アイランド公式について

関西地域セミナー

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In this talk,

- I will continue the discussion on BH information loss problem.
- A recent progress : the island formula for $S(\rho_R)$
- (1) How it works (Page curve, replica wormhole)

  (2) Microscopic origin of the formula.

  (3) Application to the information recovery. (Petz map)

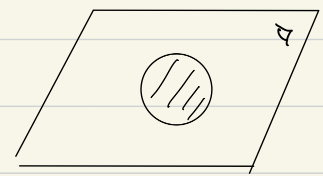
情報喪失問題とは？

⇒ ブラックホールのホーキング放射についてのパラドックス

<u>ブラックホール</u>：強い重力によって，ひどくゆがんだ時空

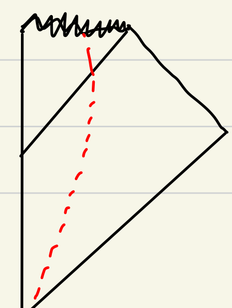
・ $ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

・ $r = 2GM$ ：事象の地平面



・ 星の重力崩壊によってうまれる。

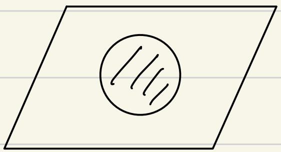
（ペンローズ図）



ブラックホールと熱力学

ブラックホールは事象の地平面の面積に比例する
熱力学的エントロピーを持っている。

$$\boxed{S_{BH} = \frac{A}{4G_N}}$$

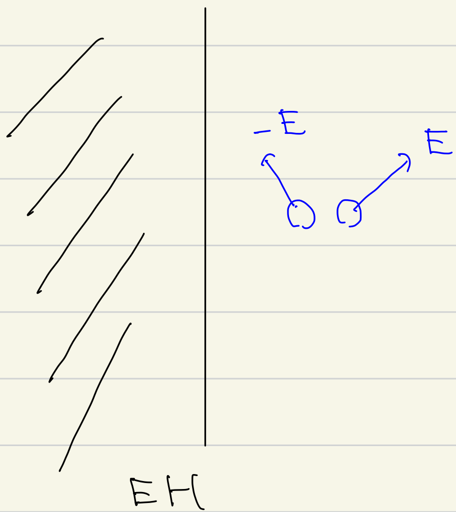


∵ もし BH がエントロピーを持っていないとすると ⇒ 第二法則の破れ

⇒ 量子重力理論において、BH は $\dim \mathcal{H}_{BH} = e^{S_{BH}}$ の
微視的な状態からなる。

BHのホーキング放射

・BHは温度をもつ ⇒ 熱的な放射を出している。
（ホーキング放射）



量子論的
・事象の地平面近傍での粒子対生成

・負のエネルギーをもった粒子はEHの内側に入ることで、安定になる

・無限遠にいる観測者はBHが放射を出して、その質量を失った、と思う。

⇒ **情報喪失問題**： ブラックホールのホーキング放射による蒸発は量子論のユニタリー性に矛盾する？

量子相関について

2つのスピン $\frac{1}{2}$ の粒子 A, B が、シングレット状態

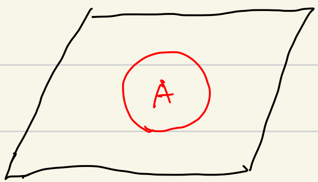
- $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$

$\Rightarrow$ Aの状態とBの状態は相関している。
(量子論的な相関 $\Rightarrow$ Bellの不等式の破れ)

- 相関の度合いを測る量 $\Rightarrow$ エンタングルメントエントロピー

$$\boxed{\rho_A = \mathrm{tr}_B |\Psi\rangle\langle\Psi| , \qquad S_{EE} = -\mathrm{tr}\,\rho_A \log \rho_A}$$

QFTにおけるEE



Σ

時間一定面を領域Aとその補集合に分ける。

$$H_{tot} = H_A \otimes H_{\bar{A}} \Rightarrow S(\rho_A)$$

・レプリカ法：$\boxed{S(\rho_A) = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{tr}\, \rho_A^n}$

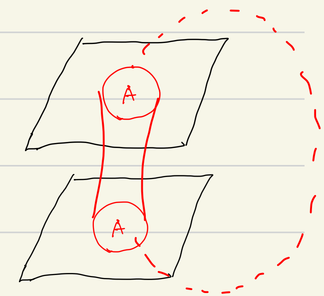
・$\mathrm{tr}\,\rho_A^n$ は path integral によって計算できる

$$\mathrm{tr}\,\rho_A^n = \int_{\Sigma_A^{(n)}} D\phi\, e^{-S[\phi]}$$

$\Sigma_A^{(n)}$：レプリカ多様体

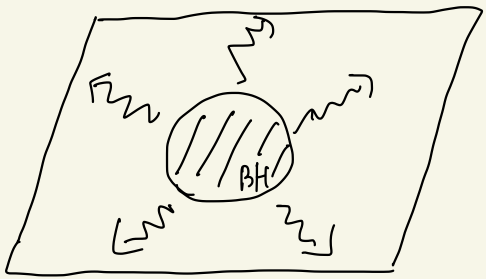
nこの時空へコピーを

Aに沿ってはりあわせる。



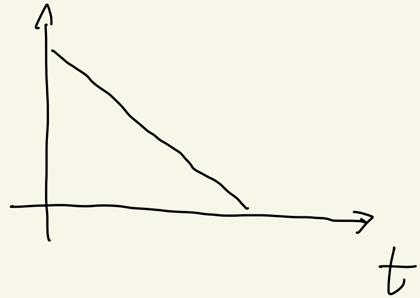
$\mathrm{tr}\,\rho_A^2$

Hawking Radiation

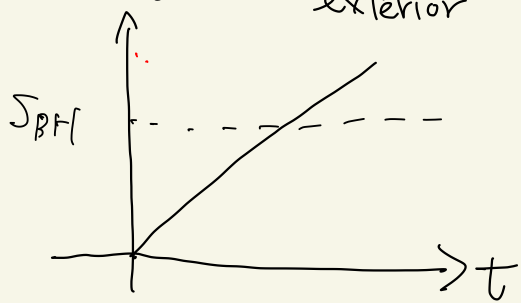


⇒

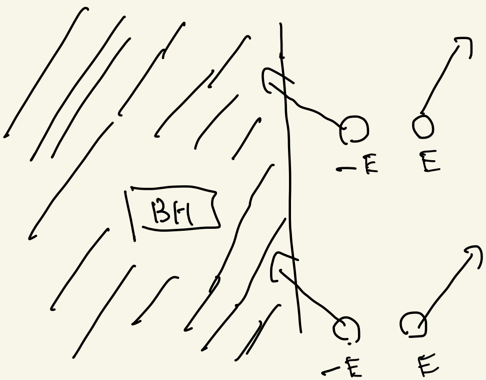
Size of the black hole



Entanglement between Interior and exterior



Horizon



: Pair creations

$\Rightarrow \; S_{EE} \leq \log \dim \{ \mathcal{H}_{\text{interior}}, \mathcal{H}_{\text{exterior}} \}$

という性質に矛盾.

ここまでのまとめ

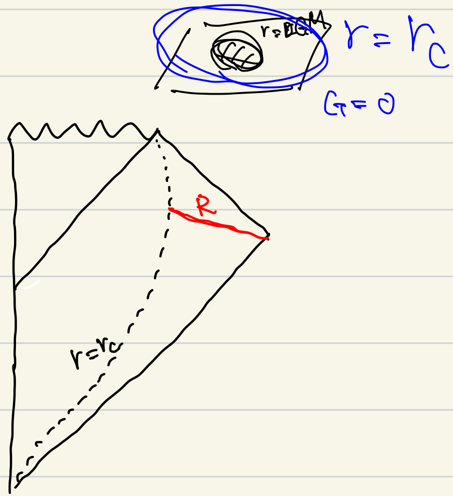
- ブラックホールは熱力学エントロピー、温度を持つ。
- BHは熱的な放射を出している。(ホーキング放射)
- ホーキングの計算によると、BHの内側と外側のエンタングルメントエントロピーは増え続ける。

⇒ 量子論のユニタリー性に矛盾 (情報喪失問題)

- 最近の進展：S_{EE}を正しく計算する公式の発見

⇒ アイランド公式 (Penington, Almheiri et al, ...)

The Set up



$$\boxed{R : r_c < r < \infty}$$

= a Bath collecting the Hawking Quanta

(1) An evaporating BH, due to Hawking radiation.

(2) Semi-classical : $\left(g_{\mu\nu} ,\ |\psi\rangle_{QFT} \right)$

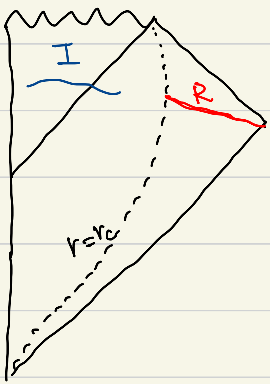
(3) Compute

the entropy of Hawking radiation $S(\rho_R)$

$$\|$$

The entanglement entropy of $|\psi\rangle_{QFT}$ on R in the presence of dynamical gravity.

The island formula (Pennigton, Almheiri et al, ...)



$$\boxed{S(\rho_R) = \min_I \operatorname{ext} \left[\frac{A(\partial I)}{4G} + S_{QFT}(I \cup R) \right]}$$

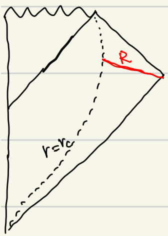
Here : $\begin{cases} I : \text{ some region in the BH.} \\ S_{QFT}(I \cup R) : \text{ EE of the state } |\psi\rangle \text{ in QFT} \end{cases}$

The island : the region which extremize the entropy functional

How it works

(1) $\boxed{t < t_{page} = S_{BH}/2}$

$\Rightarrow$ I : empty

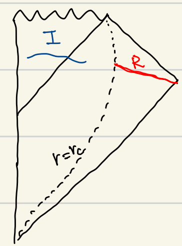


$$S(\rho_R) = S_{QFT}(R)$$

: the Hawking's result

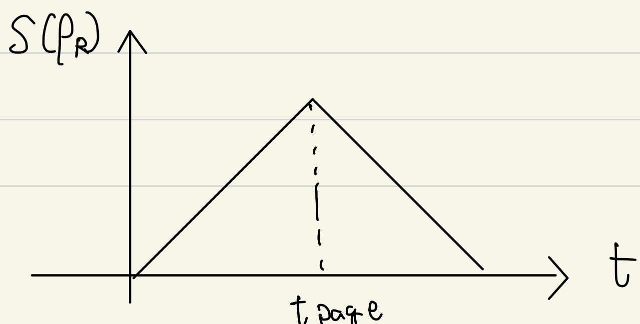
(2) $\boxed{t > t_{page}}$

$\Rightarrow$ Nontrivial island forms



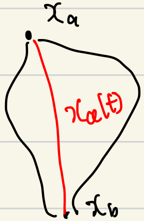
$$S(\rho_R) = \frac{A(\partial I)}{4G_N} = S_{BH}.$$

$\Rightarrow$



This entropy curve is Consistent with Unitarity

Path integral and its saddle



In QM: $\langle x_a | e^{iHt} | x_b \rangle = \int Dx \; e^{+iS_{particle}[x(t)]/\hbar}$

$$\xrightarrow{\hbar \to 0} \; e^{iS_{particle}[x_{cl}(t)]/\hbar}$$

where $\left.\dfrac{\delta S}{\delta x}\right|_{x=x_{cl}(t)} = 0$ (EoM)

- In gravity $\int Dg_{\mu\nu} D\phi \; e^{-S_{EH}[g_{\mu\nu}] - S_{matter}[\phi]}$

$$= \sum_{\{g_{cl}\}} e^{-S_E[g_{cl}]} \; Z_{QFT}[g_{cl}] \qquad \left.\frac{\delta S_{tot}}{\delta g}\right|_{g_{cl}} = 0$$

A derivation of the island formula

(1) Use the replica trick,

$$\boxed{\underset{-\mathrm{tr}\,\rho_R \log \rho_R}{\underset{\shortparallel}{S(\rho_R)}} = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{tr}\, \rho_R^n} = \int Dg_{\mu\nu}\, e^{-S[g_{\mu\nu}]}$$

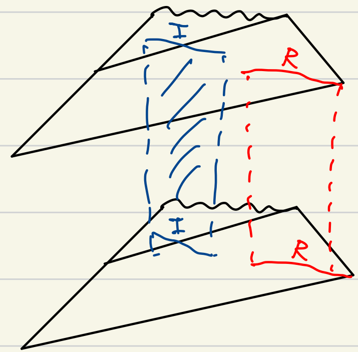
$$\overset{G_N\to 0}{=} \sum_{\{g_{cl}\}} e^{-S[g_{cl}]}$$

(2) $\mathrm{tr}\,\rho_R^n$ has a path integral expression on

Ex : $\mathrm{tr}\,\rho_R^2$:

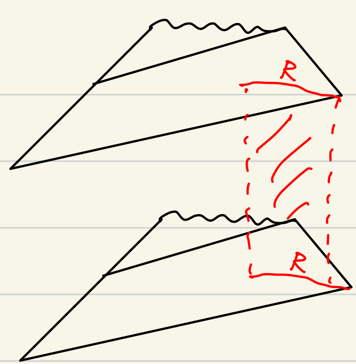
Two copies are glued along R

(3) In the presence of dynamical gravity, we also need to include the contribution of the replica wormhole,



$\mathrm{tr}\,\rho_R^2$: The sum of these two contributions.

$\mathrm{tr}\,\rho_R^2 \simeq$



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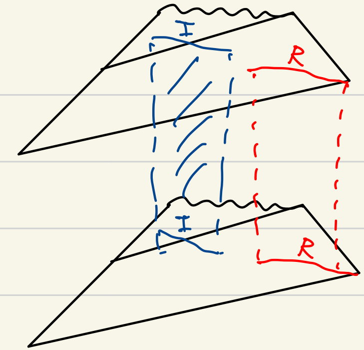
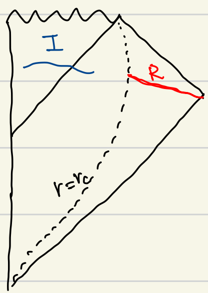


Hawking's result + Corrections

- A similar computation for $\mathrm{tr}\,\rho_R^n$ $\Rightarrow$ the Island formula

an Interpretation of the island

(1) The region in the BH reconstructable from Hawking radiation



$\Rightarrow$ $\begin{cases} \text{A geometric manifestation of Hayden Preskill} \\ \text{Actual recovery involves Petz map} \end{cases}$

(2) Microscopically, it captures $\boxed{\text{random fluctuations in the DM of HR}}$

$\Rightarrow$ In gravity, such random fluctuations

$\Rightarrow$ (Euclidean) Wormholes (in our case, replica WH)

A microscopic model (1)

- In a microscopic theory (Quantum gravity) : $\mathcal{H}_{BH} \otimes \mathcal{H}_R$

$$\boxed{|\Psi\rangle = \sum_{\alpha=1}^{d_{BH}} \sum_{i=1}^{d_R} \underbrace{C_{\alpha i}}_{d_{BH} \times d_R} |\psi_\alpha\rangle_{BH} \otimes |i\rangle_{Rad}}$$

- $C_{\alpha i}$ is unknown but

$\Rightarrow$
- ◎ Dynamics is chaotic $\Rightarrow$ $C_{\alpha i} = \left(e^{iHt}\right)_{\alpha i}$ is random
- ◎ Integrating out short distance physics $\Rightarrow$ averaging over C

$\Rightarrow$ $\boxed{\text{We are interested in } \overline{S(\rho_R)}}$

A microscopic model (1)

- The DM of Hawking radiation

$$\boxed{\rho_R = \frac{1}{d_R\, d_{BH}} \sum_{i,j=1}^{d_R} \sum_{\alpha,\beta=1}^{d_{BH}} C_{\alpha,i}\, C_{\alpha,j}^{\dagger}\, |i\rangle_R\langle j|}$$

- So the average of $n=2$ Rényi entropy is,

$$\boxed{\overline{\mathrm{tr}\,\rho_R^2} = \frac{1}{d_R^2\, d_{BH}^2} \sum_{i,j=1}^{d_R} \sum_{\alpha,\beta=1}^{d_{BH}} \overline{C_{\alpha i}\, C_{\alpha j}^{\dagger}\, C_{\beta j}\, C_{\beta i}^{\dagger}}}$$

Averaging the entropy (1)

· Since $C_{\alpha i}$ is Gaussian Random

$$
\begin{cases}
\overline{C_{\alpha i} C^{\dagger}_{\beta j}} = \delta_{\alpha\beta}\,\delta_{ij} , \\
\overline{C_{\alpha i} C^{\dagger}_{\beta j} C_{\gamma k} C^{\dagger}_{\delta m}} = C_{\alpha i} C^{\dagger}_{\beta j} C_{\gamma k} C^{\dagger}_{\delta m} + C_{\alpha i} C^{\dagger}_{\beta j} C_{\gamma k} C^{\dagger}_{\delta m}
\end{cases}
$$

$$
\overline{\mathrm{tr}\,\rho_R^2} = \frac{1}{d_R^2\, d_{BH}^2} \sum_{i,j=1}^{d_R} \sum_{\alpha,\beta=1}^{d_{BH}} \overline{C_{\alpha i} C^{\dagger}_{\alpha j} C_{\beta j} C^{\dagger}_{\beta i}} = \frac{1}{d_R} + \frac{1}{d_{BH}}
$$

Hawking's result

a non perturbative correction $e^{-S_{BH}}$

Averaging the entropy (2)

$\mathcal{H}_R \otimes \mathcal{H}_{BH}$

- $$\overline{\operatorname{tr}\rho_R^2} = \frac{1}{d_R} + \frac{1}{d_{BH}} = \begin{cases} \dfrac{1}{d_R} & : d_R \ll d_{BH} \\ \dfrac{1}{d_{BH}} & : d_R \gg d_{BH} \end{cases}$$

$\Rightarrow$
- A phase transion when $d_R = d_{BH}$ (Page curve)
- Can not be captured by the Hawking's calculation.

- Similary :

$$\boxed{\overline{S(\rho_R)} = -\overline{\operatorname{tr}\rho_R \log \rho_R} = \begin{cases} \log d_R & : d_R \ll d_{BH} \\ \log d_{BH} & : d_R \gg d_{BH} \end{cases}}$$

Averaging the entropy (2)

- Why the behavior of the entropy changes after the Page time?

- $\boxed{\overline{S(\rho_R)} \neq S(\overline{\rho}_R)}$

$$\overline{\rho}_R = \frac{1}{d_R}\sum_{i=1}^{d_R} |i\rangle_R\langle i| \Rightarrow S(\overline{\rho}_R) = \log d_R.$$

$$\rho_R = \frac{1}{d_R d_{BH}}\sum_{i=1}^{d_R}\sum_{\alpha=1}^{d_{BH}} C_{\alpha,i}\, C^{\dagger}_{\beta,j}\ |i\rangle_R\langle j|$$

$$= \overline{\rho}_R + \delta\rho_R$$

Random fluctuation of the DM

: Tiny $O(e^{-S_{BH}})$, but it can be accumulated

Averaging the entropy (2)

- why the behavior of the entropy changes after the Page time?

- $\boxed{\overline{S(\rho_R)} \neq S(\overline{\rho}_R)}$

$$\overline{\rho}_R = \frac{1}{d_R} \sum_{i=1}^{d_R} |i\rangle_R \langle i| \Rightarrow S(\overline{\rho}_R) = \log d_R .$$

$$\rho_R = \frac{1}{d_R d_{BH}} \sum_{i=1}^{d_R} \sum_{\alpha=1}^{d_{BH}} C_{\alpha,i} C^{\dagger}_{\beta,j} |i\rangle_R \langle j|$$

$$= \overline{\rho}_R + \delta \rho_R$$

the Accumulation of random fluctuations changes the entropy $t > t_{Page}$

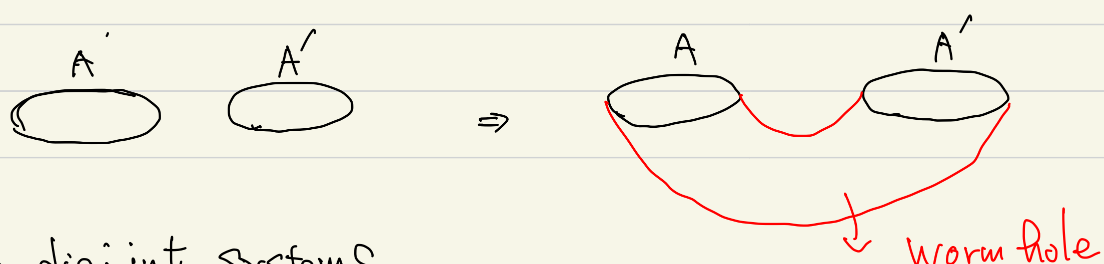
Going back to gravity (1)

· In a theory of gravity,

$$\boxed{\text{The random fluctuations} = \text{Wormholes}}$$

(Coleman, Hawking, GS, MM ... SSS)

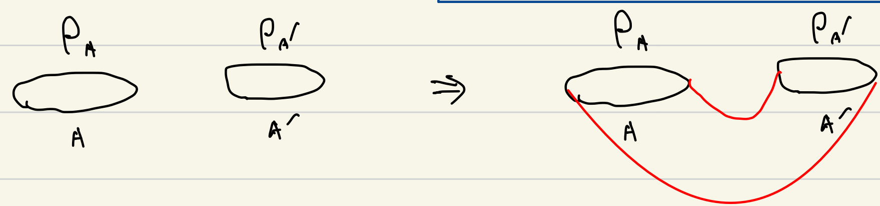
⊙ What is a wormhole? ⇒ A geometric connection between two systems



Two disjoint systems

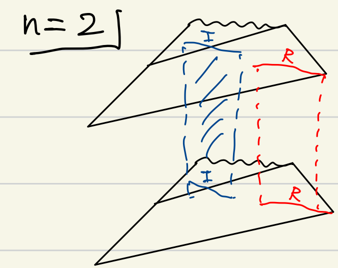
Going back to gravity (2)

- An intuitive understanding : $\boxed{\text{RFs} \Leftrightarrow \text{Correlations} \Leftrightarrow \text{Wormholes}}$



$$\overline{\mathrm{tr}\left[(\rho_A\otimes\rho_{A'})(O_A\otimes O_{A'})\right]} \neq \overline{\mathrm{tr}[\rho_A O_A]}\;\overline{\mathrm{tr}[\rho_{A'} O_{A'}]} \Rightarrow$$ A geometric connection between A and A'

- $\boxed{\text{RFs in } \overline{\mathrm{tr}\,\rho_R^n} \Rightarrow \text{a Replica wormhole}}$



This justifies the island formula.

$$\overline{\mathrm{tr}\,\rho_R^2} = \frac{1}{d_R^2\, d_{BH}^2} \sum_{i,j=1}^{d_R} \sum_{\alpha,\beta=1}^{d_{BH}} \overline{C_{\alpha i}\, C^\dagger_{\alpha j}\, C_{\beta j}\, C^\dagger_{\beta i}}$$

$$= \frac{1}{d_R^2\, d_{BH}^2} \sum_{i,j=1}^{d_R} \sum_{\alpha,\beta=1}^{d_{BH}} C_{\alpha i}\, C^\dagger_{\beta j}\, C_{\gamma k}\, C^\dagger_{\delta m} + C_{\alpha i}\, C^\dagger_{\beta j}\, C_{\gamma k}\, C^\dagger_{\delta m}$$

g

in gravity

$\simeq$

(Trivial Saddle)

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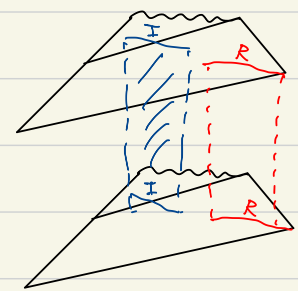
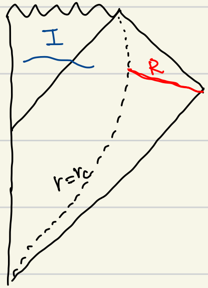


(Replica wormhole)

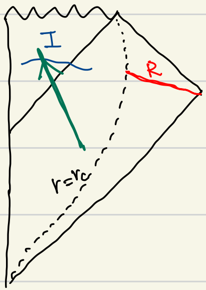
$$= \frac{1}{d_R} + \frac{1}{d_{BH}}$$

Information recovery
through the island.

An interpretation of the island



an Island : a region in the BH reconstructable from Hawking radiation H_R when $t > t_{page}$.



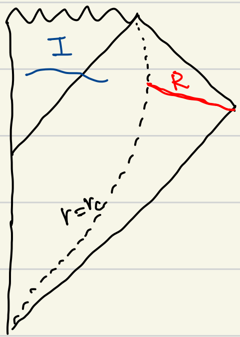
- A gedanken experiment : Sending a diary into the BH

⇒ It will enter to the island ⇒ belongs to H_R

⇒ **Geometric understanding of Hayden Preskill**

time delay between I and R ⇒ The scrambling time

Information recovery



- How does the reconstruction works?

- Take a QFT on the fix BH background

Any operator O_{QFT} in the island region must be reconstructable from $\mathcal{H}_R$

This is archived by **Petz map**.

Petz map (1)

$$\overset{\mathcal{H}_{sub}}{\cup}$$

(I) Embed $\mathcal{H}_{QFT}$ to the larger space $\mathcal{H}_{QG} = \mathcal{H}_R \otimes \mathcal{H}_{BH}$

$$\boxed{\begin{aligned} V : \quad & \mathcal{H}_{QFT} \to \mathcal{H}_{BH} \otimes \mathcal{H}_R \\ & |a\rangle_{QFT} \to |\Psi_a\rangle = \sum_{i=1}^{d_R} |\psi_{ia}\rangle_{BH} \otimes |i\rangle_R . \end{aligned}}$$

Remarks

(1) the QFT state is a slightly excited state on the BH

(2) $$\sum_{a=1}^{d_{BH}} \sum_{i=1}^{d_R} C_{ia} |\psi_a\rangle_{BH} |i\rangle \equiv \sum_{i=1}^{d_R} |\psi_i\rangle_{BH} \otimes |i\rangle_R$$

The new basis is not orthogonal, $\langle \psi_i | \psi_j \rangle = \sum_{a=1}^{d_{BH}} C_{ia} C_{ja} = \delta_{ij} + R_{ij}$

Petz map (2)

(II) Under the embedding, a QFT operator is mapped to

$$V : H_{QFT} \longrightarrow H_{BH} \otimes H_R$$

$$O_{QFT} = \sum_{a,b=1}^{d_{code}} \langle a| O_{QFT} |b\rangle \, |a\rangle\langle b| \longrightarrow \mathcal{O} = \sum_{a,b}^{d_{code}} \langle a| O_{QFT} |b\rangle \, |\Psi_a\rangle\langle\Psi_b|$$

acts on both H_R and H_B

The goal : Construct $\mathcal{O}_R$ acting only on H_R, st

$$\langle \Psi_a | \mathcal{O}_R | \Psi_b \rangle = {}_{QFT}\langle a| O_{QFT} |b\rangle$$

A comment

- This is just a usual quantum error correction procedure,

with the Quantum channel : $\mathcal{H}_{code} \xrightarrow{V} \mathcal{H}_{BH} \otimes \mathcal{H}_R \xrightarrow{\text{Erasure}} \mathcal{H}_R$

$\mathcal{H}_{QFT\,sub}$

$$\boxed{\begin{array}{ccc} \text{• Information recovery} & & \text{Quantum error correction} \\ \text{of the BH interior} & \Longleftrightarrow & \text{against the erasure} \\ \text{from Hawking radiation} & & \text{(tracing out the BH dof)} \\ \mathcal{H}_R & & \end{array}}$$

Petz map

$$\begin{array}{ccc} H_R \times H_{BH} & \longrightarrow & H_R \\ O & & O_R \end{array}$$

$$H_{QFT} \xrightarrow{V} H_R \times H_{BH}$$

O_{QFT}

Such O_R can be constructed,

$$\boxed{O_R = \sigma_R^{-\frac{1}{2}} \, \mathrm{Tr}_{BH}[\,O\,] \, \sigma_R^{-\frac{1}{2}}}$$

$$\begin{cases} O_{BH\cup R} = \displaystyle\sum_{a,b}^{d_{code}} \langle a| O_{QFT} |b\rangle \, |\Psi_a\rangle\langle\Psi_b| \\ \sigma_R = \mathrm{tr}_{BH}[VV^\dagger] = \mathrm{tr}_{BH}[\,\Pi_{proj}\,] \, . \end{cases}$$

$\Rightarrow$

- Acting only on H_R .
- Satisfy $\langle \Psi_a | O_R | \Psi_b \rangle = \langle a | O_{QFT} | b \rangle$

Gravity point of view

"west coast paper" (1911.11977)

- $$O_R = \frac{1}{d_R} \sum_{ij} |i\rangle_R \langle j| \sum_{ab=1}^{d_{code}} \langle \psi_{ia} | \psi_{jb} \rangle_{BH} \; O_{ab}$$

- $$\langle \psi_{ia} | \psi_{jb} \rangle_{BH} = \sum_{\alpha=1}^{d_{BH}} C_{\alpha (ia)} \, C^{\dagger}_{\alpha (jb)} \;\Rightarrow\; \text{We need a wormhole}$$

- $$\langle \Psi_a | O_R | \Psi_b \rangle \to \langle \Psi_a | \sigma_R^{n} \, \mathrm{Tr}[O] \, \sigma_R^{n} | \Psi_b \rangle \,, \quad n \to -\frac{1}{2}$$

$$= \frac{1}{d_R^{2n+2}} \sum_{\{i_k, I_k\}} \sum_{\{a_k, b_k\}} \left(\prod_{k=1}^{n} \langle \psi_{i_k a_k} | \psi_{i_{k+1}, a_k} \rangle \right) \left(\sum_{a', b'} \langle \psi_{i_n a'} | \psi_{I_1 b'} \rangle \, O_{a'b'} \right)$$

$$\left(\prod_{k=1} \langle \psi_{I_k b_k} | \psi_{I_{k+1}, b_k} \rangle \right) \;\Rightarrow\; \mathbb{Z}_{2n+2} \text{ symmetric replica wormhole}$$

$$\Rightarrow \begin{cases} d_R < d_{BH} : & \text{no wormholes} \Rightarrow \text{the Petz map fails} \\ d_R > d_{BH} : & \text{wormholes} \Rightarrow \text{succeed} \end{cases}$$

$$\langle \Psi_a | O_R | \Psi_b \rangle = \langle a | O_{in} | b \rangle$$

Summary

(1) Random fluctuation of ρ_R is important to obtain a Page curve.

(2) In a theory of gravity, averaging over random fluctuation is captured by including worm holes to the gravitational path integral

$\Rightarrow$ Island Formula

(3) Information recovery $\Rightarrow$ Petz map

Future prospects

(1) Can we see unitary time evolution from other quantites ?? (Ex : S matrix ?)

(2) How the geometry of the BH interior is encoded in H_R ? ⇒ We need a detailed study of Petz map

(3)