Nonperturbative Aspects of QFT and Generalized Global symmetry

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Quantum Field Theory

- Systematic framework to combine Quantum Mechanics & Special Relativity

  Hilbert space & Hamiltonian

  $\longleftrightarrow$ Correlation functions $\langle \phi(x_1) \cdots \phi(x_n) \rangle$

  $\Rightarrow$ S-matrix, etc.

- Universal low-energy description of many-body quantum systems.

  Non-QFT systems

  $\xrightarrow{\text{RG flow}}$ $\exists$ QFT

Minkowski QFT vs Euclidean QFT

In usual textbooks, we consider relativistic QFT in Minkowski space $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1)$.

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int \mathcal{D}\phi \, \exp\left(i \int d^d x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\right) \phi(x_1) \cdots \phi(x_n).$$

$\Rightarrow$ Wightman reconstruction theorem. (You have to treat distributions. (超函数))

(correlation functions $\longrightarrow$ Hilbert sp. & Hamiltonian)

It is also convenient to consider QFT in Euclidean convention $g_{\mu\nu} = \mathrm{diag}(+1, +1, \cdots, +1)$.

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int \mathcal{D}\phi \, \exp\left(- \int d^d x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V(\phi)\right)\right) \phi(x_1) \cdots \phi(x_n).$$

$\Rightarrow$ Osterwalder – Schrader reconstruction theorem. (Analytic functions if $x_i \neq x_j$)

Physically, you compute thermal systems $Z = \mathrm{tr}[\exp(-\beta \hat{H})]$. $\left(-it \overset{\text{Wick rotation}}{\Rightarrow} -\beta\right)$.

QFT $\longleftrightarrow$ Statistical systems

Perturbative QFT

We can compute $Z_0 = \int \mathcal{D}\phi \exp\left(-\int d^dx \left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2\right)\right)$.

What we really want is

$$Z(g) = \int \mathcal{D}\phi \exp\left(-\int d^dx \left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right)\right).$$

$\Longrightarrow$ Formally expand in terms of g,

$$Z(g) = \sum_n \frac{a_n}{n!} g^n,$$

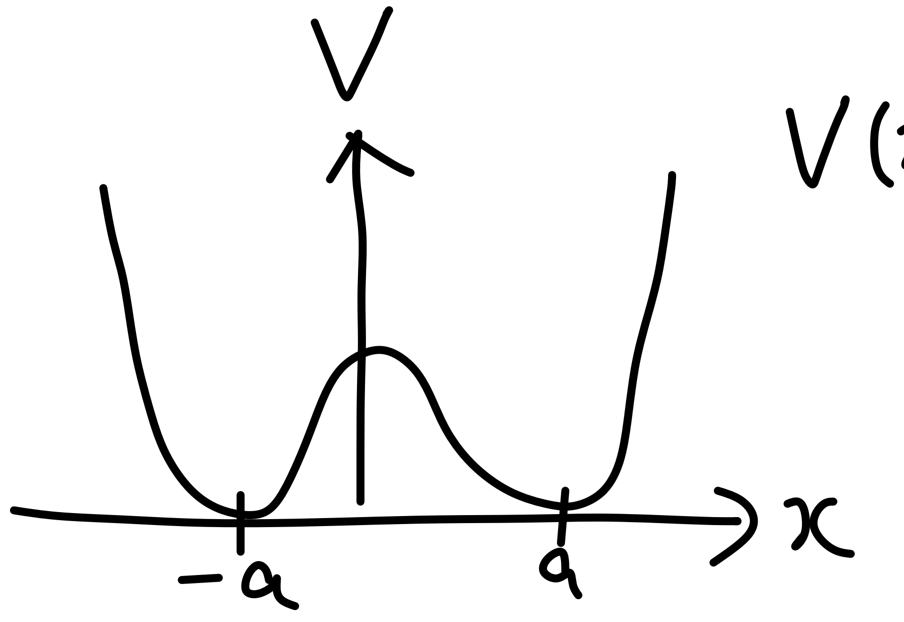
where $a_n = \int \mathcal{D}\phi \left(\frac{1}{4!}\int dx\, \phi^4(x)\right)^n \exp(-S_0)$.

This is the essence of perturbative QFT.
In the actual computations, we encounter UV divergence, so we need a remedy, renormalization.
Then, we get a formal power series in terms of renormalized g.

Perturbative expansion is a formal expansion at $g=0$.

$\longrightarrow$ It cannot capture, e.g., $\exp\left(-\frac{\#}{g}\right)$.

$V(x) = \frac{g}{4!}(x^2-a^2)^2$

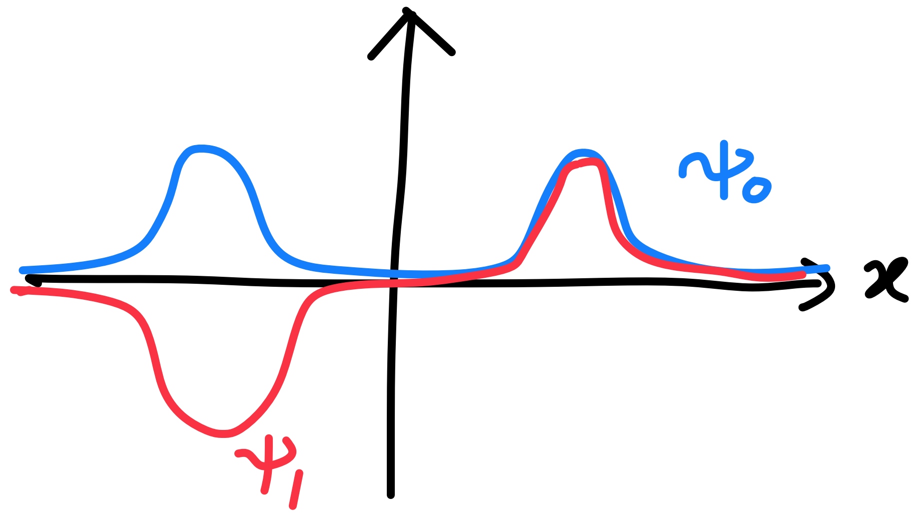


Example QM in double-well potential

Classical minima $x = \pm a$. ($\hbar \to 0$ (Here $\hbar \sim \frac{1}{\sqrt{g}}$))

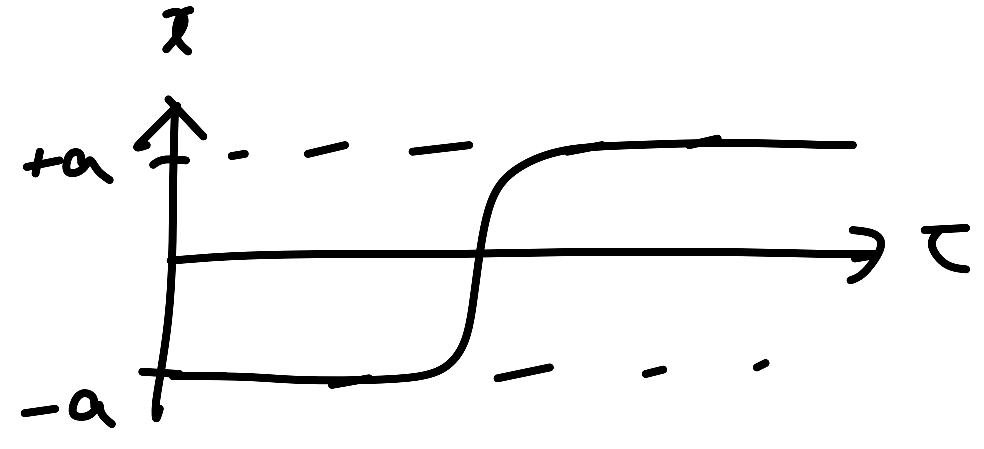
$\Rightarrow$ Within perturbation theory $\langle x \rangle_\pm = \pm a(1+\mathcal{O}(\hbar))$

However, $\langle x \rangle = \langle \psi_0 | \hat{x} | \psi_0 \rangle = 0$.



To reproduce this, we need "instanton":

$$S = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V(x)\right) = \int d\tau \, \frac{1}{2}\left(\dot{x} \mp \sqrt{2V(x)}\right)^2 \pm \int_{-a}^{+a} \sqrt{V(x)}\, dx \geq \underbrace{\sqrt{\frac{2g}{4!}}\, \frac{2}{3} a^3}_{= S_I}.$$



$\dot{x} = \pm\sqrt{2V(x)}$

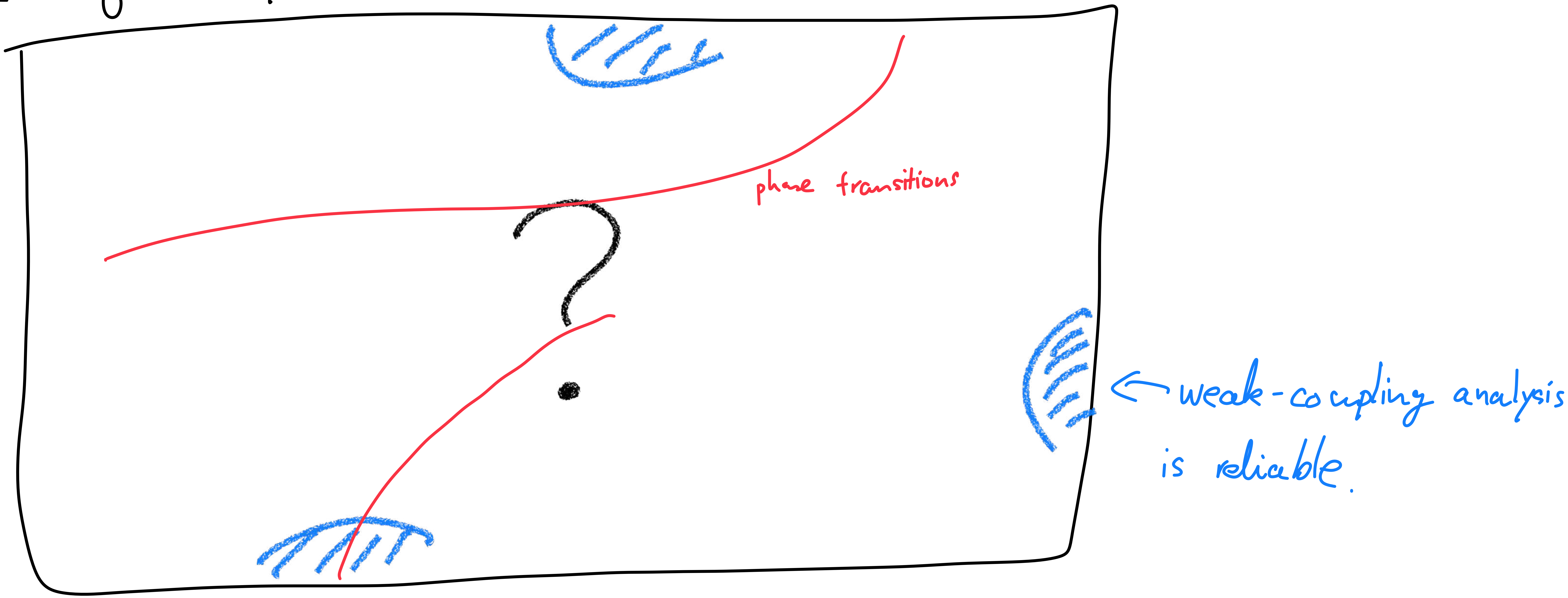
$\longrightarrow$ This contribution is roughly $\exp\left(-\frac{S_I}{\hbar}\right)$, which is exponentially small.

It is consistent with $E_1 - E_0 \sim \exp\left(-\frac{S_I}{\hbar}\right)$ by WKB analysis.

Nonperturbative effect can give huge difference.

What is phase diagram of QFT ?

Space of couplings

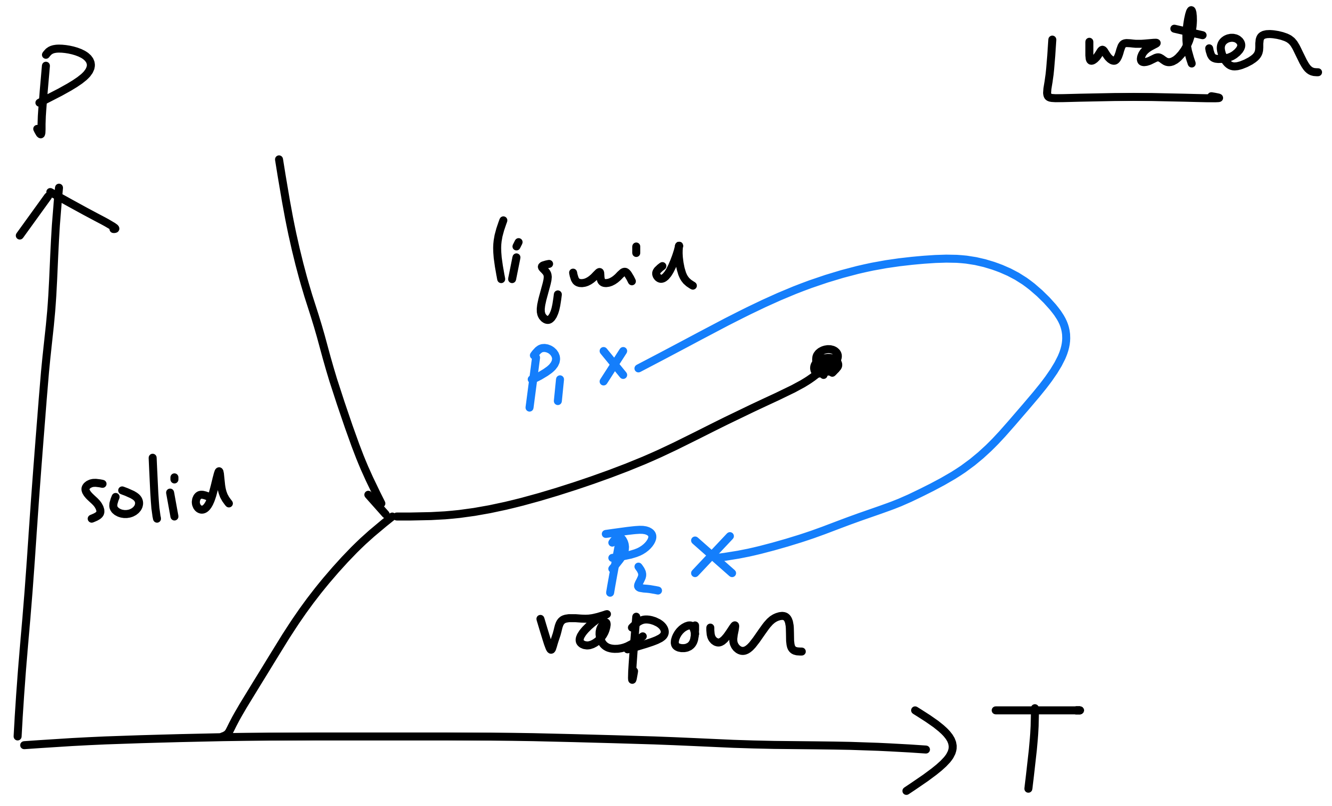


⇒ • It's important to find weak-coupling regimes of QFTs.
(You may find a nontrivial limit by some dualities!)
• We should guess the "reasonable" phase diagram.

Symmetry & Landau criterion

Phases of matter

Free energy $F = -\ln Z$ has a singularity in its thermodynamic variables $\Rightarrow$ Phase transition.



Two points P_1, P_2 are in the same phase if they can be connected without phase transitions.

($\Rightarrow$ liquid & vapour are in the same phase, but solid is different).

How do you know?

Landau criterion

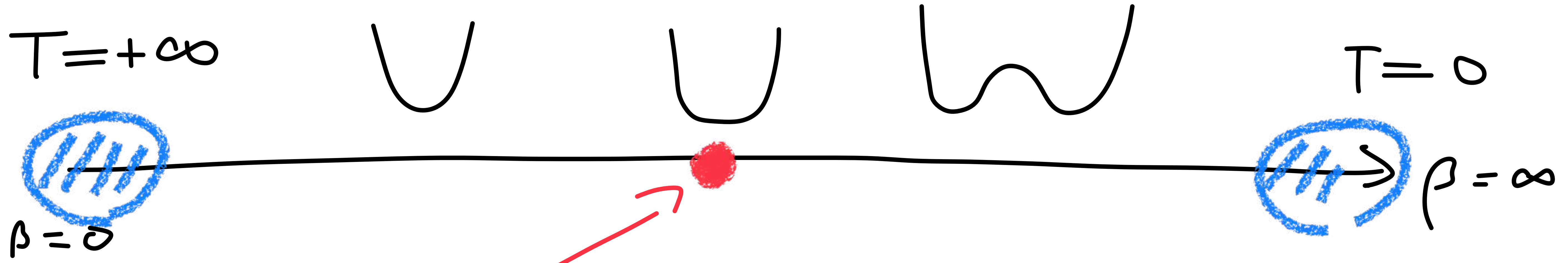
Assume that the system has a symmetry G.

If G is spontaneously broken to H_1 ($G \xrightarrow{SSB} H_1$) at P_1, and $G \xrightarrow{SSB} H_2$ at P_2, with $H_1 \neq H_2$, then P_1 & P_2 are in the different phases.

Example : Ising model

$$Z(\beta) = \sum_{\{s_i = \pm 1\}} e^{-\beta\left(-\sum_i s_i s_{i+1}\right)} \qquad \left(\approx \int \mathcal{D}\phi \, e^{-\int\{|d\phi|^2 + g(\phi^2 - v^2)^2\}}\right)$$

$\Rightarrow$



$(d \geq 2)$

$\mathbb{Z}_2$ ✓ $\qquad\qquad\qquad \mathbb{Z}_2 \xrightarrow{SSB} 1$

These are in different phases

You can predict the existence of a phase transition, even if you do not know where it is.

SSB, local order parameter

Let me remind how SSB is defined.

- (Naive def.) $\mathcal{O}(x)$: a local operator $\left(g \in G : \mathcal{O}(x) \mapsto g \cdot \mathcal{O}(x). \right)$

$$\langle \mathcal{O}(x) \rangle = v \neq 0 \quad \text{in } V \to \infty \implies G \text{ is spontaneously broken.}$$

local order parameter

(* If $M_d = S^d, T^d$, etc., then $\langle \mathcal{O}(x) \rangle = 0$ at any volume.

$\Rightarrow \lim_{V \to \infty} \langle \mathcal{O}(x) \rangle = 0$, so no SSB??)

2 rigorous defs of SSB :

① (Symmetry breaking field) Modify $H \to H + \varepsilon \int d^d x \, \mathcal{O}(x)$.

$$\lim_{\varepsilon \to 0} \lim_{V \to \infty} \langle \mathcal{O}(x) \rangle_{V, \varepsilon} = v.$$

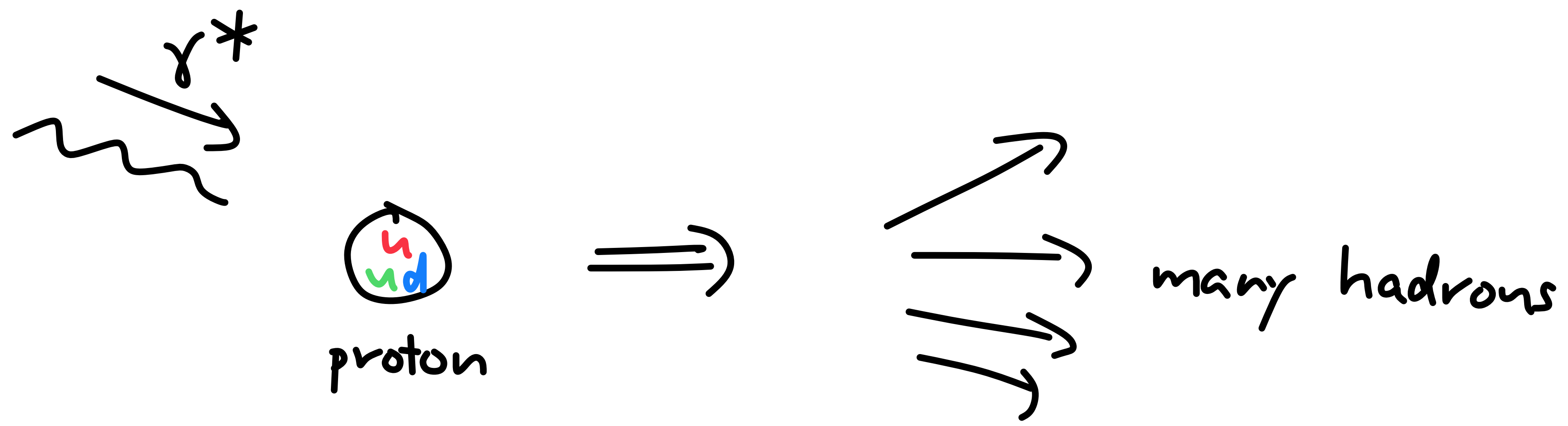
② (Violation of cluster decomposition) Compute 2-point function, and observe

$$\lim_{\substack{|x| \to \infty \\ V \to \infty}} \langle \mathcal{O}(x)^* \mathcal{O}(0) \rangle_V = |v|^2.$$

$$\Big(\text{i.e. } \underbrace{\langle \mathcal{O}^*(x) \mathcal{O}(0) \rangle}_{= |v|^2} \not\to \big| \underbrace{\langle \mathcal{O}(0) \rangle}_{= 0} \big|^2 \Big)$$

Gauge theory & Confinement

From deep inelastic scattering (DIS), we know there are freely propagating "quarks" inside proton.



$\Leftarrow$ $SU(3)$ gauge theory + Dirac fermion.

$$S = -\frac{1}{g^2}\int \mathrm{tr}(f \wedge * f) + \int \bar{\Psi}\left(\gamma^\mu(\partial_\mu + a_\mu) + m\right)\psi .$$

($f = da + a\wedge a$: $SU(3)$ gauge field strength. ψ : quark fields)

ψ is in the $\underbrace{\text{defining rep.}}_{\text{3-dim.}}$ of $SU(3)$.

3-dim. : red, green, blue

At low energy, however, we only observe the $SU(3)$-singlet states.

$\Longrightarrow$ Phenomenological def. of color confinement.

Confinement as phases of gauge theories

Can we define confinement as a phase of gauge theories?

$\Longrightarrow$ Center symmetry, or 1-form symmetry.

Ordinary symmetry is not enough!!

We need loop operators, defect operators, etc. to define confinement as a phase.

Why is this so complicated?

Fradkin - Shenker's (non-) complementarity.

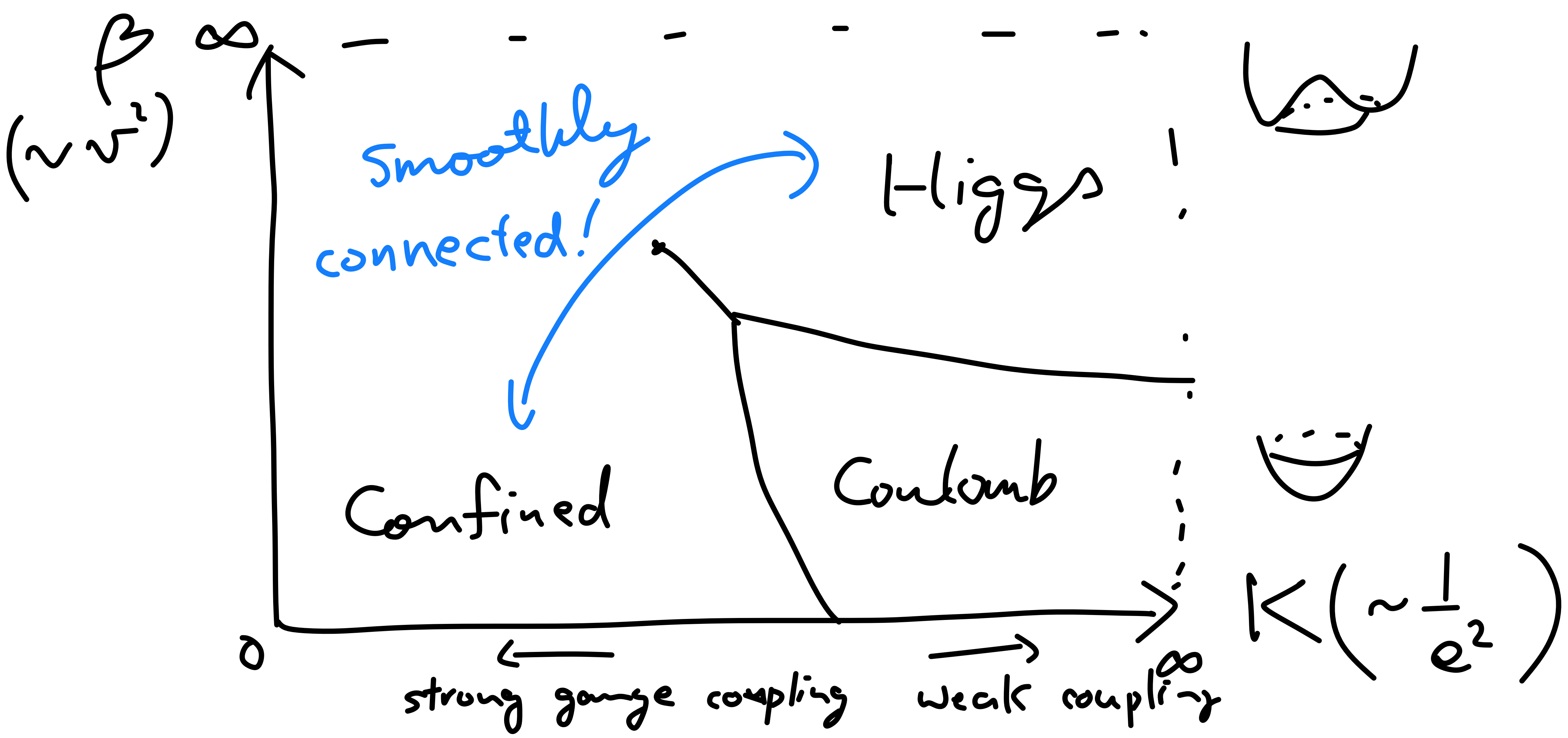
Consider the compact $U(1)$ gauge theory coupled to charge-q scalar ($q=1,2,3,\cdots$)

$$S = \frac{1}{2e^2}\int da\wedge * da + \int\left\{\left|(\partial_\mu + i q a_\mu)\phi\right|^2 + g(|\phi|^2 - v^2)^2\right\}.$$

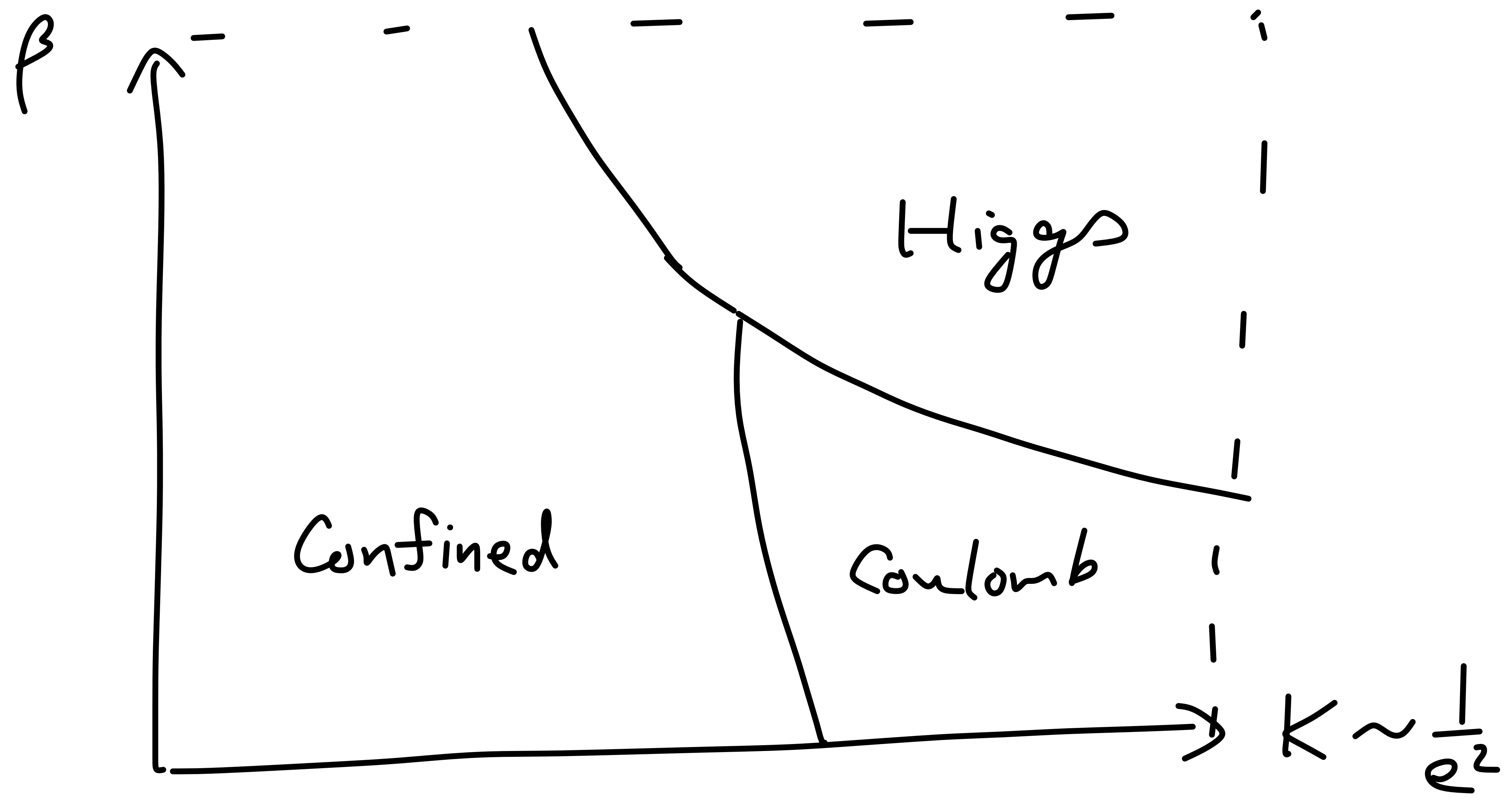
(In Fradkin, Shenker's paper, the lattice version is considered, $(\phi \leftrightarrow e^{i\theta})$

$$S = \beta \sum_i \cos(\partial_\mu \theta + q a_\mu) + K \sum_\Box \cos(f_{\mu\nu}).$$
)

$q=1$ (Confined & Higgs phases are the same)



$q\geq 2$ (They are different)



Essence of symmetry in relativistic QFT

Let's start from continuous symmetry $\phi(x) \to \phi(x) + \varepsilon^a \delta_a \phi(x)$.

Noether current $j^\mu_a(x)$ satisfies $\partial_\mu j^\mu_a(x) = 0$. (with EoM.)

$\Longrightarrow$ Define $(d-1)$-form $j_a = \frac{1}{(d-1)!} \varepsilon_{\mu_1 \mu_2 \cdots \mu_d} j^{\mu_1}_a(x) \, dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_d}$,

then $dj = 0 \iff \partial_\mu j^\mu_a = 0$.

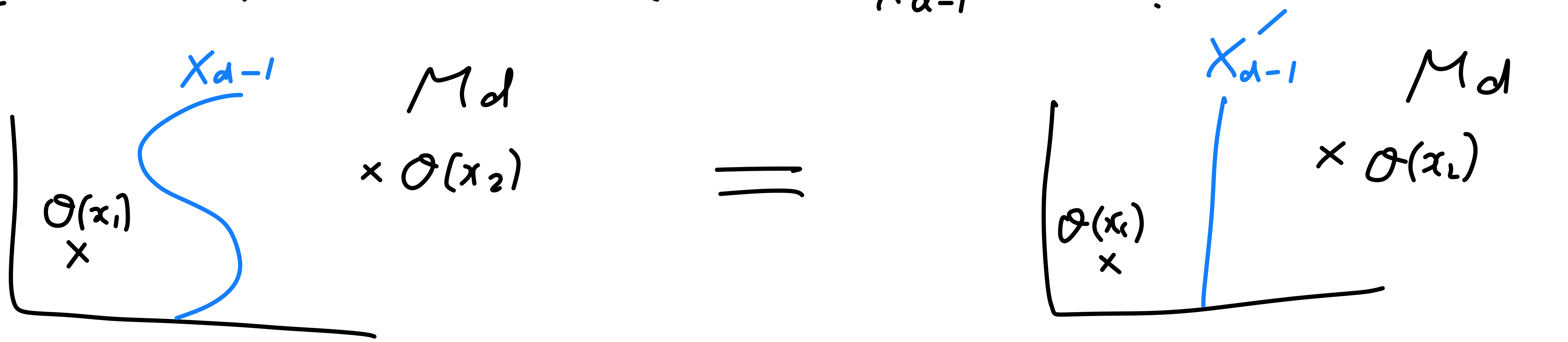
spacetime

M_d

$\cup$

Let us define a unitary operator defined on a codim-1 submanifold X_{d-1} :

$$U_\varepsilon(X_{d-1}) = \exp\left(i \, \varepsilon_a \int_{X_{d-1}} j \right).$$



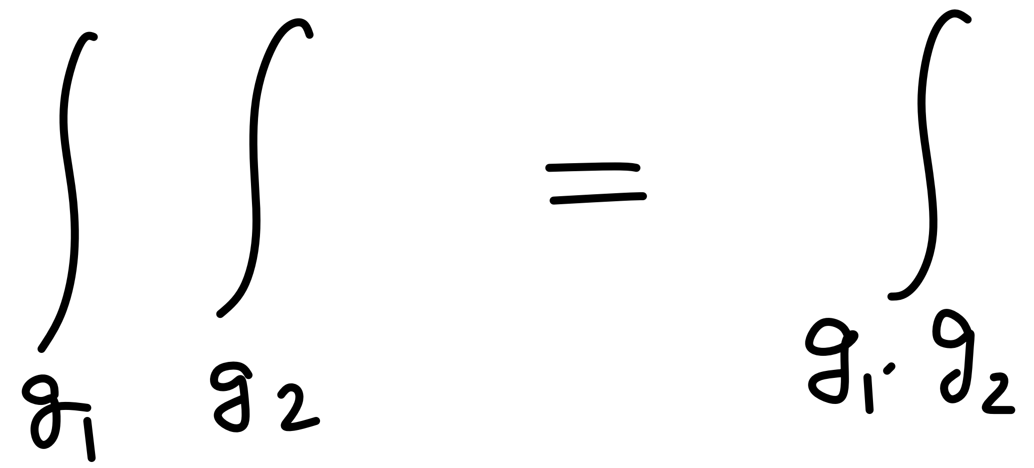
(i.e. $U_\varepsilon(X_{d-1})$ is a **topological** operator)

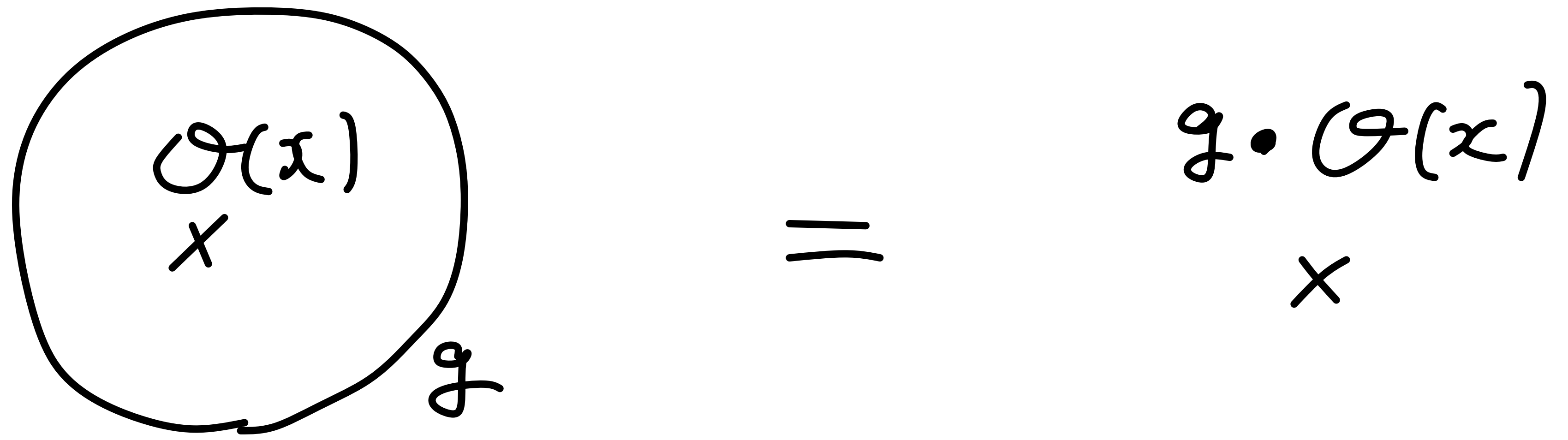
Def. (Symmetry)

d-dim (relativistic) QFT has a sym. G if

① $\exists\, U_g(X_{d-1})$: a topological codim-1 operator for each $g \in G$.

= conservation law.

②


③


(This definition applies both to continuous/discrete symmetries.)

Generalized Global Symmetry

Recently, people noticed that generalization of symmetry is very useful.

Motto

- Symmetry $\longleftrightarrow$ Conservation law

  $\Big\updownarrow$

  $\exists$ topological operators.

- "Symmetry" = Data set of topological operators consistent with QFT axioms (such as locality).

Higher-form symmetry

p-form symmetry is a simplest generalized symmetry [Gaiotto, Kapustin, Seiberg, Willet '14] ([Pantev, Sharpe, '05, ...])

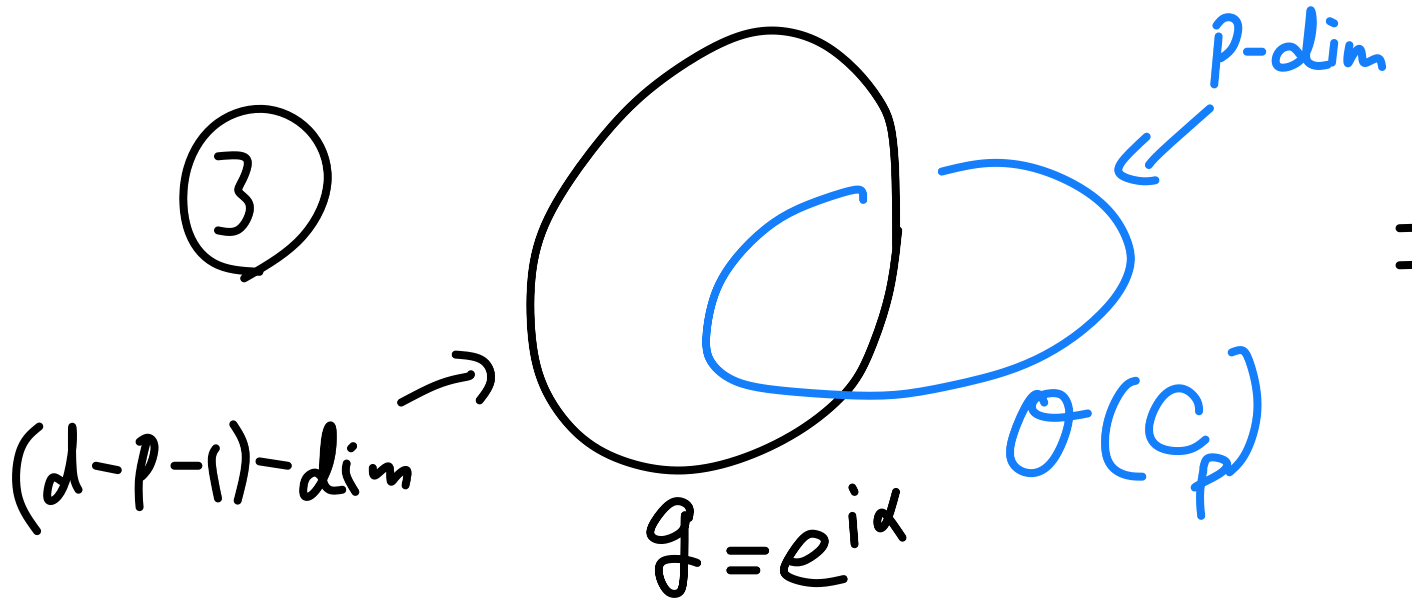
& $p=1$ is relevant for gauge theories :

Def (p-form symmetry)

d-dim QFT has a p-form symmetry G $(= U(1), \mathbb{Z}_N, \cdots)$ if

① $\exists\, U_g(X_{d-p-1})$: topological codim-$(p+1)$ operator for $g \in G$.

② $\overset{\bullet}{g_1} \quad \overset{\bullet}{g_2} \quad = \quad \overset{\bullet}{g_1 g_2}$

③ 

$= e^{i n \alpha \cdot \mathrm{Link}(X, C)} \times$ 

Example : 4d Maxwell theory

We want to revisit Fradkin & Shenker's result in view of generalized symmetry.

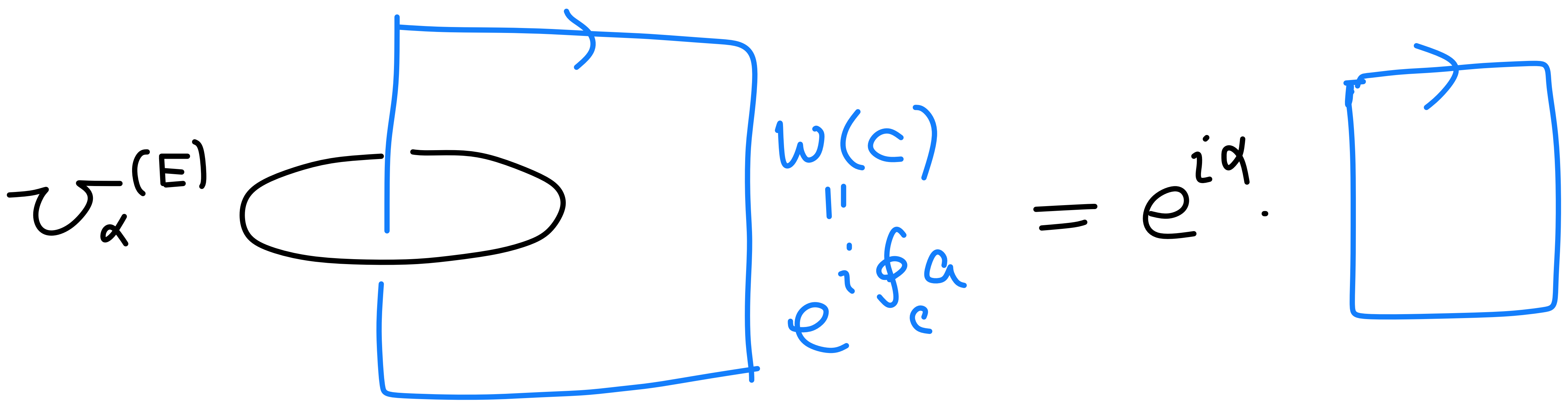
Let's consider 4d Maxwell theory as a warm up :

$$S = \frac{1}{2e^2}\int da \wedge * da .$$

<u>EoM</u>

$$\begin{cases} d\left(* \frac{1}{e^2} da \right) = 0 & \Rightarrow \; U_\alpha^{(E)}(X_2) = \exp\left(i\alpha \frac{1}{e^2}\int_{X_2} * da \right). \\ d\left(\frac{1}{2\pi} da \right) = 0 & \Rightarrow \; U_\beta^{(M)}(X_2) = \exp\left(i\beta \cdot \frac{1}{2\pi}\int_{X_2} da \right). \end{cases}$$

Theory has $U(1)_E^{[1]} \times U(1)_M^{[1]}$ symmetry.



← wilson loop.

(Similarly, $U_\beta^{(M)}$ acts on 't Hooft loop.)

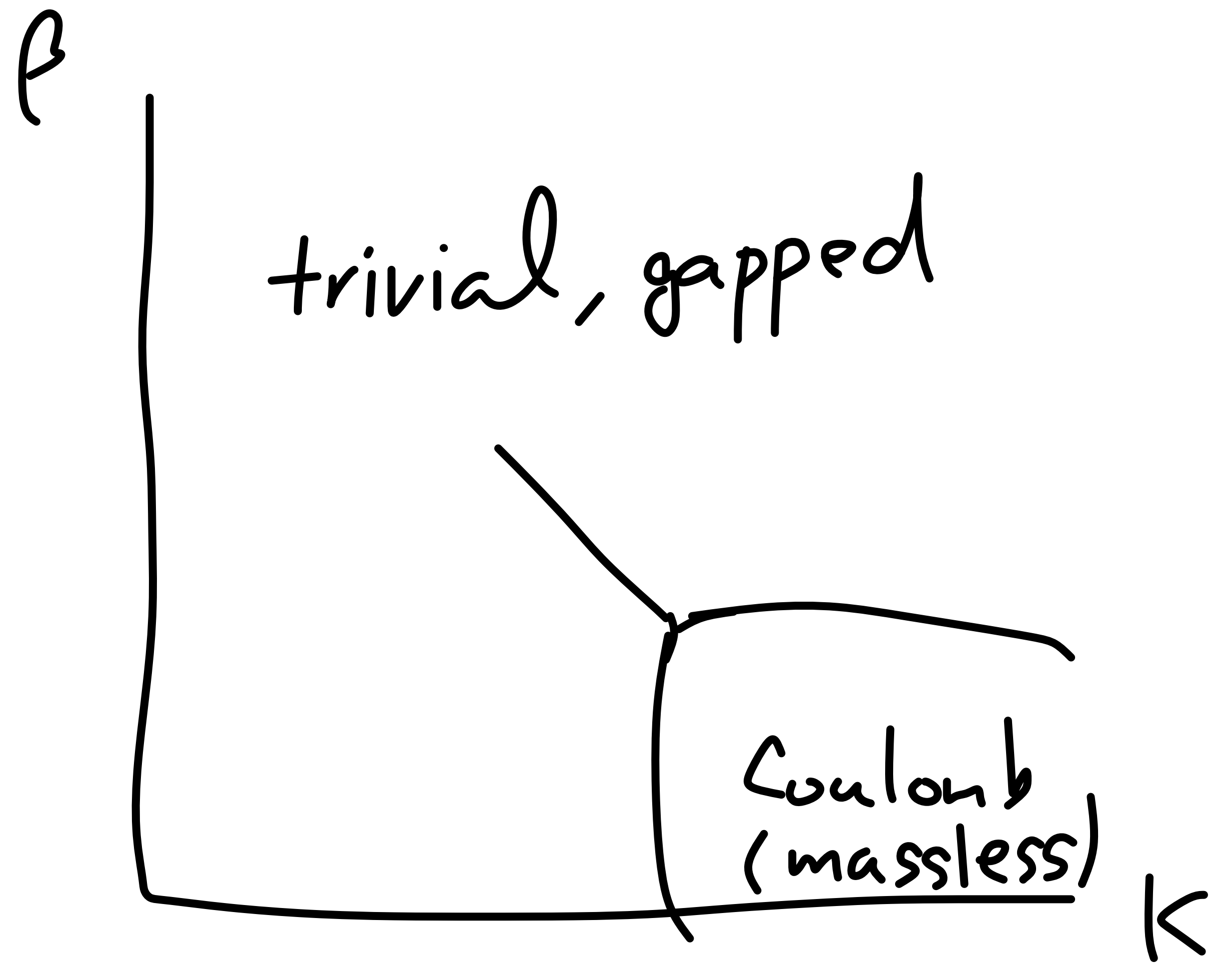
Fradkin - Shenker revisited

They consider charge-q $U(1)$-Higgs model on a lattice

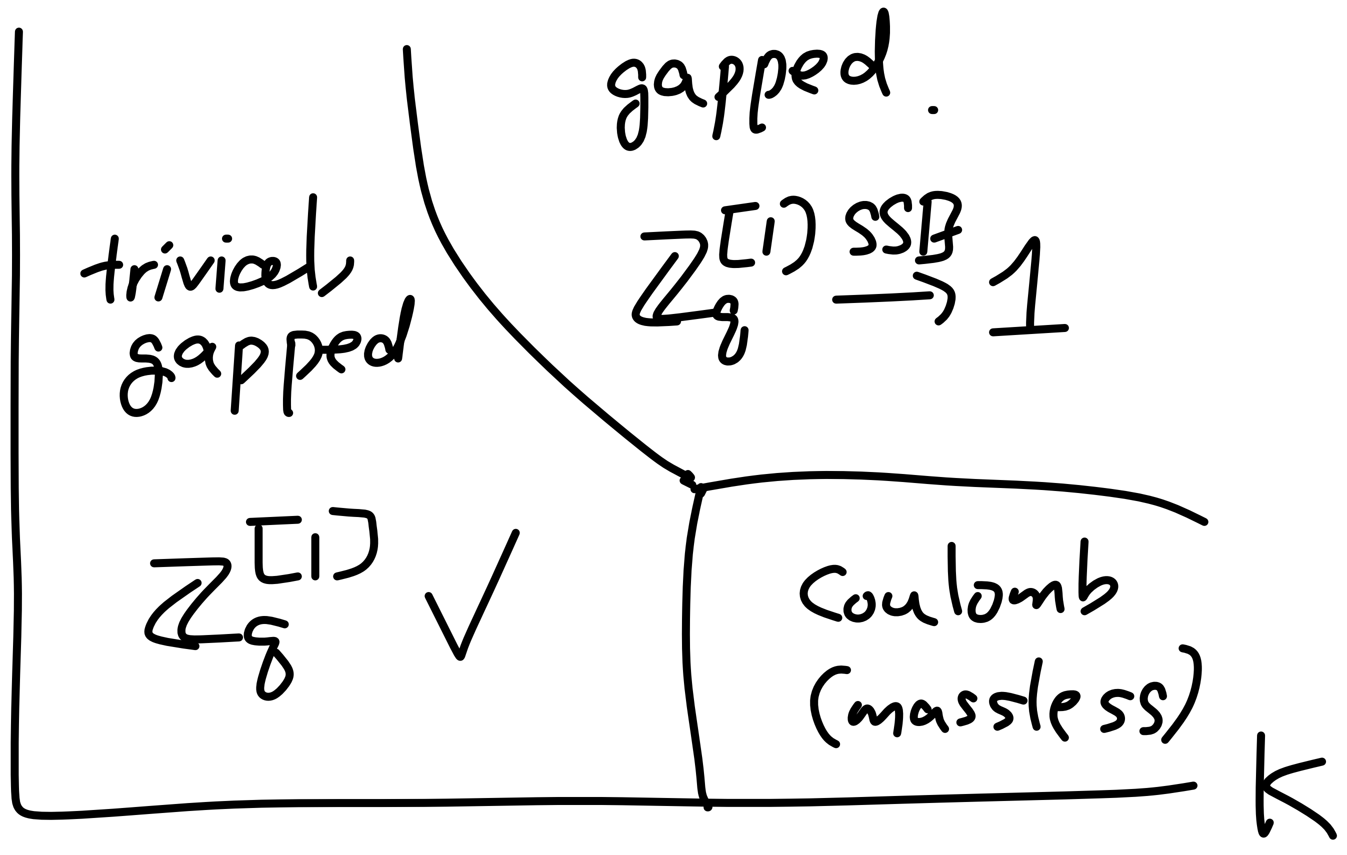
$$S = \beta \sum_{i,\mu} \cos(\partial_\mu \theta + q a_\mu) + K \sum_{\square} \cos(f_{\mu\nu})$$

$$\left(\longleftrightarrow \quad S = \frac{1}{2e^2}\int |da|^2 + \underbrace{\int \left\{ |(\partial_\mu + i q a_\mu)\phi|^2 + \quad \right\}}_{U(1)_E^{[1]} \xrightarrow{\text{explicit}} \mathbb{Z}_q^{[1]}} + \underbrace{\text{monopoles}}_{U(1)_M^{[1]} \xrightarrow{\text{explicit}} \times} \right).$$

$\underline{q = 1}$ (No symmetry)



$\underline{q \geq 2}$ ($\mathbb{Z}_q^{[1]}$ symmetry)



SU(N) gauge theory

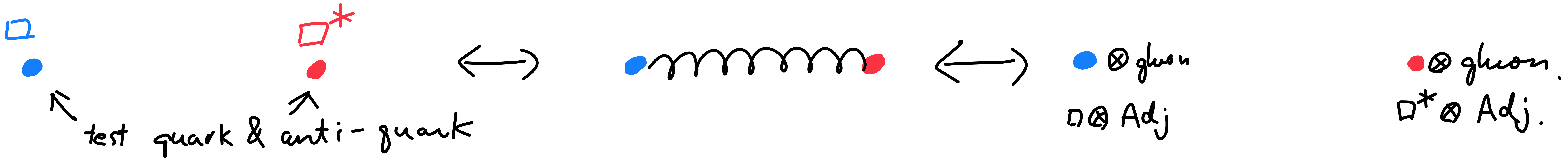
Facts SU(N) pure YM theory (+ adjoint matters) have $\mathbb{Z}_N^{[1]}$ as a generalized symmetry.

Wilson loop $W_R(C) = \mathrm{tr}_R\left[P\exp\left(\oint_C a\right)\right]$ is an order parameter to define confinement.

Explanation For $u: M_4 \to SU(N)$, $a \mapsto a^u = \underbrace{u^{-1}au}_{\text{adjoint transformation.}} + u^{-1}du.$

$\Rightarrow$ Gluons a carry adjoint charge $N-1\left\{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array}\right.$. $\quad (\square^* \otimes \square = \mathrm{Adj} \oplus \mathbb{1}.)$

gluons can emit gluons.



Classification for gapped phases of 4d YM theory

Wilson & 't Hooft proposed the following criterion of Higgs/confinement phases for 4d SU(N) YM + adj. matters:

① Confinement : Area law of Wilson loop (& Perimeter law of 't Hooft loop)

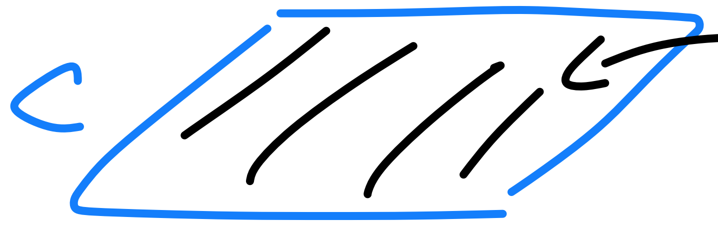
$$\langle W_\square(C)\rangle \simeq \exp(-\sigma\cdot \mathrm{Area}(C)) \iff \mathbb{Z}_N^{[1]}\ \checkmark.$$

$$(\langle H(C)\rangle \simeq \exp(-\#\cdot \mathrm{Length}(C)))$$

② Higgs :

$$\langle W_\square(C)\rangle \simeq \exp(-\#\cdot \mathrm{Length}(C)) \iff \mathbb{Z}_N^{[1]} \xrightarrow{SSB} 1$$

$$(\langle H(C)\rangle \simeq \exp(-\sigma'\,\mathrm{Area}(C))).$$

(cf. $H(C) =$ 

generator of $\mathbb{Z}_N^{[1]}$ in SU(N) YM, not a genuine line.
[Aharony, Seiberg, Tachikawa, '13])

Low-energy theorem

When $\mathbb{Z}_N^{[0]} \xrightarrow{\text{SSB}} 1$, we have N vacua by $\langle \phi(x) \rangle_k = v e^{\frac{2\pi i}{N} k}$.

When $\mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} 1$, low-energy theory contains an (intrinsic) topological order.

$\Rightarrow$ Number of vacua depends on the topology of spatial manifold:

$$\begin{cases} \#(\text{vacua for } S^3 \times \mathbb{R}) = 1 \\ \#(\text{vacua for } T^3 \times \mathbb{R}) = N^3 \quad \left(\langle \quad \rangle = e^{\frac{2\pi i}{N} k} \right) \end{cases}$$

Also, massless photons of 4d Maxwell can be regarded as Nambu-Goldstone bosons of $U(1)_E^{[1]} \times U(1)_M^{[1]}$.

Confinement with fundamental matters ?

I have to note (unfortunately...) that confinement of our QCD is still ill-defined as a phase.

Fundamental matters explicitly break $\mathbb{Z}_N^{[1]}$ to nothing.

$\Longleftrightarrow$ $\langle W_{\square}(C) \rangle$ always obey perimeter law because pair production of dynamical quarks can break confining strings.

(*) In the 't Hooft large-N limit (g^2N : fixed, N_f : fixed, $N \to \infty$), the string breaking is $\frac{1}{N}$ - suppressed.

<u>Q.</u> Can we make this rigorous in view of 1-form symmetry ?

It's an interesting question for future work... [see Nguyen, YT, Ünsal, '21 for related topics in pure SU(N) YM.]

More subtle quantum phases

Generalized symmetry & its "spontaneous breaking" $\longrightarrow$ (intrinsic) topological order.

Do we now have enough power to characterize quantum phases?

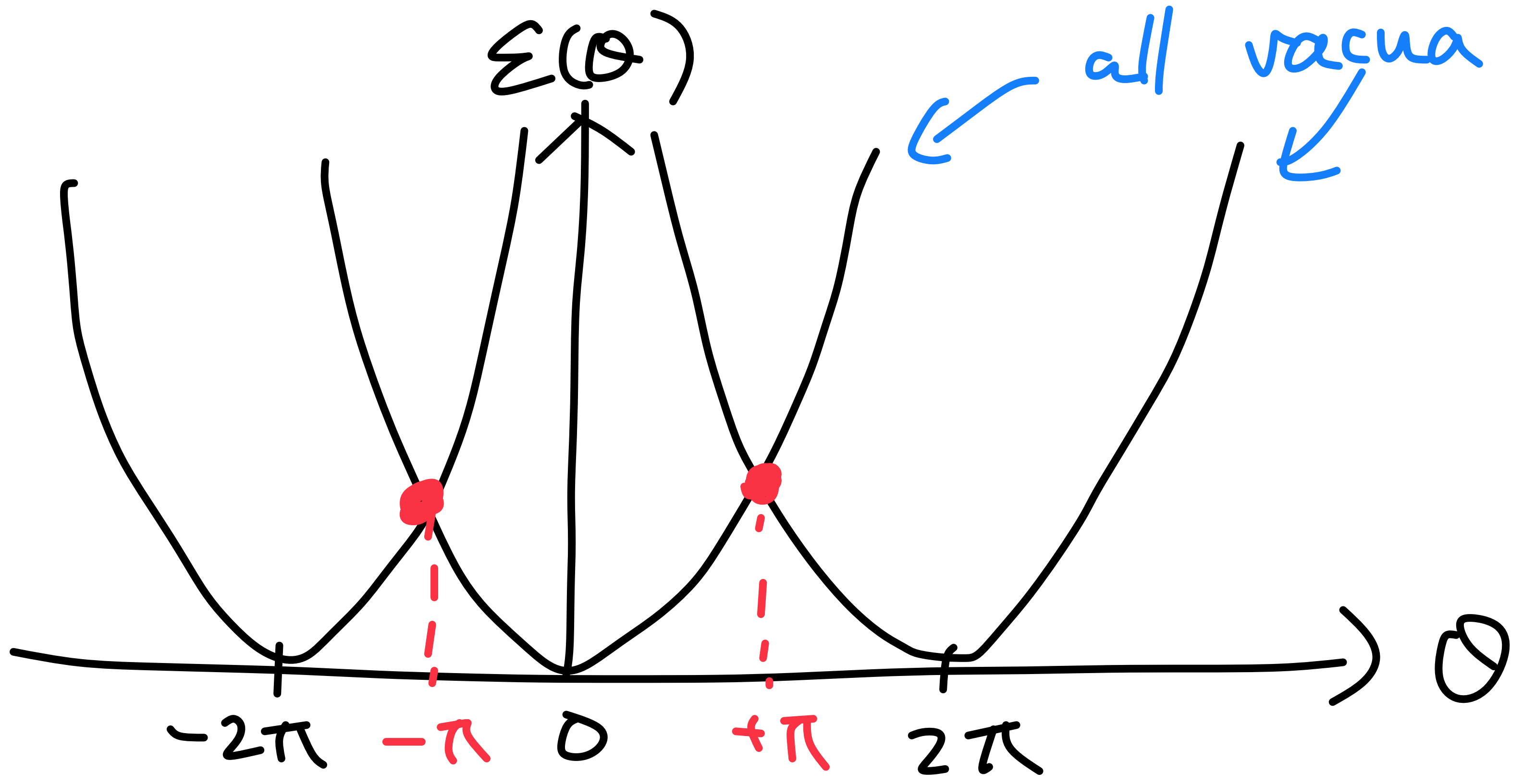
$\uparrow$ More interesting story goes on!

Example (4d $SU(N)$ YM + θ-term) ['t Hooft, Witten, Di Vecchia-Veneziano, ... ~'80]

$$S = -\frac{1}{g^2}\int tr[f \wedge * f] + i\frac{\theta}{8\pi^2}\int tr[f\wedge f].$$

If $N \gg 1$, the G.S. energy looks like.

all vacua are confined phases, i.e., $\mathbb{Z}_N^{[1]}$ ✓.



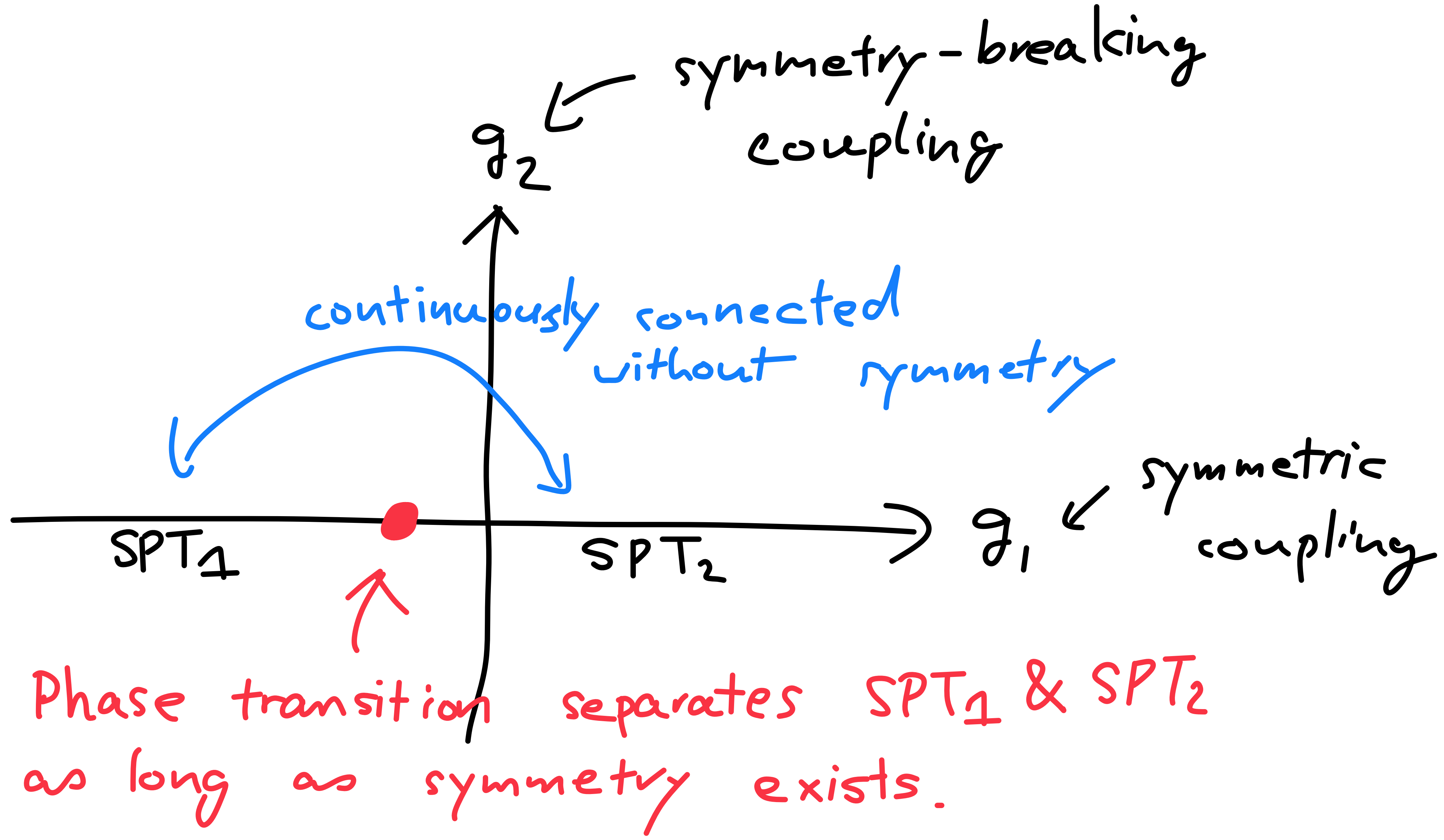
$\Longrightarrow$ Is this phase transition accidental? (like liquid-vapour)

If not, what orders it?

Symmetry-protected topological phase (or invertible TQFT)

G.S. is unique

Two trivial gapped states may not be continuously connected if some symmetry G exists.



$\rightsquigarrow$ A : G-gauge field (background)

$$\frac{Z[A]}{|Z[A]|} = \exp\left(i\, S_{top}[A]\right).$$

S_{top} : (classical) topological G-gauge theory.

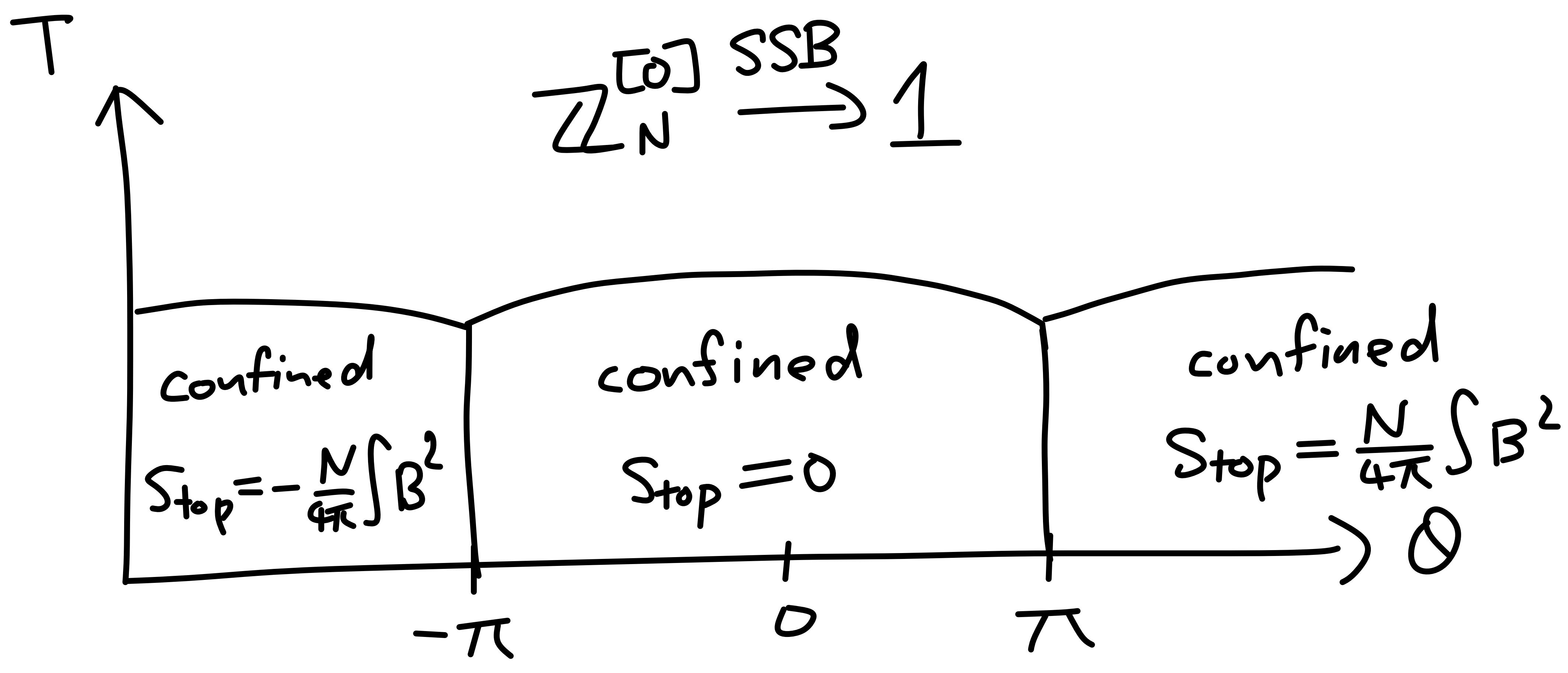
If SPT_1 & SPT_2 have different S_{top}, they are distinguished as SPT with G.

(Generically, if G is explicitly broken, they can be continuously connected)

[cf. Kitaev, Wen, Kapustin, … . Freed & Hopkins, '16. Yonekura]

YM phase diagram in (T, θ)

$N \gg 1$ phase diagram of $SU(N)$ YM with finite-temperature T & θ.



$$\mathbb{Z}_N^{[1]}\Big|_{4d} \longrightarrow \mathbb{Z}_N^{[1]}\Big|_{3d} \times \mathbb{Z}_N^{[0]}\Big|_{3d}.$$

[cf. Gaiotto, Kapustin, Komargodski, Seiberg '17
YT, Kikuchi '17, ...]

(This is now expected to be true even for $N=2$. [cf.) Kitano, Matsudo, Yamada, Yamazaki '21])

Let me explain this phenomenon for 2d pure Maxwell theory.
(It is much simpler!)

2d pure Maxwell theory

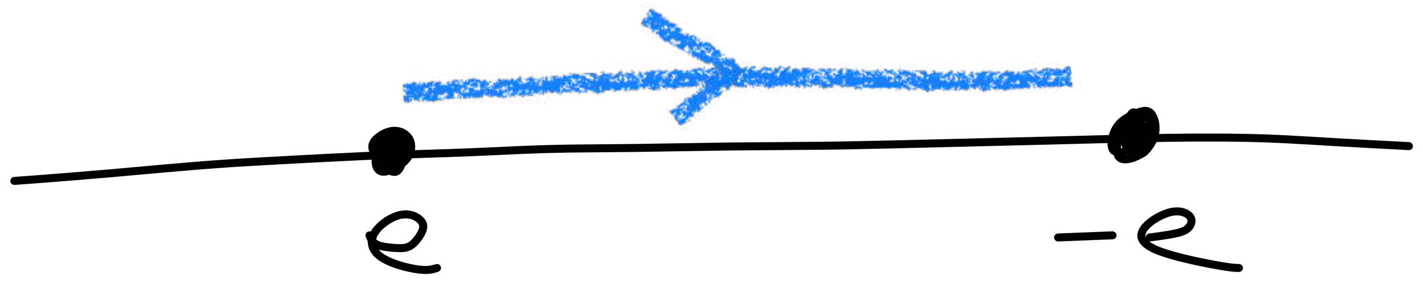
$$S = \frac{1}{2e^2}\int da \wedge * da + i\frac{\theta}{2\pi}\int da .$$

Dirac quantization says $\int_{M_2} da \in 2\pi\mathbb{Z}$ for closed M_2, so $\theta \sim \theta + 2\pi$.

This has $U(1)^{[1]}_E$ as a generalized global symmetry.

$U(1)^{[1]}_E$ is unbroken, i.e. this system is confining.

$\Leftarrow$ This can be seen classically.



$V(x)$: electrostatic potential of charge e,

$$-\frac{d^2}{dx^2}V(x) = e\,\delta(x) \implies V(x) = -e\frac{|x|}{2}$$

$$\langle W(C_{L\times T})\rangle = e^{-T\cdot(-e)V(L)} = e^{-\frac{e^2}{2}T\cdot L} \rightsquigarrow \text{Area law.}$$

[cf. Coleman '76]

Canonical quantization of 2d Maxwell.

$$\mathcal{L}_{\text{real-time}} = \frac{1}{2e^2}(\partial_0 a_1 - \partial_1 a_0)^2 + \frac{\theta}{2\pi}(\partial_0 a_1 - \partial_1 a_0).$$

Take temporal gauge $a_0 = 0$. (Our Minkowski space is $S^1 \times \mathbb{R}_{\text{time}}$.)

$$\begin{cases} \pi = \dfrac{\partial \mathcal{L}_{\text{real-time}}}{\partial \dot{a}_1} = \dfrac{1}{e^2}\dot{a}_1 + \dfrac{\theta}{2\pi} \\ H = \pi \dot{a}_1 - \mathcal{L}_{\text{real-time}} = \dfrac{e^2}{2}\left(\pi - \dfrac{\theta}{2\pi}\right)^2 \end{cases}$$

- Canonical quantization: $\pi \longrightarrow \frac{1}{i}\frac{\partial}{\partial a_1}$.

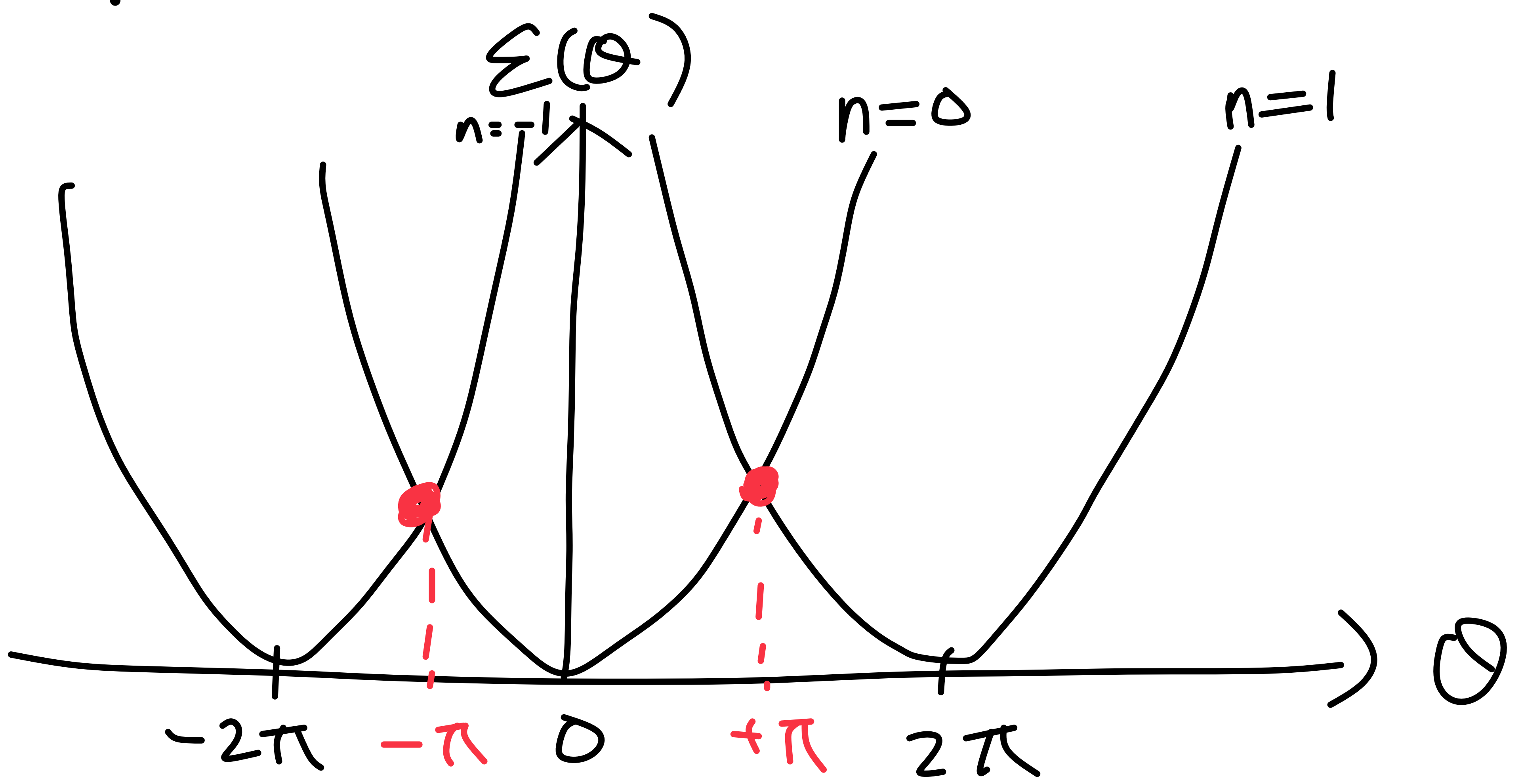
Gauss law constraint $\dfrac{\partial \mathcal{L}_{\text{real-time}}}{\partial a_0} = 0 \iff \partial_1 \pi = 0$.

physical state $\Psi[a_1]$ must be invariant under $a_1 \to a_1 + \partial_x \lambda$.

$$\rightsquigarrow \quad \Psi_n = e^{in\oint_0^L dx\, a_1(x)}, \quad E_n(\theta) = \frac{e^2}{2}\left(n - \frac{\theta}{2\pi}\right)^2.$$

($n \in \mathbb{Z}$ is required by large gauge invariance.)

Spectrum & SPT

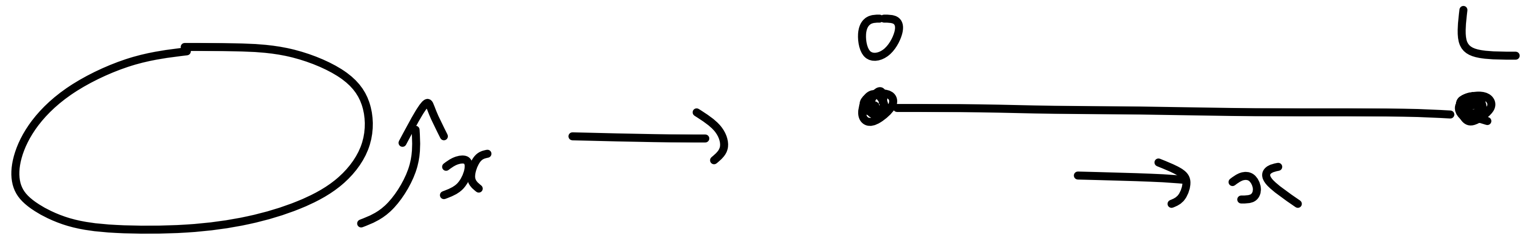


$$E_n(\theta) = \frac{e^2}{2}\left(n - \frac{\theta}{2\pi}\right)^2.$$

$$\Psi_n[a_1] = e^{in\oint a_1 dx}.$$

These are confined for any n.

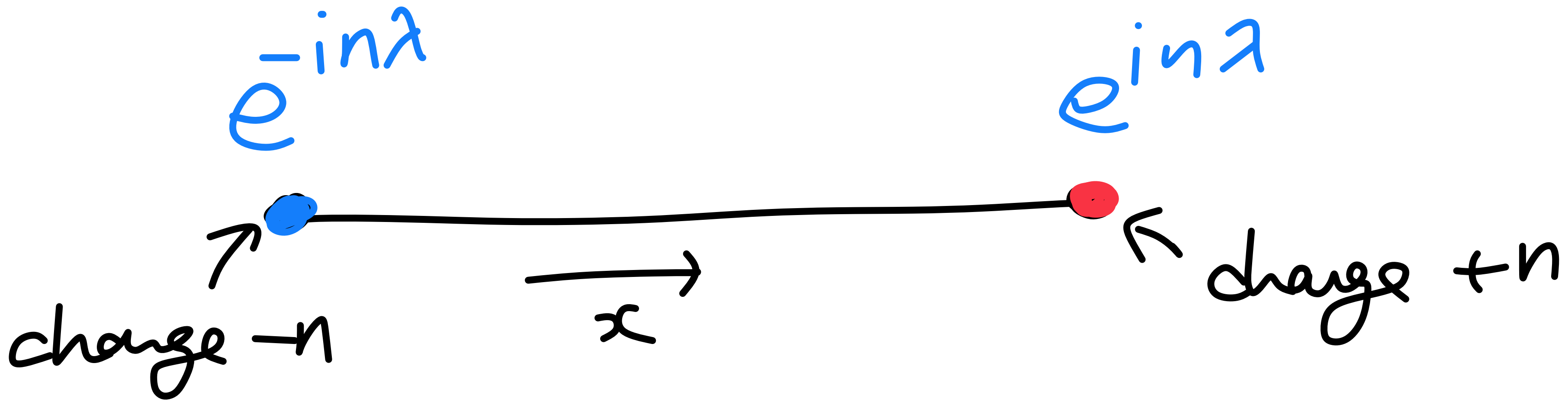
Let's cut $S^1_L \longrightarrow [0, L]$



Apply the gauge trans. $a_1 \to a_1 + \partial_x \lambda$, then

$$\Psi_n[a_1] \longrightarrow \Psi_n[a_1 + \partial_x \lambda] = e^{in(\lambda(L) - \lambda(0))}\, \Psi_n[a_1].$$

boundary charges.



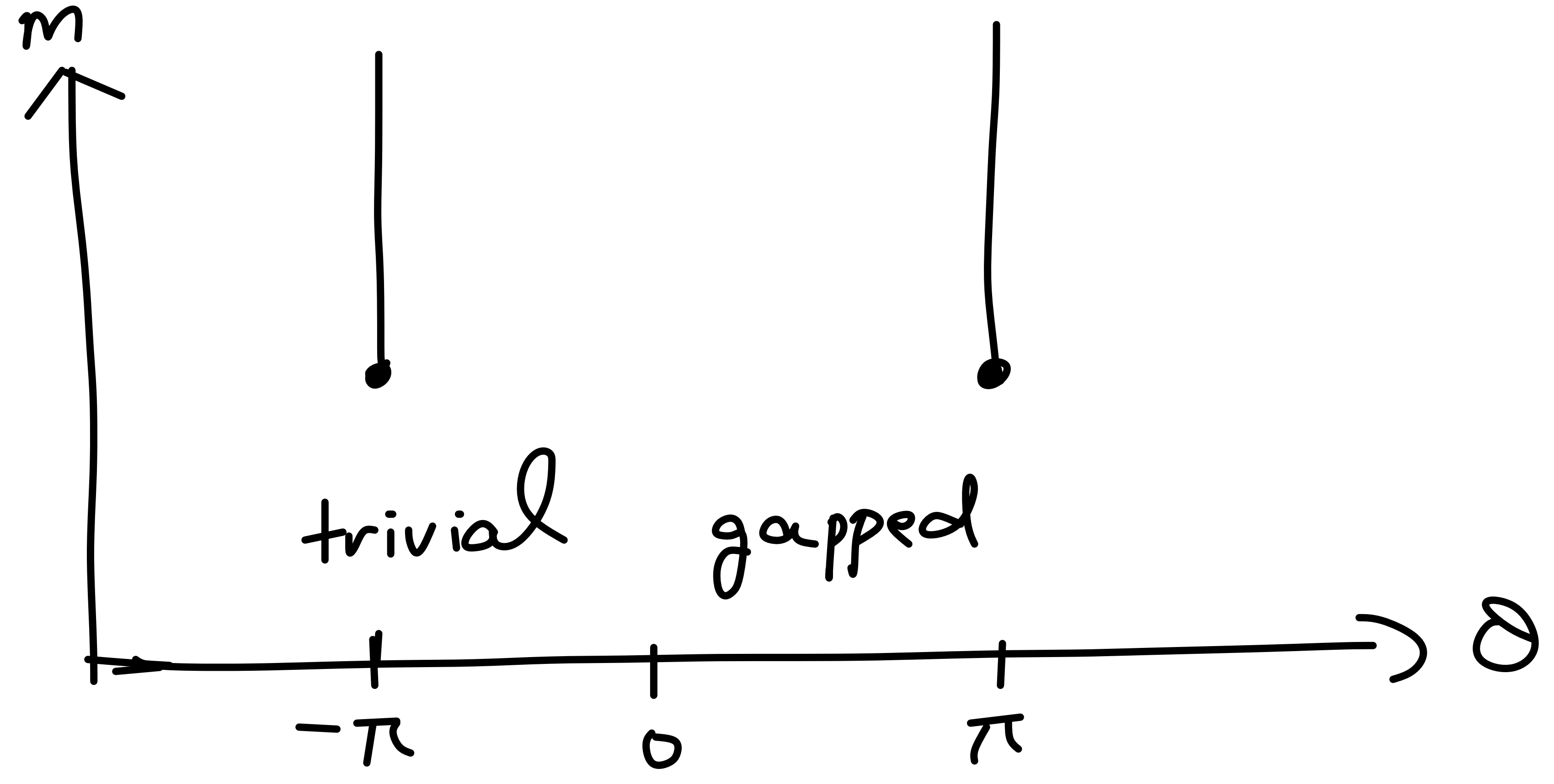
(This is another feature of SPT by anomaly inflow!)

2d Maxwell + charge-q Dirac fermion

Consider the charge-q Schwinger model. $\left(\begin{array}{l} m\to\infty : \text{2d pure Maxwell} \\ m\to 0 : \text{free massive boson} + \alpha \end{array}\right)$

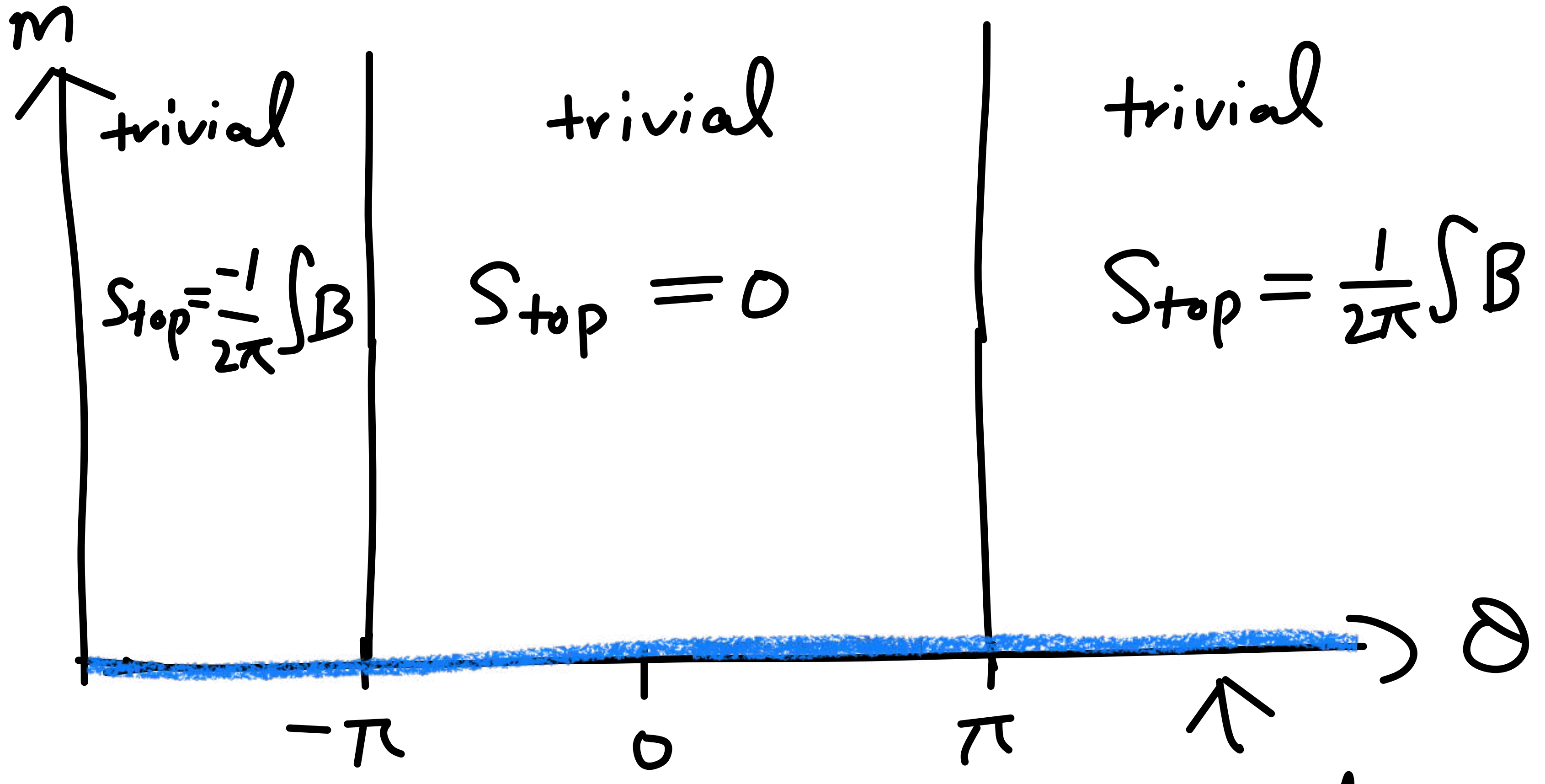
$$S = \frac{1}{2e^2}\int |da|^2 + i\frac{\theta}{2\pi}\int da + \int \bar{\Psi}\left(\gamma^\mu(\partial_\mu + iq a_\mu) + m\right)\Psi \qquad (m>0).$$

<u>$q=1$</u>



$U(1)^{[1]} \xrightarrow{\text{explicit}} \times$

<u>$q\geq 2$</u>



$U(1)^{[1]} \xrightarrow{\text{explicit}} \mathbb{Z}_q^{[1]}$

For $m>0$, theory is always confined.

Nontrivial SPTs with $\mathbb{Z}_q^{[1]}$.

(At $m=0$, theory is deconfined)

q-degenerate vacua. (* Similar story holds for 4d QCD).

Summary

Symmetry gives a powerful tool to classify phases.

Many generalizations are now available.

Symmetry = Topological operators.

$\Rightarrow$ Topological phases, SPT ... & Anomaly!