Chiral Fermion on Curved Domain-wall

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格子ゲージ理論

- 場の量子論の非摂動的扱いの一つ 。厳密な場の量子論の定式化。
- 時空を格子にして、無限自由度を有限自由度で近似する。数値計算が可能 。
- 運動量に自然に上限ができて、紫外発散が起こらない。

クォークの閉じ込めやハドロンの質量などが計算可能！

Domain-wall and edge states

質量の符号を反転させる．

$$H = -i\sigma_1\left(\frac{\partial}{\partial x} + \sigma_3\epsilon M\right)$$

$$\epsilon(x) = \begin{cases} -1 & (x < 0) \\ 1 & (x > 0) \end{cases},$$

このとき，$x = 0$ に局在した chiral 0-mode

$$\psi = e^{-M|x|}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

が存在する．

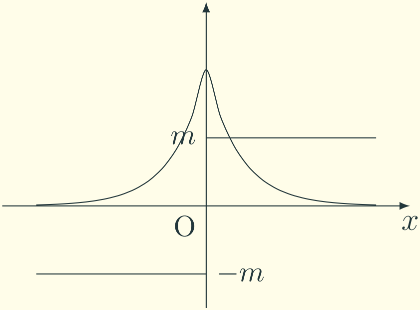
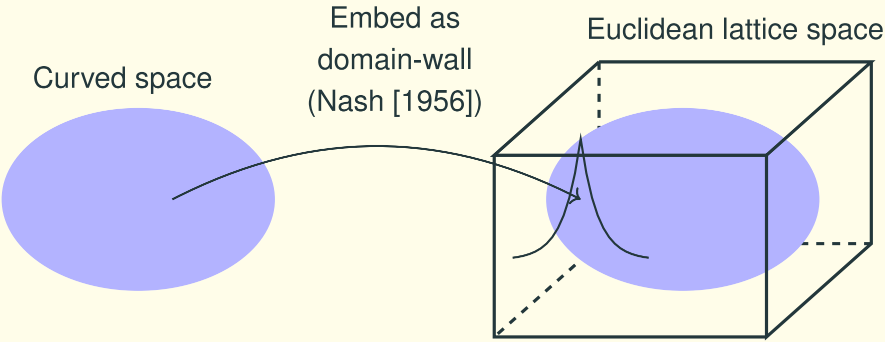


Fig 1: Edge state localized at the domain-wall.

Motivation

平坦な正方格子に曲がった domain-wall を埋め込むことで，曲がった空間の fermion を格子で定式化する．



Cf. Brower et al. [2017] and Ambjørn et al. [2001] studied on triangle lattices.

Embedding a curved space

任意の n 次元 Riemann 多様体 (M^n, g) について，埋め込み $f : M^n \to \mathbb{R}^m$ $(m \gg n)$ があって，M^n は

$$x^\mu = f^\mu(\tilde{x}^1, \cdots, \tilde{x}^n) \; (\mu = 1, \cdots, m) \tag{1}$$

$$\left(\begin{array}{ll} x^\mu & : \text{Cartesian coordinates of } \mathbb{R}^m \\ \tilde{x}^i & : \text{coordinates of } M^n \end{array} \right.$$

と同一視され，計量 g は

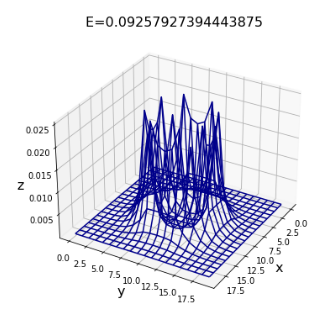
$$g_{ij} = \sum_{\mu\nu} \frac{\partial f^\mu}{\partial \tilde{x}^i} \delta_{\mu\nu} \frac{\partial f^\nu}{\partial \tilde{x}^j}. \tag{2}$$

したがって，(M^n, g) は $\mathbb{R}^m$ の部分多様体と考えられる．

cf. Nash [1956].

Our Work

- 曲がった domain-wall に局在する状態を発見した.
- それらは重力を感じていた (スピン接続を通じて).



Cf. Similar studies in condensed matter physics [Imura et al. [2012], Parente et al. [2011]].

Contents

Plan of this section

S^1 domain-wall を $\mathbb{R}^2$ に埋め込んで

- Dirac 演算子の固有値
- エッジ状態
- その有効 Dirac 演算子

を連続，格子で調べた．

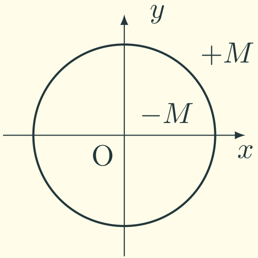


Fig 2: Continuum Case

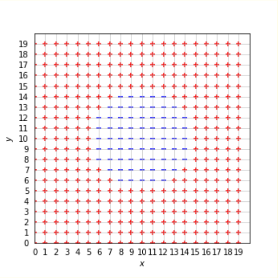
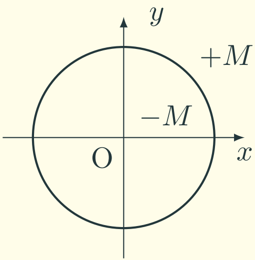


Fig 3: Lattice case

Dirac op in Continuum case

domain-wall は

$$\epsilon_A(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$



で与えたれ，Ditac 演算子は

$$H = \sigma_3 \left(\sum_{i=1,2} \left(\sigma_i \frac{\partial}{\partial x^i} \right) + M\epsilon \right) = \begin{pmatrix} M\epsilon & e^{-i\theta}(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}) \\ -e^{i\theta}(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}) & -M\epsilon \end{pmatrix}. \tag{3}$$

Edge states

M が十分大きい時に，エッジ状態は

$$\psi^{E,j}_{\text{edge}} \simeq \sqrt{\frac{M}{4\pi r}} e^{-M|r-r_0|} \begin{pmatrix} e^{i(j-\frac{1}{2})\theta} \\ e^{i(j+\frac{1}{2})\theta} \end{pmatrix}.$$

しかも

$$\gamma_{\text{normal}} := \sigma_1 \cos\theta + \sigma_2 \sin\theta = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix},$$

の固有値 $+1$ を持つ．

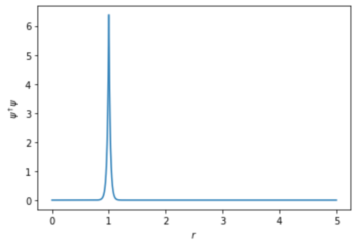


Fig 4: Edge state when $M = 5, r_0 = 1$

Effective Dirac op

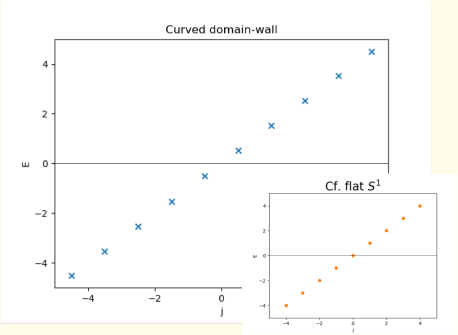


Fig 5: Eigenvalue of edge states at $M = 5, r_0 = 1$

重力は固有値の gap に現れる.

chiral edge state の有効 Dirac 演算子は

$$H_{S^1} = \frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + \frac{1}{2}\right) \quad (4)$$

$$E = \frac{j}{r_0}\left(j = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots\right). \quad (5)$$

$\frac{1}{2}$ はスピン接続 (または Berry 位相).

Lattice domain-wall fermion

$(\mathbb{Z}/n\mathbb{Z})^2$ を 2 次元格子とする．
domain-wall は

$$\epsilon(x) = \begin{cases} -1 & (|x| < r_0) \\ 1 & (|x| \geq r_0) \end{cases},$$

で与えられ，(Wilson) Dirac 演算子は

$$H = \sigma_3 \left(\sum_{i=1,2} \left[\sigma_i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{r}{2} \nabla_i^f \nabla_i^b \right] + \epsilon M \right),$$

$$(\nabla_i^f \psi)_x = \psi_{x+\hat{i}} - \psi_x,\ (\nabla_i^b \psi)_x = \psi_x - \psi_{x-\hat{i}}$$

ここで x, y 方向に周期境界条件を貸した．

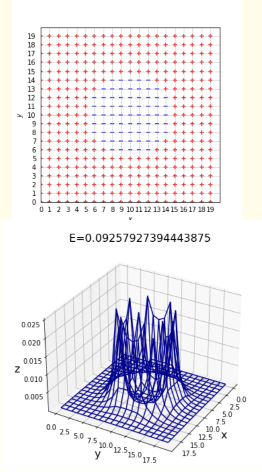


Fig 6: Edge state

Cf. Kaplan [1992] studied a flat domain-wall in $\mathbb{R}^{2m+1}$

Continuum Limit

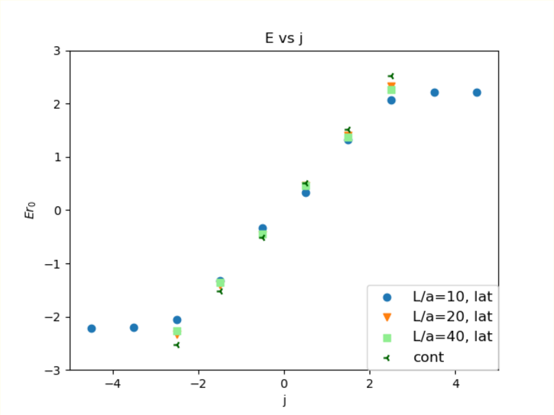


Fig 7: the Dirac eigenvalue spectrum normalized by the circle radius when $Ma = 0.7, r_0 = L/4$.

L:Lattice size

a:lattice spacing

Relative Error

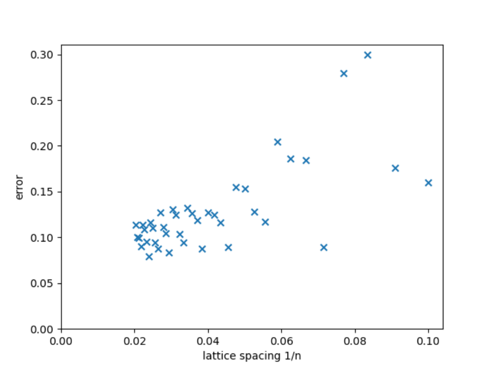


Fig 8: error $= \left| E^{\text{con}}_{\frac{1}{2}} - E^{\text{lat}}_{\frac{1}{2}} \right| / E^{\text{con}}_{\frac{1}{2}}$ is a relative error of $E_{1/2}$ between continuum and lattice when $r_0 = \frac{L}{4}$. a is lattice distance and $n \to \infty$ means continuum limit.

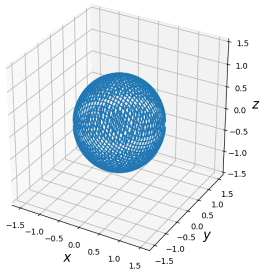
連続極限は単調ではないが，減少する．

Contents

S^2 domain-wall

domain-wall は

$$\epsilon(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$



で与えられ，Dirac 演算子は

$$H = \gamma^0 \left(\gamma^j \frac{\partial}{\partial x^j} + M\epsilon \right) = \begin{pmatrix} M\epsilon & \sigma^j \partial_j \\ -\sigma^j \partial_j & -M\epsilon \end{pmatrix} \quad (6)$$

$$\gamma^0 = \sigma_3 \otimes 1, \gamma^j = \sigma_1 \otimes \sigma^j \quad (7)$$

Edge states and Their spectrum

M が十分大きい時，エッジ状態は

$$\psi_{\text{edge}}^{\pm E,j,j_3} \simeq \sqrt{\frac{M}{2}} \frac{e^{-M|r-r_0|}}{r} \begin{pmatrix} \chi_{j,j_3}^{(\pm)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{r} \chi_{j,j_3}^{(\pm)} \end{pmatrix}.$$

$$E \simeq \frac{j+\frac{1}{2}}{r_0} \quad \left(j = \frac{1}{2}, \frac{3}{2}, \cdots\right)$$

さらに"chiral"なエッジ状態で

$$\gamma_{\text{normal}} := \sum_{i=1}^{3} \frac{x^i}{r} \gamma^i = \begin{pmatrix} 0 & \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r} \\ \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r} & 0 \end{pmatrix} \tag{8}$$

の固有値 $+1$ を持つ．

有効 Dirac 演算子は

$$H_{S^2} = \frac{1}{r_0}(\boldsymbol{\sigma}\cdot\boldsymbol{L} + 1), \tag{9}$$

で与えられ，2 成分 spinor χ に作用する．

Effective Dirac op and Dirac op. of S^2

有効 Dirac 演算子は

$$H_{S^2} = \frac{1}{r_0}(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1) \tag{10}$$

$$\Big\downarrow \quad s = \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$ で gauge 変換すると

$$\begin{aligned} s^{-1} H_{S^2} s &= -\frac{\sigma_3}{r_0}\left(\sigma_1 \frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2\right)\right) \\ &= -\frac{\sigma_3}{r_0} \mathbb{D}_{S^2}. \end{aligned} \tag{11}$$

Spin conn. of S^2

エッジ状態は S^2 domain-wall の重力を感じる

Cf. [Takane and Imura [2013]].

Euler number of S^2

スピン接続

$$\omega_\Delta = -\frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2\sin\theta d\phi = -\frac{1}{2}i\sigma_3\cos\theta d\phi, \tag{12}$$

から Levi-Civita 接続 ω，Riemann 曲率 R が得られる．

$$\omega = \begin{pmatrix} 0 & -\cos\theta d\phi \\ \cos\theta d\phi & 0 \end{pmatrix} \tag{13}$$

$$\frac{R}{2\pi} = \frac{d\omega + \omega^2}{2\pi} = \begin{pmatrix} 0 & \frac{\sin\theta}{2\pi}d\theta d\phi \\ -\frac{\sin\theta}{2\pi}d\theta d\phi & 0 \end{pmatrix} \tag{14}$$

Euler class of S^2

S^2 の Euler 数は

$$\chi(S^2) = \int_{S^2}\frac{\sin\theta}{2\pi}d\theta d\phi = 2. \tag{15}$$

で与えられる．

Induced gravity makes a gap in the spectrum.

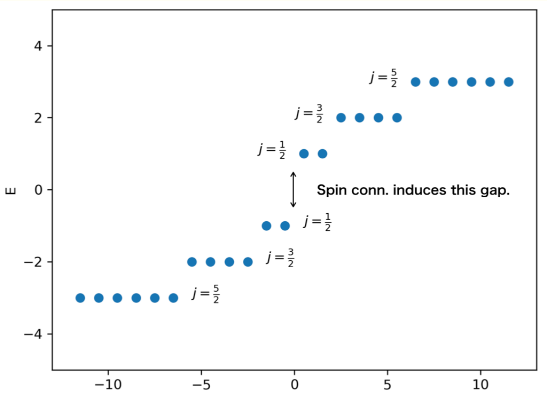


Fig 9: Spectrum of edge states when $M = 5, r_0 = 1$

Eigenvalue

$$E \simeq \pm \frac{j + \frac{1}{2}}{r_0} \tag{16}$$

Degeneracy

$$2j + 1 \tag{17}$$

$(\text{Euler number of } S^2) = 2$

Lattice Domain-wall Fermion

$(\mathbb{Z}/n\mathbb{Z})^3$ を 3 次元格子とする．

domain-wall は

$$\epsilon(x) = \begin{cases} -1 & (|x| < r_0) \\ 1 & (|x| \geq r_0) \end{cases} ,$$

で与えられ，(Wilson) Dirac 演算子は

$$H = \gamma_3 \left(\sum_{i=1,2} \left[\gamma_i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{r}{2} \nabla_i^f \nabla_i^b \right] + \epsilon M \right).$$

$$(\nabla_i^f \psi)_x = \psi_{x+\hat{i}} - \psi_x, \; (\nabla_i^b \psi)_x = \psi_x - \psi_{x-\hat{i}}$$

である．

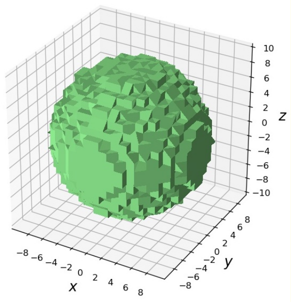


Fig 10: S^2 Domain-wall on lattice

Edge states

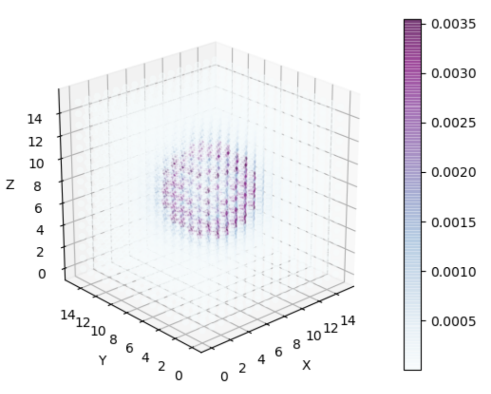


Fig 11: Edge state localized at S^2 when M=0.7 and lattice size $= 16^3$

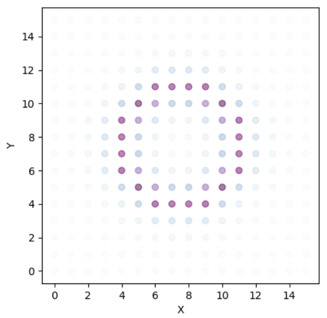


Fig 12: Slice at $z = 7$

Spectrum in Lattice case

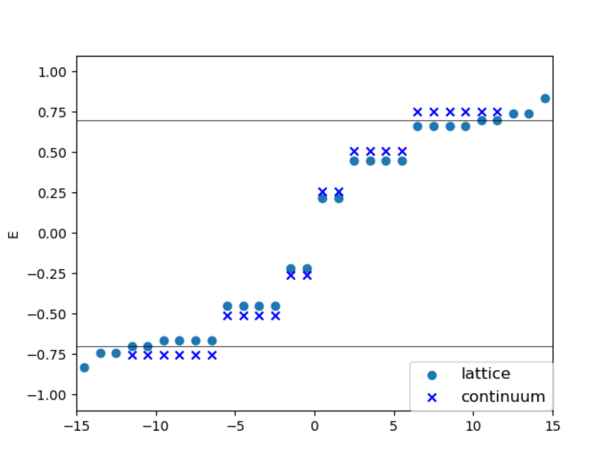


Fig 13: Spectrum of edge states at S^2 when $n = 16, M = 0.7$.

連続の固有値を再現!

Contents

Summary

我々は S^1, S^2 を正方格子上の曲がった domain-wall として考えた，
そして

- chiral エッジ状態が domain-wall に現れ
- それらは誘導されたスピン接続を通じて重力を感じていた．

Outlook

- 連続極限を確立する.
- 重力アノマリーの流入を調べる.
- 非自明な曲率を持つ指数定理に応用する.

Thank You

Reference i

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Nash, J. (1956). The imbedding problem for riemannian manifolds. Annals of Mathematics, 63(1):20–63.

Parente, V., Lucignano, P., Vitale, P., Tagliacozzo, A., and Guinea, F. (2011). Spin connection and boundary states in a topological insulator. Phys. Rev. B, 83:075424.

Takane, Y. and Imura, K.-I. (2013). Unified description of dirac electrons on a curved surface of topological insulators. Journal of the Physical Society of Japan, 82(7):074712.

Contents

Effective Dirac op

We consider a normalized edge state as

$$\psi_{\text{edge}} = \rho(r)\begin{pmatrix}\chi(\theta)\\ e^{i\theta}\chi(\theta)\end{pmatrix},\ \chi(\theta+2\pi)=\chi(\theta) \tag{18}$$

$$\int_0^\infty dr 2r\rho^2 = 1,\ \int_0^{2\pi} d\theta\chi^\dagger\chi = 1 \tag{19}$$

and let $2r\rho^2 \to \delta(r-r_0)$ $(M\to\infty)$. Then we obtain

$$\int dxdy\psi_{\text{edge}}^\dagger H\psi_{\text{edge}} \to \int_0^{2\pi} d\theta\chi^\dagger \underbrace{\frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta}+\frac{1}{2}\right)}_{\text{Effective Dirac op } H_{S^1}\ !!}\chi \tag{20}$$

The factor $\frac{1}{2}$ means induced spin connection.

Effective Dirac op

We consider a normalized edge state as

$$\psi_{\text{edge}} = \rho(r) \begin{pmatrix} \chi(\theta, \phi) \\ \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r} \chi(\theta, \phi) \end{pmatrix} \tag{21}$$

$$\int_0^\infty dr r^2 2\rho^2 = 1, \ \int_{S^2} \chi^\dagger \chi = 1, \tag{22}$$

and we assume $2r^2\rho^2 \to \delta(r - r_0)$ $(M \to \infty)$. Thus

$$\int dx^3 \psi_{\text{edge}}^\dagger H \psi_{\text{edge}} = \int_0^\infty dr 2r^2\rho^2 \int_{S^2} \chi^\dagger \frac{1}{r}(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1)\chi$$
$$\to \int_{S^2} \chi^\dagger \underbrace{\frac{1}{r_0}(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1)}\chi \ (M \to \infty), \tag{23}$$

Effective Dirac op H_{S^2} !!

where $\boldsymbol{L}$ is an orbital angular momentum.

Effective Dirac op and Dirac op. of S^2

The gauge transformation using

$$s = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{24}$$

changes $\chi \to s^{-1}\chi$ and

$$\begin{aligned} H_{S^2} \to & s^{-1} H_{S^2} s \\ = & \frac{1}{r_0}\begin{pmatrix} 0 & -\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{1}{2}\frac{\cos\theta}{\sin\theta} \\ \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{2}\frac{\cos\theta}{\sin\theta} & 0 \end{pmatrix} \\ = & -\frac{\sigma_3}{r_0}\left(\sigma_1\frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \underbrace{\frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2}_{\text{Spin conn. of } S^2}\right)\right) \\ = & -\frac{\sigma_3}{r_0}\not{D}_{S^2}. \end{aligned} \tag{25}$$

Edge states are affected by the spin connection of the spherical domain-wall [Takane and Imura [2013]].

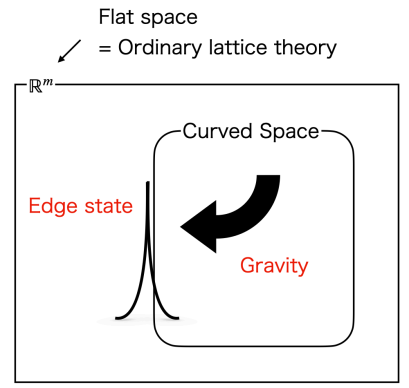
Goal

Embed S^1, S^2 into a square lattice.

Curved domain-wall

↓

- Edge states appear !
- They feel gravity !



Motivation

It is too difficult to consider a lattice theory on a curved space.
If we use

- A square lattice
  $\longrightarrow$ A curved space can NOT be approximated by it.
- Triangulation [Ambjørn et al. [2001]]
  $\longrightarrow$ Lattice regularization is different from of lattice gauge theory.

Fig 14: Triangulation of a toy[1]

[1] https://12px.com/blog/2014/02/delaunay/

Main result

- Edge states appear at the curved domain-wall,
- They feel gravity or curvature through the induced spin connection.

Cf. Similar studies in condensed matter physics.[Imura et al. [2012], Parente et al. [2011]].

Domain-wall and edge states

If the sign of mass is flipped as

$$\epsilon(x) = \begin{cases} -1 & (x < 0) \\ 1 & (x > 0) \end{cases},$$

then localized states appear at $x = 0$.

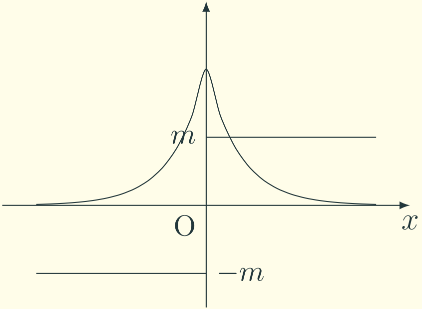


Fig 15: Edge state localized at the domain-wall.