Comments on the Atiyah-Patodi-Singer index theorem, domain wall, and Berry phase

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APS index theorem

[M.F.Atiyah, V.K. Patodi, I.M.Singer, 75]
See also [M.F.Atiyah, I.M.Singer, 68]

Statement

Dirac operator on an even-dimensional manifold with the boundary:

$$D = \bar{\gamma} \not{D} = \bar{\gamma}\gamma^{\mu}(\partial_{\mu} + iA_{\mu})$$

where the APS boundary condition is imposed

Def. APS index:

$$\mathrm{ind}_{\mathrm{APS}} D = \dim \ker D_{+} - \dim \ker D_{-}$$

APS index theorem:

$$\mathrm{ind}_{\mathrm{APS}} D = \int_{\mathcal{M}} \mathrm{ch}(F) + \frac{1}{2}\eta(i\not{\nabla})$$

$$\eta(i\not{\nabla}) = \sum_{n} \mathrm{sign}\, \lambda_n$$

Recent topics

Non-perturbative generalization of anomaly inflow etc.

e.g. [E. Witten, 00, 16]
[E. Witten, K. Yonekura, 19]

Domain wall APS index theorem

[H.Fukaya et al., 17,19,20]

Reformulation of the APS index theorem

APS boundary condition
Gapless bulk

→ Localized fermion by domain wall mass

Domain wall APS index

$$\mathrm{ind}_{\mathrm{DW}} D = \frac{\eta(D + \bar{\gamma} m_{\mathrm{DW}}) - \eta(D + \bar{\gamma} m_{\mathrm{PV}})}{2}$$

Equivalence to the original APS index

$$\mathrm{ind}_{\mathrm{DW}} D = \mathrm{ind}_{\mathrm{APS}} D$$

$$= \int_{|x_d| \leq L_2/2} \mathrm{ch}(F) + \frac{\eta_{\mathrm{L}} - \eta_{\mathrm{R}}}{2}$$

Brief summary

Re-derivation of the domain wall APS index theorem

[T. Onogi, TY, 21]
[T. Onogi, TY, to appear]

The original proof was mathematically rigorous by technically complicated…

⇒

- Domain wall APS index = Berry phase
- (Domain wall APS index = Witten index)

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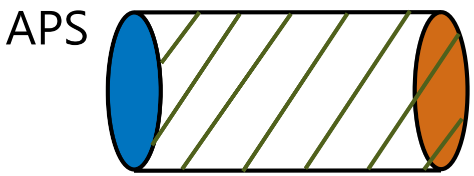
Motivations

APS index as the phase of the part. func. [E. Witten, 16]

$$Z = |Z| e^{i\pi \cdot \mathrm{ind}_{\mathrm{APS}} D}$$

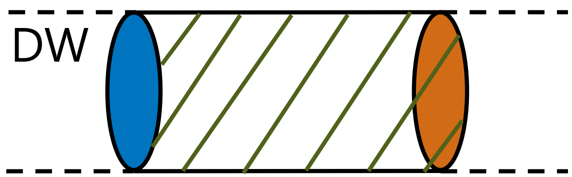
It is expected that...

$$i\pi \cdot \mathrm{ind}_{\mathrm{APS}} D = \ln Z \Bigg(\quad \Bigg)$$



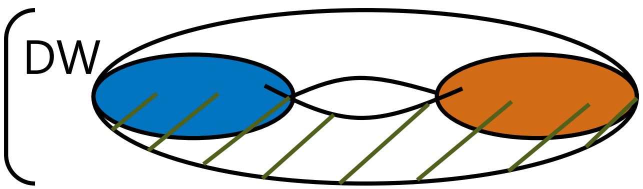
Motivated by [E. Witten, 16]

$$= \ln Z \Bigg(\quad \Bigg)$$



Motivated by [H.Fukaya et al., 17,19,20]

$$= \ln Z \Bigg(\quad \Bigg)$$



$$= \ln \mathrm{Tr}\left[(-1)^F \mathcal{P} e^{-\oint \mathrm{d}x_2 \hat{h}(x_2)}\right]$$

Conjecture

What to show:

[T. Onogi, TY, 21]

$$\pi \cdot \mathrm{ind}_{\mathrm{APS}} D = \vartheta$$

The trace is over the Dirac sea states

$$|\Psi^-\rangle \propto \sum_{\sigma \in \mathrm{perm.}} (-1)^\sigma \prod_i \Psi^-_{\sigma(i)} |0\rangle$$

There are contributions from Domain wall fermion and ghost

$$\vartheta = \vartheta_{\mathrm{DW}} - \vartheta_{\mathrm{PV}}$$

Snapshot Hamiltonian in two-dimensions

Dirac equation

$$D\Psi = 0, \quad D = \bar{\gamma}\left(\not{D} + m\right)$$

For simplicity

$$A_\mu = A_\mu(x_2)$$

which is equivalent to

$$-\frac{\partial}{\partial x_2}\Psi_{p_1}(x_2) = h(x_2)\Psi_{p_1}(x_2)$$

$$h(x_2) = h'(x_2) + iA_2(x_2)$$

$$h'(x_2) = (p_1 + A_1(x_2))\sigma_1 + m(x_2)\sigma_2$$

Eigenstates

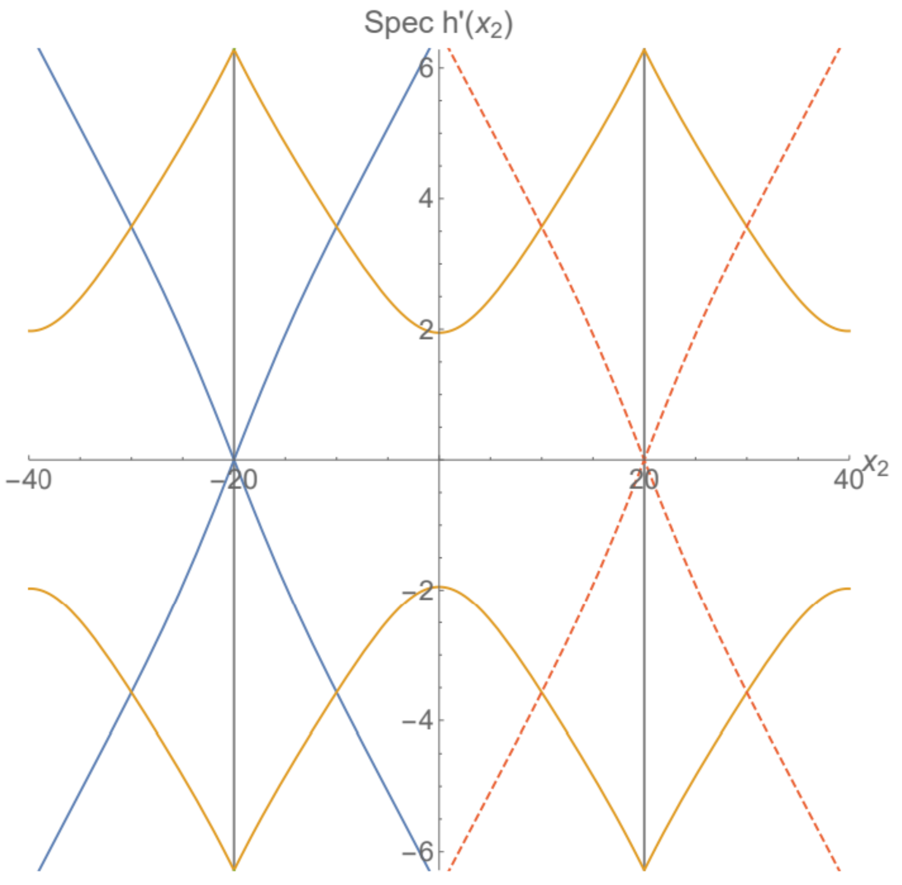
$$h'(x_2)\Psi^{\pm}_{p_1}(x_2) = \pm\varepsilon_{p_1}(x_2)\Psi^{\pm}_{p_1}(x_2)$$

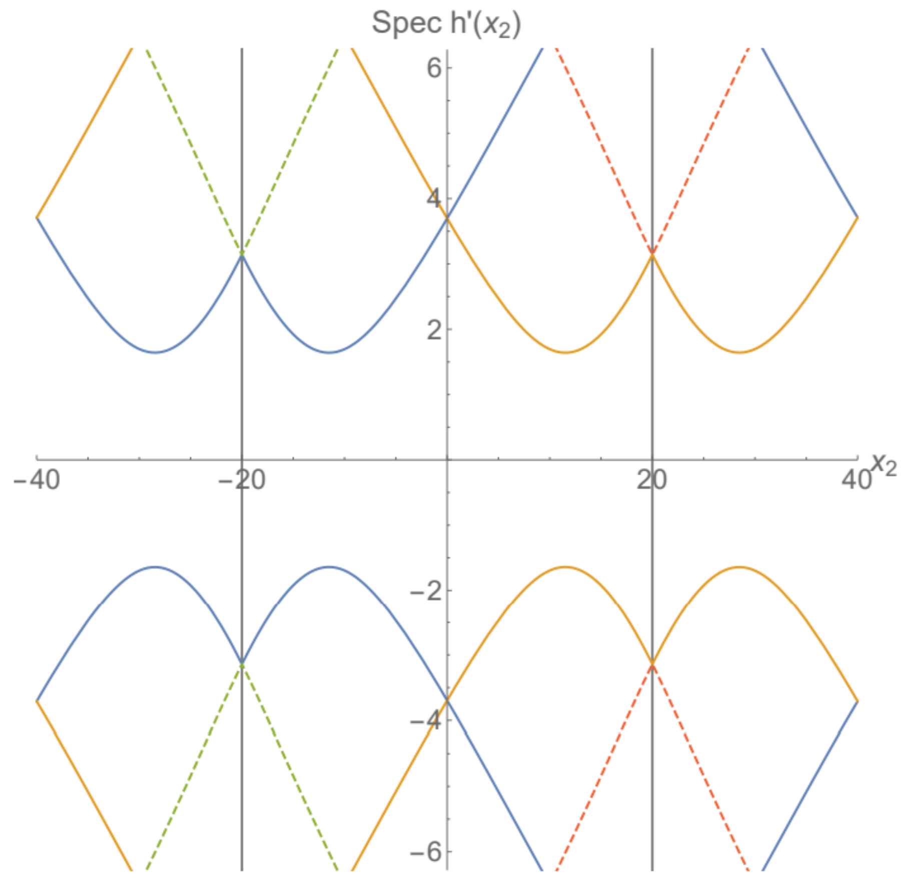
$$\varepsilon_{p_1}(x_2) = \sqrt{\tilde{p}_1(x_2)^2 + m(x_2)^2}$$

$$\Psi^{+}_{p_1}(x_2) = e^{i\alpha^+}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix},$$

$$\Psi^{-}_{p_1}(x_2) = e^{i\alpha^-}\begin{pmatrix} \sin\theta/2 \\ e^{i(\pi+\phi)}\cos\theta/2 \end{pmatrix},$$

Adiabatic approximation





Adiabatic approx. is valid as long as

$$\frac{|m|}{\epsilon} \ll L_1^{-2} \ll |m|^2 \qquad a \neq \mathbb{Z}$$

Berry phase

Euclidean time development

$$\Psi^-_{p_1}(0) \to \exp\left[-\int_0^{x_2} \mathrm{d}x_2 \left(-\varepsilon_{p_1} + i\left\langle \Psi^-_{p_1} \middle| (-i\partial_2 + A_2) \middle| \Psi^-_{p_1} \right\rangle\right)\right] \Psi^-_{p_1}(0)$$

With normalization so that $\vartheta_{\mathrm{DW}} = 0 \text{ for } A = 0$

$$\vartheta_{\mathrm{DW}} \simeq \frac{1}{2}\sum_{p_1} \oint \mathrm{d}x_2\, \partial_2\phi(1+\cos\theta) = \frac{1}{2}\sum_{p_1}\oint \mathrm{d}x_2\, \partial_2\phi$$

$$= \frac{1}{2}\sum_{p_1}\oint \mathrm{d}x_2\, \frac{\tilde{p}_1 m}{\tilde{p}_1^2 + m^2}\left(-\frac{\partial_2 \tilde{p}_1}{\tilde{p}_1} + \frac{\partial_2 m}{m}\right).$$

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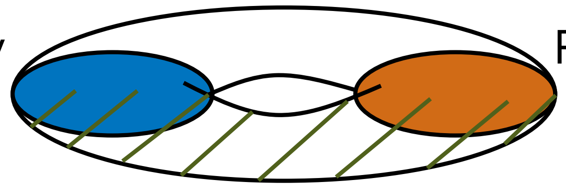
Decomposition of the Berry phase

$$\vartheta_{\mathrm{DW}} = \vartheta_{\mathrm{DW}}^{\mathrm{inside}} + \vartheta_{\mathrm{DW}}^{\mathrm{outside}}$$

$$\vartheta_{\mathrm{DW}}^{\mathrm{inside}} \simeq \vartheta_{\mathrm{blk}} + \vartheta_{\text{L-bdy}} + \vartheta_{\text{R-bdy}}$$



$$\vartheta_{\mathrm{blk}} = \frac{1}{2}\sum_{p_1}\int_{\mathrm{blk}} \mathrm{d}x_2 \, \frac{|m|}{\tilde{p}_1(x_2)^2 + |m|^2}(-\partial_2 \tilde{p}_1(x_2)),$$

$$\vartheta_{\text{L-bdy}} = \frac{1}{2}\sum_{p_1}\int_{\text{L-bdy}} \mathrm{d}x_2 \, \frac{\tilde{p}_1(-L_2/2)}{\tilde{p}_1(-L_2/2)^2 + m(x_2)^2}\partial_2 m(x_2),$$

$$\vartheta_{\text{R-bdy}} = \frac{1}{2}\sum_{p_1}\int_{\text{R-bdy}} \mathrm{d}x_2 \, \frac{\tilde{p}_1(+L_2/2)}{\tilde{p}_1(+L_2/2)^2 + m(x_2)^2}\partial_2 m(x_2).$$

Bulk contribution

Since the bulk mass is large,
the momentum sum is approximated by an integral

$$\begin{aligned}
\vartheta_{\mathrm{blk}} &= \frac{1}{2}\int_{\mathrm{blk}} \mathrm{d}^2x \int_{-\infty}^{\infty} \frac{\mathrm{d}p_1}{2\pi} \frac{1/|m|}{1+(\tilde{p}_1(x_2)/|m|)^2}(-\partial_2 A_1(x_2)) \\
&= \frac{1}{2}\int_{\mathrm{blk}} \mathrm{d}^2x \int_{-\infty}^{\infty} \frac{\mathrm{d}(\tilde{p}_1/|m|)}{2\pi} \frac{1}{1+(\tilde{p}_1/|m|)^2}(-\partial_2 A_1(x_2)) \\
&= \frac{1}{4}\int_{\mathrm{blk}} \mathrm{d}^2x\,(-\partial_2 A_1(x_2)). \\
&= \frac{\pi}{2}\cdot\frac{1}{2\pi}\int \mathrm{d}^2x\, F_{12}
\end{aligned}$$

Boundary contribution

Since the mass is small,
the momentum sum is NOT approximated by an integral

$$\vartheta_{\text{L-bdy}} = \frac{1}{2}\sum_{p_1}\int_{\text{L-bdy}} \mathrm{d}x_2 \, \frac{\tilde{p}_1(-L_2/2)}{\tilde{p}_1(-L_2/2)^2 + m(x_2)^2}\partial_2 m(x_2)$$

$$= \frac{1}{2}\sum_{p_1}\int_{\text{L-bdy}} \mathrm{d}m \, \frac{\tilde{p}_1(-L_2/2)}{\tilde{p}_1(-L_2/2)^2 + m^2}.$$

$$= \frac{1}{2}\lim_{|b|\to\infty}\sum_{n}\int_0^{|b|} \mathrm{d}b \, \frac{n + a_{\text{L}}}{(n + a_{\text{L}})^2 + b^2}$$

Rough evaluation

$$\begin{aligned}
\vartheta_{\text{L-bdy}} &= \frac{1}{2} \lim_{|b|\to\infty} \sum_n \int_0^{|b|} \mathrm{d}b \, \frac{n + a_{\text{L}}}{(n + a_{\text{L}})^2 + b^2} \\
&= \frac{1}{2} \sum_n \left[\tan^{-1} \frac{b}{n + a_{\text{L}}} \right]_0^\infty \\
&= \frac{\pi}{2} \cdot \frac{1}{2} \left[\sum_{n + a_{\text{L}} > 0} 1 + \sum_{n + a_{\text{L}} < 0} (-1) \right] \\
&= \frac{\pi}{2} \cdot \frac{\eta_{\text{L}}}{2}.
\end{aligned}$$

Evaluation of the boundary contribution

$$\vartheta_{\text{L-bdy}} = \frac{1}{2}\lim_{|b|\to\infty}\sum_n \int_0^{|b|} \mathrm{d}b\, \frac{n+a_{\mathrm{L}}}{(n+a_{\mathrm{L}})^2+b^2}$$

$$= \lim_{|b|\to\infty}\frac{1}{4}\sum_n \int_0^{|b|} \mathrm{d}b \left[\frac{1}{n+a_{\mathrm{L}}-ib}+\frac{1}{n+a_{\mathrm{L}}+ib}\right]$$

$$= \lim_{|b|\to\infty}\frac{i}{4}\sum_n \ln\frac{n+a_{\mathrm{L}}-[a_{\mathrm{L}}]-i|b|}{n+a_{\mathrm{L}}-[a_{\mathrm{L}}]+i|b|}$$

$$= \lim_{|b|\to\infty}\frac{i}{4}\ln\frac{\sin\pi(a_{\mathrm{L}}-[a_{\mathrm{L}}]-i|b|)}{\sin\pi(a_{\mathrm{L}}-[a_{\mathrm{L}}]+i|b|)}$$

$$= \frac{i}{4}\ln e^{2\pi i(a_{\mathrm{L}}-[a_{\mathrm{L}}]-1/2)} = \frac{\pi}{2}\left(\frac{1}{2}-a_{\mathrm{L}}+[a_{\mathrm{L}}]\right)$$

$$= \frac{\pi}{2}\cdot\frac{\eta_{\mathrm{L}}}{2}$$

Combining all contributions

Other contributions are computed in a similar way

$$
\begin{aligned}
\vartheta &= \vartheta_{\mathrm{DW}} - \vartheta_{\mathrm{PV}} \\
&= \left[\vartheta_{\mathrm{DW}}^{\mathrm{inside}} + \vartheta_{\mathrm{DW}}^{\mathrm{outside}}\right] - \left[\vartheta_{\mathrm{PV}}^{\mathrm{inside}} + \vartheta_{\mathrm{PV}}^{\mathrm{outside}}\right] \\
&= \frac{\pi}{2}\left[\left(+\frac{1}{2\pi}\int_{\mathrm{inside}} \mathrm{d}^2x\, F_{12} + \frac{\eta_{\mathrm{L}} - \eta_{\mathrm{R}}}{2}\right) + \left(-\frac{1}{2\pi}\int_{\mathrm{outside}} \mathrm{d}^2x\, F_{12} + \frac{\eta_{\mathrm{L}} - \eta_{\mathrm{R}}}{2}\right)\right] \\
&\qquad - \frac{\pi}{2}\left[\left(-\frac{1}{2\pi}\int_{\mathrm{inside}} \mathrm{d}^2x\, F_{12} + 0\right) + \left(-\frac{1}{2\pi}\int_{\mathrm{outside}} \mathrm{d}^2x\, F_{12} + 0\right)\right] \\
&= \pi\left[\frac{1}{2\pi}\int_{\mathrm{inside}} \mathrm{d}^2x\, F_{12} + \frac{\eta_{\mathrm{L}} - \eta_{\mathrm{R}}}{2}\right] = \pi \cdot \mathrm{ind}_{\,\mathrm{APS}} D.
\end{aligned}
$$

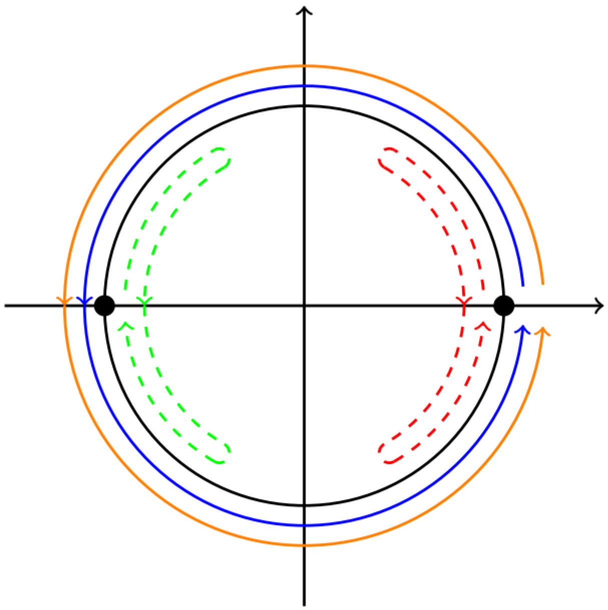
The conjecture is confirmed in a two-dimensional case!

Contents

Homotopy of the wave func.

Recall the wave func.

$$\Psi^-_{p_1}(x_2) = e^{i\alpha^-}\begin{pmatrix} \sin\theta/2 \\ e^{i(\pi+\phi)}\cos\theta/2 \end{pmatrix}$$



$$\cos\phi = \frac{\tilde{p}_1(x_2)}{\sqrt{\tilde{p}_1(x_2)^2 + m(x_2)^2}}, \quad \sin\phi = \frac{m(x_2)}{\sqrt{\tilde{p}_1(x_2)^2 + m(x_2)^2}}$$

Evaluation of the Berry phase

Berry phase = the number of homotopically non-trivial states

$$\vartheta_{\mathrm{DW}} = \frac{1}{2}\sum_{p_1}\oint \mathrm{d}x_2\, \partial_2 \phi$$
$$= \pi\left([a_{\mathrm{L}}] - [a_{\mathrm{R}}]\right)$$
$$= \pi \cdot \mathrm{ind}_{\mathrm{APS}} D.$$

$$\vartheta_{\mathrm{PV}} = 0.$$

$$\Rightarrow \quad \vartheta = \vartheta_{\mathrm{DW}} - \vartheta_{\mathrm{PV}}$$
$$= \pi \cdot \mathrm{ind}_{\mathrm{APS}} D.$$

The conjecture is confirmed in a two-dimensional case!

Contents

Conclusion and future works

- Conjecture:

  APS index = Berry phase associated with domain wall Dirac op.
- The conjecture is confirmed in a two-dimensional case

- The simplicity of the derivation allows generalization to other exotic systems?