{12}}(N)\,\big(\lambda_{11}(N+\hat{1})-\lambda_{11}(N-\hat{1})\big)+\bar{\lambda_{21}}(N)\,\big(\lambda_{11}(N+\hat{2})-\lambda_{11}(N-\hat{2})\big) \\
& +\bar{\lambda_{11}}(N)\,\big(\lambda_{12}(N+\hat{1})-\lambda_{12}(N-\hat{1})\big)-\bar{\lambda_{22}}(N)\,\big(\lambda_{12}(N+\hat{2})-\lambda_{12}(N-\hat{2})\big) \\
& +\bar{\lambda_{11}}(N)\,\big(\lambda_{21}(N+\hat{2})-\lambda_{21}(N-\hat{2})\big)+\bar{\lambda_{22}}(N)\,\big(\lambda_{21}(N+\hat{1})-\lambda_{21}(N+\hat{1})\big) \\
& -\bar{\lambda_{12}}(N)\,\big(\lambda_{22}(N+\hat{2})-\lambda_{22}(N-\hat{2})\big)+\bar{\lambda_{21}}(N)\,\big(\lambda_{22}(N+\hat{1})-\lambda_{22}(N-\hat{1})\big) \\
& +m\big(\bar{\lambda_{11}}(N)\lambda_{11}(N)+\bar{\lambda_{12}}(N)\lambda_{12}(N)+\bar{\lambda_{21}}(N)\lambda_{21}(N)+\bar{\lambda_{22}}(N)\lambda_{22}(N)\big)\Bigg] .
\end{aligned}
\tag{3.32}
$$

In this representation, however, there are ∂^3 type hopping terms and there is no hopping term inside the unit lattice. To realize the staggered fermion without additional doublers, the hopping term should be the nearest-neighbor type. The length of the hopping term should be half of the large lattice space. Therefore, this type of action should have additional doublers, so this attempt is failed.

3.3 Gauge part

The lattice action for gauge part is below.

$$S = \sum_n \left[-\frac{1}{g^2}\cos(A_p(n)) - \frac{i\theta}{2\pi}\tilde{A}_p(n) \right] \tag{3.33}$$

This action is the same as the usual plaquette action because,

$$\begin{aligned}
-\frac{1}{g^2}\cos(A_p(n)) =& -\frac{1}{2g^2}\left[\mathrm{e}^{iA_p(n)} + \mathrm{e}^{-iA_p(n)}\right] \\
=& -\frac{1}{2g^2}\left[U_p(n) + U_p^\dagger(n)\right] \\
=& \frac{1}{2g^2}\left[(1 - U_p(n)) + h.c.\right] - \frac{1}{g^2} \\
\simeq& \frac{1}{4g^2}F_{\mu\nu}F_{\mu\nu} - \frac{1}{g^2} + \text{(higher order term)}
\end{aligned}$$

where $U_p(n) \equiv \mathrm{e}^{iA_p(n)}$.

Theta term is realized the second term of (3.33), because this $\tilde{A}_p(n)$ counts the winding number how many the gauge field wind to the each plaquettes. It is understood by the following relation.

$$\begin{aligned}
\exp\left(\sum_n -\frac{i\theta}{2\pi}A_p(n)\right) =& \prod_n \mathrm{e}^{-\frac{i\theta}{2\pi}A_p(n)} \\
=& \left(\prod_n U_p(n)\right)^{-\frac{\theta}{2\pi}} \\
=& 1 \\
\exp\left(\sum_n -\frac{i\theta}{2\pi}\tilde{A}_p(n)\right) =& \prod_n \mathrm{e}^{-\frac{i\theta}{2\pi}(A_p(n) - 2\pi N(n))} \\
=& \left(\prod_n U_p(n)\mathrm{e}^{-2\pi i N(n)}\right)^{-\frac{\theta}{2\pi}} \\
=& \exp\left(i\theta \sum_n N(n)\right) \qquad (3.34) \\
\to& \exp\left(\int d^2x \frac{i\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\right)
\end{aligned}$$

$N(n)$ is the instanton number (or winding number) on the plaquette at site n. $\sum_n N(n)$ is the instanton number for the whole spacetime, so that (3.34) is θ term[4]. In this formulation, we can only consider the instanton number (or winding number) for each plaquettes from -2 to $+2$. However, we can treat large instanton number density in the continuous limit.

We get the continuous action (3.1) from the lattice action (3.2), by conbining fermion part (3.18) and gauge part.

4 Initial tensor

In this section, we derive the initial tensor from the lattice action (3.2).

To calculate TRG, we need to rewrite the lattice partition function to the product of local tensors $T_{abcd}(n)$.

$$
\begin{aligned}
Z =& \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi \ \mathrm{e}^{-S} \\
=& \mathrm{tr}\Big[\sum_n T_{abcd}(n)\Big]
\end{aligned} \tag{4.1}
$$

The labels a, b, c, d are called as bond. Trace in (4.1) means trace for each bonds, and it means the periodic boundary condition. If we define the local partition function $z(n)$, the tensor is almost same as z.

$$
Z \sim \sum_n z(n) \sim \sum_n T_{abcd}(n)
$$

In this section, we derive the initial tensor for the Schwinger model.

4.1 Fermion part

Lattice fermion action is (3.6). The partition function of fermions with bachground $U(1)$ gauge field is,

$$
\begin{aligned}
Z = \int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\bigg\{ -\frac{1}{2}\sum_{n,\mu} \Big[\eta_\mu(n)\big\{\bar{\chi}(n)U_\mu(n)\chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu})U_\mu^\dagger(n)\chi(n)\big\} \\
+ m\bar{\chi}(n)\chi(n)\Big]\bigg\}
\end{aligned} \tag{4.2}
$$

[4]For 4dim U(1) gauge theory, θ term is not well-defined because $\pi_3(U(1))$ is trivial. However, we treat U(1) gauge theory in 2dim spacetime, so θ term is well-defined because of $\pi_1(U(1)) = \mathbb{Z}$

where $U_\mu(n) = \mathrm{e}^{iA_\mu(n)}$. This partition function includes the staggered phase η_μ, but we can erase this phase by the redifinition of gauge fields. We write the new gauge fields as $\hat{A}_\mu(n)$. This change of variable also works for the gauge part, because the relation between the plaquettes written by A_μ and $\hat{A}_\mu$ are just,

$$\begin{aligned}
\hat{U}_\mu(n) \equiv& \eta_\mu(n) U_\mu(n) = \eta_\mu(n) \exp\left(iA_\mu(n)\right) \\
\hat{U}_p(n) \equiv& \hat{U}_1(n)\hat{U}_2(n+\hat{1})\hat{U}_1^\dagger(n+\hat{2})\hat{U}_2^\dagger(n) \\
=& -U_1(n)U_2(n+\hat{1})U_1^\dagger(n+\hat{2})U_2^\dagger(n) \\
=& -U_p(n).
\end{aligned} \tag{4.3}$$

The staggered phase η_μ just changes sign of plaquettes, because each plaquettes include only one link which $\eta_\mu = -1$. If we replace U_p to $\hat{U}_p$, then staggered phase disappear.

Next, we create local partition function. We follow the Grassmann TRG method written in [6].The partition function (4.2) cannot be written in the product of local partition function directly, because fermion action includes hopping term between site n and $n+\hat{\mu}$. To create local partition function, we introduce auxiliary fields ζ and ξ. These auxiliary fields lives on each links, and path integrals for them are correapond to the contraction of these bonds. The basic way to introduce the auxiliary field is,

$$\begin{aligned}
\mathrm{e}^{-a\bar{\chi}(n)\hat{U}_\mu(n)\chi(n+\hat{\mu})} =& \left\{ \int d\bar{\zeta}_\mu(n) d\zeta_\mu(n)\ \mathrm{e}^{-\left(\bar{\zeta}_\mu(n)+\sqrt{a}\bar{\chi}(n)\hat{U}_\mu(n)\right)\left(\zeta_\mu(n)-\sqrt{a}\chi(n+\hat{\mu})\right)} \right\} \mathrm{e}^{-a\bar{\chi}(n)\hat{U}_\mu(n)\chi(n+\hat{\mu})} \\
=& \int d\bar{\zeta}_\mu(n) d\zeta_\mu(n)\ \mathrm{e}^{-\bar{\zeta}_\mu(n)\zeta_\mu(n)} \mathrm{e}^{-\sqrt{a}\bar{\chi}(n)\hat{U}_\mu(n)\zeta_\mu(n)} \mathrm{e}^{\sqrt{a}\bar{\zeta}_\mu(n)\chi(n+\hat{\mu})}.
\end{aligned} \tag{4.4}$$

a is just a factor for this term. In our lattice action (3.2), $a = 1/2$. For the other terms, we can create similar relation.

$$\begin{aligned}
\mathrm{e}^{a\bar{\chi}(n+\hat{\mu})\hat{U}^\dagger_\mu(n)\chi(n)} =& \left\{ \int d\bar{\xi}_\mu(n) d\xi_\mu(n)\ \mathrm{e}^{-\left(\bar{\xi}_\mu(n)+\sqrt{a}\hat{U}^\dagger_\mu(n)\chi(n)\right)\left(\xi_\mu(n)-\sqrt{a}\bar{\chi}(n+\hat{\mu})\right)} \right\} \mathrm{e}^{a\bar{\chi}(n+\hat{\mu})\hat{U}^\dagger_\mu(n)\chi(n)} \\
=& \int d\bar{\xi}_\mu(n) d\xi_\mu(n)\ \mathrm{e}^{-\bar{\xi}_\mu(n)\xi_\mu(n)} \mathrm{e}^{-\sqrt{a}\hat{U}^\dagger_\mu(n)\chi(n)\xi_\mu(n)} \mathrm{e}^{\sqrt{a}\bar{\xi}_\mu(n)\bar{\chi}(n+\hat{\mu})}.
\end{aligned} \tag{4.5}$$

Then, the partition function (4.2) becomes,

$$
\begin{aligned}
Z =& \int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\Bigg\{ -\frac{1}{2}\sum_{n,\mu}\Big[\bar{\chi}(n)\hat{U}_\mu(n)\chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu})\hat{U}^\dagger_\mu(n)\chi(n) \\
& + m\bar{\chi}(n)\chi(n)\Big]\Bigg\} \\
=& \prod_n \int d\bar{\chi}(n)d\chi(n)\ \mathrm{e}^{-\frac{1}{2}m\bar{\chi}(n)\chi(n)} \prod_\mu \mathrm{e}^{-\frac{1}{2}\bar{\chi}(n)\hat{U}_\mu(n)\chi(n+\hat{\mu})}\mathrm{e}^{\frac{1}{2}\bar{\chi}(n+\hat{\mu})\hat{U}^\dagger_\mu(n)\chi(n)} \\
=& \prod_n \int d\bar{\chi}(n)d\chi(n)\ \mathrm{e}^{-\frac{1}{2}m\bar{\chi}(n)\chi(n)}\cdot \\
& \prod_\mu \left(\int d\bar{\zeta}_\mu(n)d\zeta_\mu(n)\ \mathrm{e}^{-\bar{\zeta}_\mu(n)\zeta_\mu(n)}\mathrm{e}^{-\frac{1}{\sqrt{2}}\bar{\chi}(n)\hat{U}_\mu(n)\zeta_\mu(n)}\mathrm{e}^{\frac{1}{\sqrt{2}}\bar{\zeta}_\mu(n)\chi(n+\hat{\mu})}\right)\cdot \\
& \left(\int d\bar{\xi}_\mu(n)d\xi_\mu(n)\ \mathrm{e}^{-\bar{\xi}_\mu(n)\xi_\mu(n)}\mathrm{e}^{-\frac{1}{\sqrt{2}}\hat{U}^\dagger_\mu(n)\chi(n)\xi_\mu(n)}\mathrm{e}^{\frac{1}{\sqrt{2}}\bar{\xi}_\mu(n)\bar{\chi}(n+\hat{\mu})}\right) \\
=& \int\left(\prod_{n,\mu} d\bar{\zeta}_\mu(n)d\zeta_\mu(n)\ \mathrm{e}^{-\bar{\zeta}_\mu(n)\zeta_\mu(n)}\cdot d\bar{\xi}_\mu(n)d\xi_\mu(n)\ \mathrm{e}^{-\bar{\xi}_\mu(n)\xi_\mu(n)}\right)\cdot\int\left(\prod_n d\bar{\chi}(n)d\chi(n)\ \mathrm{e}^{-\frac{1}{2}m\bar{\chi}(n)\chi(n)}\right)\cdot \\
& \left(\prod_{n,\mu}\mathrm{e}^{-\frac{1}{\sqrt{2}}\bar{\chi}(n)\hat{U}_\mu(n)\zeta_\mu(n)}\mathrm{e}^{\frac{1}{\sqrt{2}}\bar{\zeta}_\mu(n-\hat{\mu})\chi(n)}\mathrm{e}^{-\frac{1}{\sqrt{2}}\hat{U}^\dagger_\mu(n)\chi(n)\xi_\mu(n)}\mathrm{e}^{\frac{1}{\sqrt{2}}\bar{\xi}_\mu(n-\hat{\mu})\bar{\chi}(n)}\right) \\
\equiv& \int\left(\prod_{n,\mu} d\bar{\zeta}_\mu(n)d\zeta_\mu(n)\ \mathrm{e}^{-\bar{\zeta}_\mu(n)\zeta_\mu(n)}\cdot d\bar{\xi}_\mu(n)d\xi_\mu(n)\ \mathrm{e}^{-\bar{\xi}_\mu(n)\xi_\mu(n)}\right)\cdot \\
& \prod_n T_{\zeta_1(n)\xi_1(n)\zeta_2(n)\xi_2(n)\bar{\zeta}_1(n-\hat{1})\bar{\xi}_1(n-\hat{1})\bar{\zeta}_2(n-\hat{2})\bar{\xi}_2(n-\hat{2}),A_1(n)A_2(n)}(n)
\end{aligned}
\tag{4.6}
$$

This is the definition of the local tensor. The contraction for this tensor includes path integral for auxiliary fields on each links. The shape of the tensor is the following.

$$
\begin{array}{ccccc}
 & & (\zeta_2(n),\xi_2(n);A_2(n)) & & \\
 & & | & & \\
\big(\bar{\zeta}_1(n-\hat{1}),\bar{\xi}_1(n-\hat{1})\big) & - & T(n) & - & (\zeta_1(n),\xi_1(n);A_1(n)) \\
 & & | & & \\
 & & \big(\bar{\zeta}_2(n-\hat{2}),\bar{\xi}_2(n-\hat{2})\big) & &
\end{array}
\tag{4.7}
$$

To calculate the partition function numerically, we need to write down the explicit form of the tensor. Fermions are Grassmann variable, so the number of each fermion fields should be 0 or 1. Therefore, if all gauge fields vanish ($A_\mu(n) = 0$), this tensor (4.6) has 2^8 components. We rewrite this tensor by using numbers of fermion fields as labels of the tensor.

$$
\begin{aligned}
&T_{\zeta_1(n)\xi_1(n)\zeta_2(n)\xi_2(n)\bar{\zeta}_1(n-\hat{1})\bar{\xi}_1(n-\hat{1})\bar{\zeta}_2(n-\hat{2})\bar{\xi}_2(n-\hat{2}),A_1(n)A_2(n)}(n) \\
&=\int \left(d\bar{\chi}(n)d\chi(n)\ \mathrm{e}^{-\frac{1}{2}m\bar{\chi}(n)\chi(n)}\right)\left(\prod_{\mu}\mathrm{e}^{-\frac{1}{\sqrt{2}}\bar{\chi}(n)\hat{U}_\mu(n)\zeta_\mu(n)}\mathrm{e}^{\frac{1}{\sqrt{2}}\bar{\zeta}_\mu(n-\hat{\mu})\chi(n)}\mathrm{e}^{-\frac{1}{\sqrt{2}}\hat{U}^\dagger_\mu(n)\chi(n)\xi_\mu(n)}\mathrm{e}^{\frac{1}{\sqrt{2}}\bar{\xi}_\mu(n-\hat{\mu})\bar{\chi}(n)}\right) \\
&=\int d\bar{\chi}(n)d\chi(n)\left(1-\frac{1}{2}m\bar{\chi}(n)\chi(n)\right)\left(1-\frac{1}{\sqrt{2}}\bar{\chi}(n)\hat{U}_1(n)\zeta_1(n)\right)\left(1-\frac{1}{\sqrt{2}}\bar{\chi}(n)\hat{U}_2(n)\zeta_2(n)\right) \\
&\quad\left(1+\frac{1}{\sqrt{2}}\bar{\zeta}_1(n-\hat{1})\chi(n)\right)\left(1+\frac{1}{\sqrt{2}}\bar{\zeta}_2(n-\hat{2})\chi(n)\right)\left(1-\frac{1}{\sqrt{2}}\hat{U}^\dagger_1(n)\chi(n)\xi_1(n)\right) \\
&\quad\left(1-\frac{1}{\sqrt{2}}\hat{U}^\dagger_2(n)\chi(n)\xi_2(n)\right)\left(1+\frac{1}{\sqrt{2}}\bar{\xi}_1(n-\hat{1})\bar{\chi}(n)\right)\left(1+\frac{1}{\sqrt{2}}\bar{\xi}_2(n-\hat{2})\bar{\chi}(n)\right) \\
&=-\sum_{\mu,\nu}\Bigg\{-\frac{1}{2}m-\frac{1}{2}\mathrm{e}^{i\hat{A}_\mu(n)}\zeta_\mu(n)\bar{\zeta}_\nu(n-\hat{\nu})-\frac{1}{2}\mathrm{e}^{i\hat{A}_\mu(n)}\zeta_\mu(n)\mathrm{e}^{-i\hat{A}_\nu(n)}\xi_\nu(n) \\
&\quad+\frac{1}{2}\bar{\zeta}_\mu(n-\hat{\mu})\bar{\xi}_\nu(n-\hat{\nu})+\frac{1}{2}\mathrm{e}^{-i\hat{A}_\mu(n)}\xi_\mu(n)\bar{\xi}_\nu(n-\hat{\nu})\Bigg\} \\
&\simeq-\sum_{iji'j'}\Bigg[-\frac{1}{2}m\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,0}+\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,2}\sum_{\mu,\nu}\Bigg\{-\frac{1}{2}\mathrm{e}^{i\pi x(a_\mu)}\delta_{i_\mu,1}\delta_{i'_\nu,1} \\
&\quad-\frac{1}{2}\mathrm{e}^{i\pi x(a_\mu)}\mathrm{e}^{-i\pi x(a_\nu)}\delta_{i_\mu,1}\delta_{j_\nu,1}+\frac{1}{2}\delta_{i'_\mu,1}\delta_{j'_\nu,1}+\frac{1}{2}\mathrm{e}^{-i\pi x(a_\mu)}\delta_{j_\mu,1}\delta_{j'_\nu,1}\Bigg\}\Bigg]\cdot \\
&\quad(\zeta_1(n))^{i_1}(\zeta_2(n))^{i_2}(\xi_1(n))^{j_1}(\xi_2(n))^{j_2}\left(\bar{\zeta}_1(n-\hat{1})\right)^{i'_1}\left(\bar{\zeta}_2(n-\hat{2})\right)^{i'_2}\left(\bar{\xi}_1(n-\hat{1})\right)^{j'_1}\left(\bar{\xi}_2(n-\hat{2})\right)^{j'_2} \\
&=\sum_{iji'j'}\frac{1}{2}\Bigg[m\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,0} \\
&\quad-\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,2}\sum_{\mu,\nu}\delta_{i_\mu+j'_\mu,1}\delta_{j_\nu+i'_\nu,1}(-1)^{i_\mu}\mathrm{e}^{i\pi(i_\mu x(a_\mu)-j_\nu x(a_\nu))}\Bigg]\cdot \\
&\quad(\zeta_1(n))^{i_1}(\zeta_2(n))^{i_2}(\xi_1(n))^{j_1}(\xi_2(n))^{j_2}\left(\bar{\zeta}_1(n-\hat{1})\right)^{i'_1}\left(\bar{\zeta}_2(n-\hat{2})\right)^{i'_2}\left(\bar{\xi}_1(n-\hat{1})\right)^{j'_1}\left(\bar{\xi}_2(n-\hat{2})\right)^{j'_2} \\
&\equiv\sum_{iji'j'}t^f_{i_1j_1i_2j_2i'_1j'_1i'_2j'_2a_1a_2}(n)\cdot \\
&\quad(\zeta_1(n))^{i_1}(\zeta_2(n))^{i_2}(\xi_1(n))^{j_1}(\xi_2(n))^{j_2}\left(\bar{\zeta}_1(n-\hat{1})\right)^{i'_1}\left(\bar{\zeta}_2(n-\hat{2})\right)^{i'_2}\left(\bar{\xi}_1(n-\hat{1})\right)^{j'_1}\left(\bar{\xi}_2(n-\hat{2})\right)^{j'_2} \\
&=\sum_{iji'j'}t^f_{i_1j_1i_2j_2i'_1j'_1i'_2j'_2a_1a_2}(n)(-1)^{P(i_1,j_1,i_2,j_2,i'_1,j'_1,i'_2,j'_2)}\cdot \\
&\quad(\zeta_1(n))^{i_1}(\xi_1(n))^{j_1}(\zeta_2(n))^{i_2}(\xi_2(n))^{j_2}\left(\bar{\xi}_1(n-\hat{1})\right)^{j'_1}\left(\bar{\zeta}_1(n-\hat{1})\right)^{i'_1}\left(\bar{\xi}_2(n-\hat{2})\right)^{j'_2}\left(\bar{\zeta}_2(n-\hat{2})\right)^{i'_2}
\end{aligned}
\tag{4.8}
$$

$x(a_\mu)$ is the "node" of Gauss-Legendre quadrature method, which is defined in the next section. $P(i_1, j_1, i_2, j_2, i'_1, j'_1, i'_2, j'_2)$ is a phase factor comes from the reordering auxiliary fields. The definition of $P(i_1, j_1, i_2, j_2, i'_1, j'_1, i'_2, j'_2)$ is the following.

$$P(i_1, j_1, i_2, j_2, i'_1, j'_1, i'_2, j'_2) = i_2 j_1 + j'_1(i'_1 + i'_2) + i'_2 j'_2 \tag{4.9}$$

This is the initial tensor for our system.

4.2 Gauge part

lattice gauge action is (3.33). The partition function is,

$$\begin{aligned} Z =& \int \mathcal{D}A \exp\left\{ -\sum_{n,\mu}\left[-\frac{1}{g^2}\cos(A_p(n)) - \frac{i\theta}{2\pi}\tilde{A}_p(n)\right]\right\} \\ =& \prod_{n,\mu}\left(\int_{-\pi}^{\pi}\frac{dA_\mu(n)}{2\pi}\right)\prod_n \exp\left\{\sum_{n,\mu}\left[\frac{1}{g^2}\cos(A_p(n)) + \frac{i\theta}{2\pi}\tilde{A}_p(n)\right]\right\} \\ =& \prod_{n,\mu}\left(\int_{-\pi}^{\pi}\frac{dA_\mu(n)}{2\pi}\right)\prod_n z(n) \end{aligned} \tag{4.10}$$

We redefine parameters and fields,

$$\beta \equiv \frac{1}{g^2}, \qquad p(n) \equiv \frac{A_p(n)}{\pi}, \qquad q(n) \equiv \frac{\tilde{A}_p(n)}{\pi}. \tag{4.11}$$

In this redifinition, we also redifine the integration range of A_μ. Then the local partition function $z(n)$ becomes,

$$z(n) = \exp\left\{\beta\cos(\pi p(n)) + \frac{i\theta}{2}q(n)\right\}. \tag{4.12}$$

The full partition function is,

$$Z = \prod_{n,\mu}\left(\int_{-1}^{1}\frac{dA_\mu(n)}{2}\right)\prod_n \exp\left\{\beta\cos(\pi p(n)) + \frac{i\theta}{2}q(n).\right\} \tag{4.13}$$

This local partition function (4.12) includes four gauge fields; $A_1(n)$, $A_2(n)$, $A_1(n+\hat{2})$, $A_2(n+\hat{1})$. These gauge fields correspond to bonds of local tensor. We get local tensor from (4.12), with four bonds,

$$\begin{aligned} T_{abcd}(n) =& \exp\left\{\beta\cos(\pi p(n)) + \frac{i\theta}{2}q(n)\right\} \\ =& \exp\left\{\beta\cos(\pi(d+a-b-c)) + \frac{i\theta}{2}q(n)\right\}. \end{aligned} \tag{4.14}$$

$$T_{abcd}(n) = \begin{array}{ccccc} & & b & & \\ & & | & & \\ c & - & T(n) & - & a \\ & & | & & \\ & & d & & \end{array} \tag{4.15}$$

$q(n)$ is also written by a, b, c, d.

$$\begin{aligned} q(n) &= p(n) - 2N(n) \\ &= d + a - b - c - 2N(n), \end{aligned} \qquad N(n) = \begin{cases} +2 & (3 \leq p(n) < 4) \\ +1 & (1 \leq p(n) < 3) \\ 0 & (-1 \leq p(n) < 1) \\ -1 & (-3 \leq p(n) < -1) \\ -2 & (-4 \leq p(n) < -3) \end{cases} \tag{4.16}$$

We contract tensors in the following way.

$$\begin{aligned} &\sum_i T_{iabc}(n) T_{deif}(n+\hat{1}) \\ &= \int_{-1}^{1} \frac{dA_1(n)}{2} \exp\left\{\beta \cos\big(\pi(c + A_1(n) - a - b)\big) + \frac{i\theta}{2}\big(c + A_1(n) - a - b - 2N(n)\big)\right\} \\ &\cdot \exp\left\{\beta \cos\big(\pi(f + d - e - A_1(n))\big) + \frac{i\theta}{2}\big(f + d - e - A_1(n) - 2N(n+\hat{1})\big)\right\} \end{aligned} \tag{4.17}$$

Index i correspond to the variable of integration $A_1(n)$.

To calculate path integral of $A_\mu(n)$ numerically, we discretize the integral by using Gauss-Legendre quadrature. In Gauss-Legendre quadrature method, we discretize the integral in the following way,

$$\int_{-1}^{1} dx f(x) \simeq \sum_{i=1}^{K} w(i) f(x(i)). \tag{4.18}$$

Here, $w(i)$ is called "weights" and $x(i)$ is called "nodes," and both of them are decided if we once decide the number of nodes K.

Then, we calculate the contraction of tensors (4.17),

$$\begin{aligned} &\sum_i T_{iabc}(n) T_{deif}(n+\hat{1}) \\ &\simeq \sum_{i=1}^{K} \frac{w(i)}{2} \exp\left\{\beta \cos\big(\pi(c + x(i) - a - b)\big) + \frac{i\theta}{2}\big(c + x_i - a - b - 2N(n)\big)\right\} \\ &\cdot \exp\left\{\beta \cos\big(\pi(f + d - e - x(i))\big) + \frac{i\theta}{2}\big(f + d - e - x(i) - 2N(n+\hat{1})\big)\right\}. \end{aligned} \tag{4.19}$$

From this contraction, we should define the tensors and contraction for our numerical calculation as the following.

$$
\begin{aligned}
t^g_{abcd} \equiv & \frac{\sqrt{w(a)w(b)w(c)w(d)}}{4} \exp\Big\{\beta\cos\big(\pi(x(d)+x(a)-x(b)-x(c))\big) \\
& + \frac{i\theta}{2}\big(x(d)+x(a)-x(b)-x(c)-2N(n)\big)\Big\} \qquad (4.20) \\
Z = & \sum_{a,b,c,\cdots} t_{abcd}(n)t_{efag}(n+\hat{1})t_{hijb}(n+\hat{2})t_{klhf}(n+\hat{1}+\hat{2})\cdots
\end{aligned}
$$

This is our setup for TRG numerical calculation of the gauge part.

4.3 Total initial tensor

To get initial tensor for the 2-flavor Schwinger model, we conbine t^f in (4.8) and t^g in (4.20).

There is one non-trivial point in this operation. In the fermion part, we use $\hat{U}_p$ instead of U_p, so we change the sign of U_p to rewrite it $\hat{U}_p$ in the gauge part. However, gauge action (3.33) is not written in the usual plaquette, because of the θ term. So we need to rewrite this gauge action carefully.

The flipping the sign of U_p corresponds to the π shift of gauge fields A_p. In the expression (4.13), this operation shifts $p(n)$ to $p(n)+1$. The kinetic term of gauge action in (4.13) is changed as,

$$
\begin{aligned}
\beta\cos(\pi p(n)) = & \frac{\beta}{2}\left[U_p(n)+U_p^\dagger(n)\right] = \frac{\beta}{2}\left[-\hat{U}_p(n)-\hat{U}_p^\dagger(n)\right] \\
\equiv & \frac{\beta}{2}\left[\mathrm{e}^{i\pi(\hat{p}(n)+1)}+\mathrm{e}^{-i\pi(\hat{p}(n)+1)}\right] \\
= & \beta\cos\left(\pi(\hat{p}(n)+1)\right), \qquad \hat{p}(n)=p(n)-1. \qquad (4.21)
\end{aligned}
$$

We define the new variable $\hat{p}(n)$. We also need to rewrite $q(n)$, which is defined in (4.16), by using $\hat{p}(n)$ as following.

$$
\begin{aligned}
q(n) = & p(n)-2N(n) = \hat{p}(n)+1-2N(n) \\
\equiv & \hat{p}(n)-2\hat{N}(n) \qquad (4.22)
\end{aligned}
$$

$\hat{N}(n)$ is the new winding number for each plaquettes, and its definition is the following.

$$
\hat{N}(n) = N(n)-\frac{1}{2}, \qquad \hat{N}(n) = \begin{cases} +\frac{3}{2} & (2\leq\hat{p}(n)<4) \\ +\frac{1}{2} & (0\leq\hat{p}(n)<2) \\ -\frac{1}{2} & (-2\leq\hat{p}(n)<0) \\ -\frac{3}{2} & (-4\leq\hat{p}(n)<-2) \end{cases} \qquad (4.23)
$$

The gauge part tensiors (4.14) and (4.20) are also written by $\hat{p}$,

$$T_{abcd}(n) = \exp\left\{\beta\cos(\pi(\hat{p}(n)+1)) + \frac{i\theta}{2}q(n)\right\}$$
$$= \exp\left\{\beta\cos(\pi(d+a-b-c+1)) + \frac{i\theta}{2}(\hat{p}(n) - 2\hat{N}(n))\right\}, \tag{4.24}$$

$$t^g_{abcd} = \frac{\sqrt{w(a)w(b)w(c)w(d)}}{4}\exp\Big\{\beta\cos\big(\pi(x(d)+x(a)-x(b)-x(c)+1)\big)$$
$$+ \frac{i\theta}{2}\big(x(d)+x(a)-x(b)-x(c)-2\hat{N}(n)\big)\Big\}. \tag{4.25}$$

The fermion part is defined in (4.8), and it is the following expression.

$$t^f_{i_1j_1i_2j_2i'_1j'_1i'_2j'_2dc}(-1)^{P(i_1,j_1,i_2,j_2,i'_1,j'_1,i'_2,j'_2)} = (-1)^{i_2j_1+j'_1(i'_1+i'_2)+i'_2j'_2}\frac{1}{2}\Big[m\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,0}$$
$$-\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,2}\delta_{i_1+j'_1,1}\delta_{j_1+i'_1,1}(-1)^{i_1}\mathrm{e}^{i\pi(i_1-j_1)x(d)}$$
$$-\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,2}\delta_{i_1+j'_1,1}\delta_{j_2+i'_2,1}(-1)^{i_1}\mathrm{e}^{i\pi(i_1x(d)-j_2x(c))}$$
$$-\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,2}\delta_{i_2+j'_2,1}\delta_{j_1+i'_1,1}(-1)^{i_2}\mathrm{e}^{i\pi(i_2x(c)-j_1x(d))}$$
$$-\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,2}\delta_{i_2+j'_2,1}\delta_{j_2+i'_2,1}(-1)^{i_2}\mathrm{e}^{i\pi(i_2-j_2)x(c)}\Big]$$
$$= (-1)^{i_2j_1+j'_1(i'_1+i'_2)+i'_2j'_2}\frac{1}{2}\Big[m\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,0}$$
$$-\delta_{i_1+j'_1+i_2+j'_2,1}\delta_{j_1+i'_1+j_2+i'_2,1}(-1)^{i_1+i_2}\mathrm{e}^{i\pi(i_1-j_1)x(d)}\mathrm{e}^{i\pi(i_2-j_2)x(c)}\Big]. \tag{4.26}$$

Finally, we get the total tensor by conbining (4.25) with (4.26). The total tensor is,

$$t_{i_1j_1i_2j_2i'_1j'_1i'_2j'_2;abcd}(n) = t^f_{i_1j_1i_2j_2i'_1j'_1i'_2j'_2dc}(-1)^{P(i_1,j_1,i_2,j_2,i'_1,j'_1,i'_2,j'_2)}t^g_{abcd}$$
$$= (-1)^{i_2j_1+j'_1(i'_1+i'_2)+i'_2j'_2}\frac{\sqrt{w(a)w(b)w(c)w(d)}}{8}$$
$$\Big[m\delta_{i_1+j_1+i_2+j_2+i'_1+j'_1+i'_2+j'_2,0}$$
$$-\delta_{i_1+j'_1+i_2+j'_2,1}\delta_{j_1+i'_1+j_2+i'_2,1}(-1)^{i_1+i_2}\mathrm{e}^{i\pi(i_1-j_1)x(d)}\mathrm{e}^{i\pi(i_2-j_2)x(c)}$$
$$\exp\Big\{\beta\cos\big(\pi(x(d)+x(a)-x(b)-x(c)+1)\big)$$
$$+\frac{i\theta}{2}\big(x(d)+x(a)-x(b)-x(c)-2\hat{N}(n)\big)\Big\}\Big]. \tag{4.27}$$

$$t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2;abcd}(n) = \begin{array}{ccccc} & & b, i_2, j_2 & & \\ & & | & & \\ c, i'_1, j'_1 & - & t(n) & - & a, i_1, j_1 \\ & & | & & \\ & & d, i'_2, j'_2 & & \end{array} \tag{4.28}$$

The definition of $\hat{N}$ is written in (4.23).

4.4 Creating large initial tensor

To understand effects of the staggered phase, we create large initial tensor which holds translation symmetry.

4.4.1 Initial tensor

To create large initial tensor, we want to use the initial tensor without the staggered phase. So our starting point is t^f of (4.8) and t^g (4.20). The initial tensor without staggered phase is the following.

$$\begin{aligned} t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2;abcd}(n) =& t^f_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2 dc} (-1)^{P(i_1, j_1, i_2, j_2, i'_1, j'_1, i'_2, j'_2)} t^g_{abcd} \\ =& (-1)^{i_2 j_1 + j'_1(i'_1 + i'_2) + i'_2 j'_2} \frac{\sqrt{w(a)w(b)w(c)w(d)}}{8} \\ & \Big[m \delta_{i_1 + j_1 + i_2 + j_2 + i'_1 + j'_1 + i'_2 + j'_2, 0} \\ & - \delta_{i_1 + j'_1 + i_2 + j'_2, 1} \delta_{j_1 + i'_1 + j_2 + i'_2, 1} (-1)^{i_1 + i_2} \mathrm{e}^{i\pi(i_1 - j_1)x(d)} \mathrm{e}^{i\pi(i_2 - j_2)x(c)} \\ & \exp\Big\{ \beta \cos\big(\pi(x(d) + x(a) - x(b) - x(c))\big) \\ & + \frac{i\theta}{2}\big(x(d) + x(a) - x(b) - x(c) - 2N(n)\big) \Big\} \Big]. \end{aligned} \tag{4.29}$$

$$t_{i_1 j_1 i_2 j_2 i'_1 j'_1 i'_2 j'_2;abcd}(n) = \begin{array}{ccccc} & & b, i_2, j_2 & & \\ & & | & & \\ c, i'_1, j'_1 & - & t(n) & - & a, i_1, j_1 \\ & & | & & \\ & & d, i'_2, j'_2 & & \end{array} \tag{4.30}$$

We use the following notation to create large tensor.

$$
T(N) = \begin{array}{ccccccc}
 & & i_4, j_4, a_4 & & i_2, j_2, a_2 & & \\
 & & | & & | & & \\
i'_1, j'_1, c_1 & - & t_2(n-\hat{1}) & -(k_1, l_1, b_1)- & t_1(n) & - & i_1, j_1, a_1 \\
 & & | & & | & & \\
 & & (k_4, l_4, b_4) & & (k_2, l_2, b_2) & & \\
 & & | & & | & & \\
i'_3, j'_3, c_3 & - & t_3(n-\hat{1}-\hat{2}) & -(k_3, l_3, b_3)- & t_4(n-\hat{2}) & - & i_3, j_3, a_3 \\
 & & | & & | & & \\
 & & i'_4, j'_4, c_4 & & i'_2, j'_2, c_2 & &
\end{array} \tag{4.31}
$$

i, j, k, l denote fermions' labels, and a, b, c denote gauge fields' labels. $T(N)$ is the large tensor we want, and N denotes the coordinate of the large lattice. This T has one-site translation inveriance.

$$
\begin{aligned}
T =& \sum_{k,l,b} t_1 t_2 t_3 t_4 \\
=& \sum_{k,l,b} t_{i_1 j_1 i_2 j_2 k_1 l_1 k_2 l_2; a_1 a_2 b_1 b_2} t_{k_1 l_1 i_4 j_4 i'_1 j'_1 k_4 l_4; b_1 a_4 c_1 b_4} t_{k_3 l_3 k_4 l_4 i'_3 j'_3 i'_4 j'_4; b_3 b_4 c_3 c_4} t_{i_3 j_3 k_2 l_2 k_3 l_3 i'_2 j'_2; a_3 b_2 b_3 c_2} \\
=& \sum_{k,l,b} (-1)^{i_2 j_1 + l_1(k_1+k_2) + k_2 l_2} \frac{\sqrt{w(a_1)w(a_2)w(b_1)w(b_2)}}{8} \\
& \Big[m\delta_{i_1+j_1+i_2+j_2+k_1+l_1+k_2+l_2,0} - \delta_{i_1+l_1+i_2+l_2,1}\delta_{j_1+k_1+j_2+k_2,1}(-1)^{i_1+i_2} \mathrm{e}^{i\pi(i_1-j_1)x(b_2)} \mathrm{e}^{i\pi(i_2-j_2)x(b_1)} \\
& \exp\left\{ \beta\cos\big(\pi(x(b_2)+x(a_1)-x(a_2)-x(b_1))\big) + \frac{\theta}{2\pi}\log\left(\mathrm{e}^{i\pi(x(b_2)+x(a_1)-x(a_2)-x(b_1))}\right)\right\}\Big] \\
& (-1)^{i_4 l_1 + j'_1(i'_1+k_4) + k_2 l_2} \frac{\sqrt{w(b_1)w(a_4)w(c_1)w(b_4)}}{8} \\
& \Big[m\delta_{k_1+l_1+i_4+j_4+i'_1+j'_1+k_4+l_4,0} - \delta_{k_1+j'_1+i_4+l_4,1}\delta_{l_1+i'_1+j_4+k_4,1}(-1)^{k_1+i_4} \mathrm{e}^{i\pi(k_1-l_1)x(b_4)} \mathrm{e}^{i\pi(i_4-j_4)x(c_1)} \\
& \exp\left\{ \beta\cos\big(\pi(x(b_4)+x(b_1)-x(a_4)-x(c_1))\big) + \frac{\theta}{2\pi}\log\left(\mathrm{e}^{i\pi(x(b_4)+x(b_1)-x(a_4)-x(c_1))}\right)\right\}\Big] \\
& (-1)^{k_4 l_3 + j'_3(i'_3+i'_4) + i'_4 j'_4} \frac{\sqrt{w(b_3)w(b_4)w(c_3)w(c_4)}}{8} \\
& \Big[m\delta_{k_3+l_3+k_4+l_4+i'_3+j'_3+i'_4+j'_4,0} - \delta_{k_3+j'_3+k_4+j'_4,1}\delta_{l_3+i'_3+l_4+i'_4,1}(-1)^{k_3+k_4} \mathrm{e}^{i\pi(k_3-l_3)x(c_4)} \mathrm{e}^{i\pi(k_4-l_4)x(c_3)} \\
& \exp\left\{ \beta\cos\big(\pi(x(c_4)+x(b_3)-x(b_4)-x(c_3))\big) + \frac{\theta}{2\pi}\log\left(\mathrm{e}^{i\pi(x(c_4)+x(b_3)-x(b_4)-x(c_3))}\right)\right\}\Big] \\
& (-1)^{k_2 j_3 + l_3(k_3+i'_2) + i'_2 j'_2} \frac{\sqrt{w(a_3)w(b_2)w(b_3)w(c_2)}}{8} \\
& \Big[m\delta_{i_3+j_3+k_2+l_2+k_3+l_3+i'_2+j'_2,0} - \delta_{i_3+l_3+k_2+j'_2,1}\delta_{j_3+k_3+l_2+i'_2,1}(-1)^{i_3+k_2} \mathrm{e}^{i\pi(i_3-j_3)x(c_2)} \mathrm{e}^{i\pi(k_2-l_2)x(b_3)} \\
& \exp\left\{ \beta\cos\big(\pi(x(c_2)+x(a_3)-x(b_2)-x(b_3))\big) + \frac{\theta}{2\pi}\log\left(\mathrm{e}^{i\pi(x(c_2)+x(a_3)-x(b_2)-x(b_3))}\right)\right\}\Big]
\end{aligned}
\tag{4.32}
$$

References

[1] H. Kanno, S. Akiyama, K. Murakami and S. Takeda, JHEP **11** (2025), 036 doi:10.1007/JHEP11(2025)036 [arXiv:2412.08959 [hep-lat]].

[2] C. Gattringer and C. B. Lang, Lect. Notes Phys. **788** (2010), 1-343 Springer, 2010, ISBN 978-3-642-01849-7, 978-3-642-01850-3 doi:10.1007/978-3-642-01850-3

[3] R. Dempsey, I. R. Klebanov, S. S. Pufu and B. Zan, Phys. Rev. Res. **4** (2022) no.4, 043133 doi:10.1103/PhysRevResearch.4.043133 [arXiv:2206.05308 [hep-th]].

[4] M. F. L. Golterman and J. Smit, Nucl. Phys. B **245** (1984), 61-88 doi:10.1016/0550-3213(84)90424-3

[5] G. W. Kilcup and S. R. Sharpe, Nucl. Phys. B **283** (1987), 493-550 doi:10.1016/0550-3213(87)90285-9

[6] S. Akiyama and D. Kadoh, "More about the Grassmann tensor renormalization group," JHEP **10** (2021), 188 doi:10.1007/JHEP10(2021)188 [arXiv:2005.07570 [hep-lat]].