Improving threshold and decoding for fault-tolerant color code quantum computing

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1. Background

2. Ising model formulation for highly accurate topological color codes decoding

3. Improving threshold for fault-tolerant color code quantum computing by flagged weight optimization

4. Conclusion

Quantum error correction (QEC)

- QEC is necessary to realize a fault-tolerant quantum computer.
- A logical qubit is encoded into multiple physical qubits using QEC codes.

Error correction procedure:

Syndrome measurement: extract information about errors

▼

Decoding: classical computation for estimating the errors given the syndrome NP-hard in general

▼

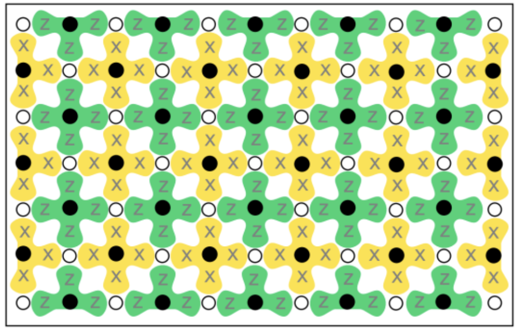
Recovery: apply correcting operations

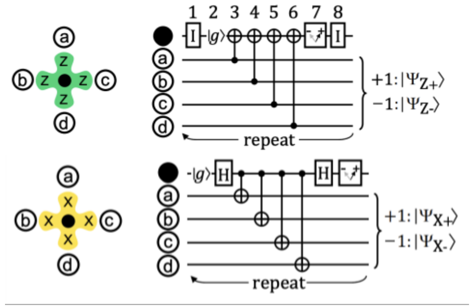
A QEC protocol should have:

- High threshold
- Efficient and accurate decoding algorithm
- Low-overhead gate implementation

Surface codes

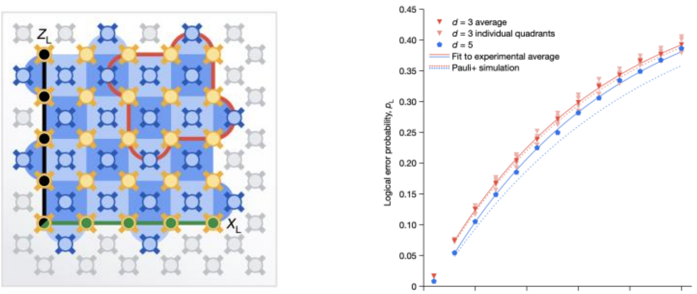
One of the most promising QEC codes





Stabilizer operators

A G. Fowler et al, Phys. Rev. A **86**, 032324 (2012).



Experiment by Google

Nature 614, 676-681 (2023)

Advantages:

- Efficiently decodable because 1 error flips only 2 syndromes

  e.g., by minimum-weight perfect matching algorithm

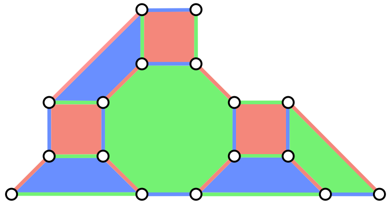
- The threshold is high (~ 1%)

Drawback:

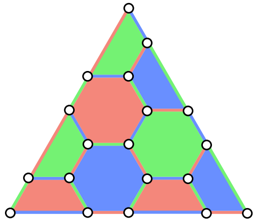
- Some (even Clifford) gates are costly to implement

Color codes

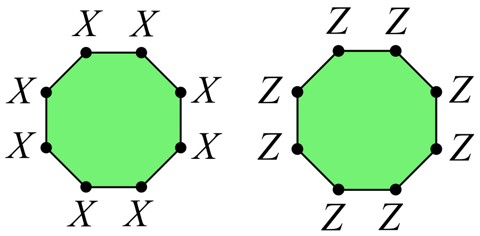
Another promising QEC codes



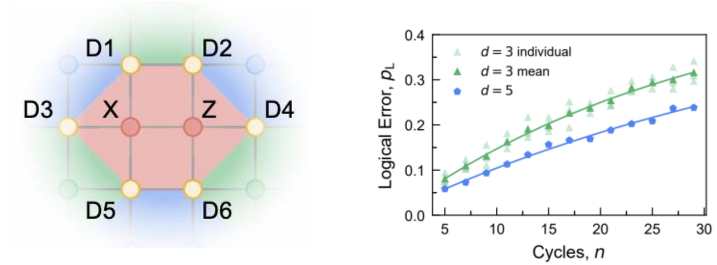
4.8.8 color code



6.6.6 color code



Stabilizer operators

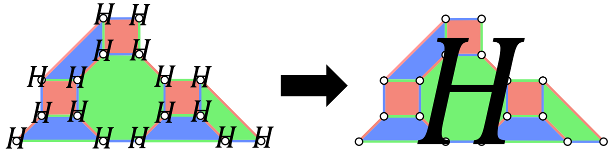


Experiment by Google (Dec. 2024)

arXiv: 2412.14256

Advantage:

- All Clifford gates can be implemented transversally → low overhead (compared to surface codes)



Drawbacks:

- Decoding is difficult because 1 error flips **3** syndromes
- Low thresholds

To be improved in our study

4.8.8 color code: 0.08% - 0.14%

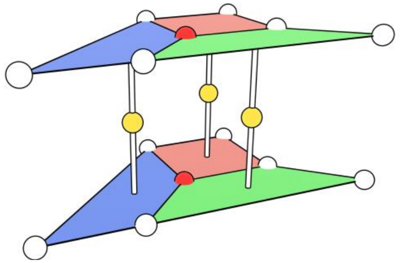
A J. Landahl et al, arXiv:1108.5738 (2011).

6.6.6 color code: 0.2% - 0.47%

C Chamberland et al, New J. Phys. 22, 023019 (2020).

My two independent studies

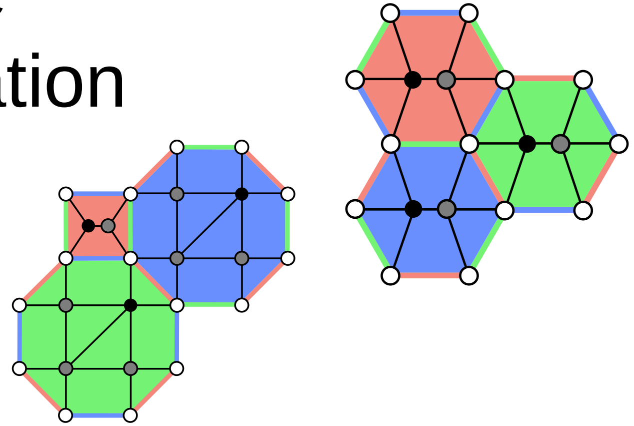
- Ising model formulation for highly accurate topological color codes decoding



Yugo Takada, Yusaku Takeuchi, Keisuke Fujii, Phys. Rev. Res. **6**, 013092 (2024).

- Improving threshold for fault-tolerant color code quantum computing by flagged weight optimization

Yugo Takada, Keisuke Fujii, PRX Quantum **5**, 030352 (2024).



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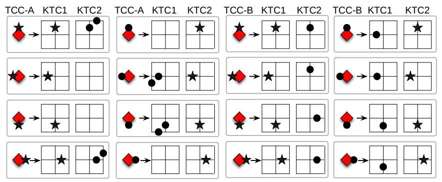
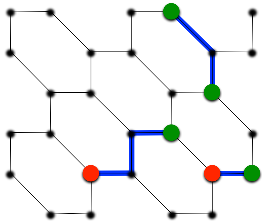
Problems with existing decoders in color codes

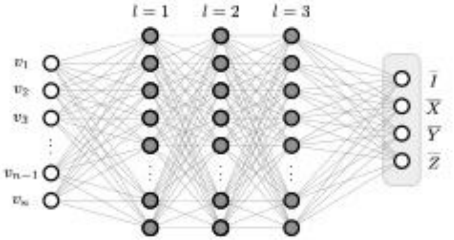
Renormalization group decoder
H. Bombin et al., New J. Phys, **14**, 073048 (2012).



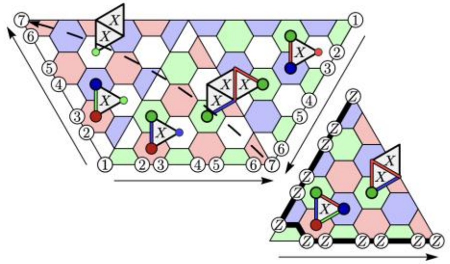
Restriction decoder
A. Kubica et al., Quantum **7**, 929 (2023).



Neural-network decoder
N. Maskara et al., Phys. Rev. A, **99**, 5 (2018).



Möbius decoder
K Sahay et al., PRX Quantum **3**, 010310 (2022).



Low accuracy

Integer programming decoder
J. Landahl et al, arXiv:1108.5738 (2011).

✓ Binary variables are assigned for **each error**

$$x_v \in \{0,1\}$$

$$\begin{cases} \min & \sum_v x_v \\ \mathrm{s.t.} & \bigoplus_{v \in f} x_v = s_f \quad \forall f \end{cases}$$

High accuracy, but requires **exponential time**

Proposed method

Main idea: formulate the decoding problem as Ising model, then solve it by simulated annealing (SA)

① Decompose Pauli error E into $E = T(S)GL$ D. Poulin, Physical Review A 74, 052333 (2006).

✓ Binary variables are assigned for **stabilizer operators**

$T(S)$: pure error	G : stabilizer operator	L : logical operator

② Formulate the number of errors as an (many-body) Ising Hamiltonian

$$H = c + \sum_i h_i \sigma_i + \sum_{i<j} J_{ij} \sigma_i \sigma_j + \sum_{i<j<k} K_{ijk} \sigma_i \sigma_j \sigma_k + \cdots$$

The syndrome constraints are imposed on the initial spin configuration

③ Solve the energy minimization problem of H by SA

✓ Use OpenJij

https://www.openjij.org/

$$\min H$$

Proposed method for bit-flip noise

$$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$$

① Decompose Pauli error E into $E = T(S)GL$

✓ Mapping binary variables

$$T(S) = \prod_i X_i^{t_i(S)}$$

$t_i(S)$: Bits representing the action of $T(S)$

$$G = \prod_i G_{Xi}^{g_i}$$

G_{Xi} : Stabilizer generators
g_i : Binary variables

$$L = L_X^l$$

L_X : A logical operator
l : A binary variable

$$x_i = t_i(S) \oplus \left(\bigoplus_{j \in B_i} g_j \right) \oplus (l_i \cdot l)$$

***X* errors**

B_i : Sets of Indices representing stabilizer operators acting on the i-th qubit

l_i : Bits defined by $L_X = \prod_i X_i^{l_i}$

Proposed method for bit-flip noise

② Formulate the number of errors as an Ising Hamiltonian

$$x_i = t_i(S) \oplus \left(\bigoplus_{j \in B_i} g_j \right) \oplus (l_i \cdot l)$$

$$\sum_i x_i$$

# of errors

Convert to Ising spin variables $\longrightarrow$

$$\sigma_j := 1 - 2g_j$$
$$J_i := (1 - 2l \cdot l_i)(1 - 2t_i(S))$$

$$H_l = -\sum_i J_i \prod_{j \in B_i} \sigma_j$$

Three-body Ising Hamiltonian

③ Solve the energy minimization problem of H_l by SA

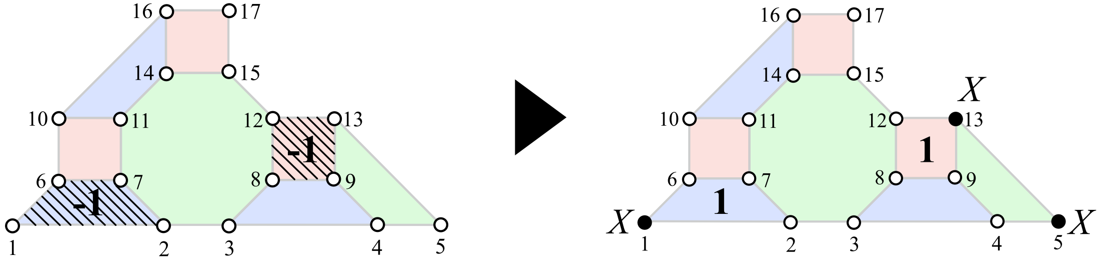
$$\min_l \left(\min H_l \right)$$

Example of proposed method for bit-flip noise

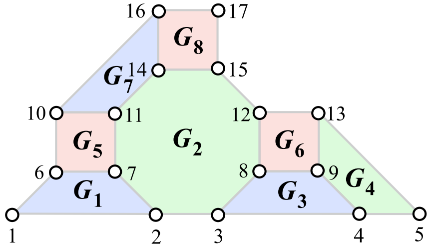
$T(S)$

$$T(S) = X_1^{(s_1 \oplus s_2 \oplus s_5 \oplus s_7)} X_2^{s_2} X_4^{s_3}\ X_5^{(s_3 \oplus s_4 \oplus s_6)} X_6^{(s_5 \oplus s_7)} X_{10}^{s_7} X_{13}^{s_6}\ X_{17}^{s_8} \qquad s_i \in \{0, 1\}$$

e.g.

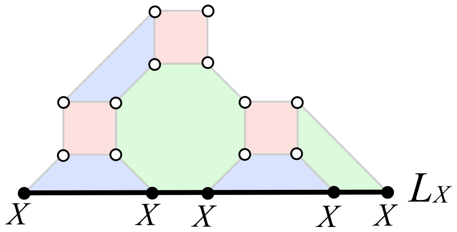


G



$$G = G_1^{g_1} G_2^{g_2} \cdots G_8^{g_8}$$

L



$$L = (X_1 X_2 X_3 X_4 X_5)^l$$

Convert to Ising spin variables

$$\sum_i x_i \;= -(s_1 s_2 s_5 s_7 g_1 l + s_2 g_1 g_2 l + g_2 g_3 l + s_3 g_3 g_4 l + s_3 s_4 s_6 g_4 l + s_5 s_7 g_1 g_5 + g_1 g_2 g_5 + g_2 g_3 g_6 + g_3 g_4 g_6 + s_7 g_5 g_7 + g_2 g_5 g_7 + g_2 g_6 + s_6 g_4 g_6 + g_2 g_7 g_8 + g_2 g_8 + g_7 g_8 + s_8 g_8)$$

# of errors **Three-body Ising Hamiltonian (variables: g_i, l)**

Proposed method for depolarizing noise

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

① Decompose Pauli error E into $E = T(S)GL$

✓ Mapping binary variables

$$T(S) = \prod_i X_i^{t_i^X(S)} \prod_j Z_j^{t_j^Z(S)}$$

$t_i^X(S), t_i^Z(S)$: Bits representing the action of X, Z of $T(S)$

$$G = \prod_i G_{Xi}^{g_i^X} \prod_j G_{Zj}^{g_j^Z}$$

G_{Xi}, G_{Zi} : Stabilizer generators

g_i^X, g_j^Z : Binary variables

$$L = L_X^{l_X} L_Z^{l_Z}$$

L_X, L_Z : Logical operators

l_X, l_Z : Binary variables

$$x_i = t_i^X(S) \oplus \left(\bigoplus_{j \in B_i} g_j^X \right) \oplus (l_i^X \cdot l_X)$$

X errors

$$z_i = t_i^Z(S) \oplus \left(\bigoplus_{j \in B_i} g_j^X \right) \oplus (l_i^Z \cdot l_Z)$$

Z errors

l_i^X, l_i^Z : Bits defined by $L_X = \prod_i X_i^{l_i^X}$, $L_Z = \prod_i Z_i^{l_i^Z}$

Proposed method for depolarizing noise

② Formulate the number of errors as an Ising Hamiltonian

$$\sum_i x_i + \sum_i z_i \boxed{- \sum_i x_i z_i}$$

Consider Y error ($Y = iXZ$)

# of errors

Convert to Ising spin variables →

$$H_{l_X,l_Z} = H^X_{l_X} + H^Z_{l_Z} + H^Y_{l_X,l_Z}$$

Six-body Ising Hamiltonian

$$H^X_{l_X} = -\sum_i J^X_i \prod_{j \in B_i} \sigma^X_j \qquad H^Z_{l_Z} = -\sum_i J^Z_i \prod_{j \in B_i} \sigma^Z_j$$

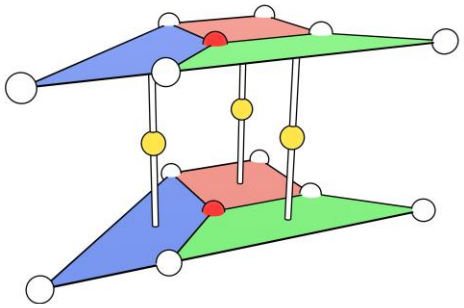
$$H^Y_{l_X,l_Z} = -\sum_i J^X_i J^Z_i \prod_{j \in B_i} \sigma^X_j \prod_{k \in B_i} \sigma^Z_k$$

③ Solve the energy minimization problem of H_{l_X,l_Z} by SA

$$\min_{l_X,l_Z} \left(\min H_{l_X,l_Z} \right)$$

Proposed method for other noise models

- Phenomenological noise



$$H_{l,\mathrm{pheno}} = -\sum_{i,t} J_i^{(t)} \prod_{j\in B_i} {\sigma'}_j^{(t)} \bar{\sigma}_i^{(t)} \bar{\sigma}_i^{(t-1)} - \sum_{f,t} \prod_{i\in f} \bar{\sigma}_i^{(t)}$$

Eight-body Ising Hamiltonian

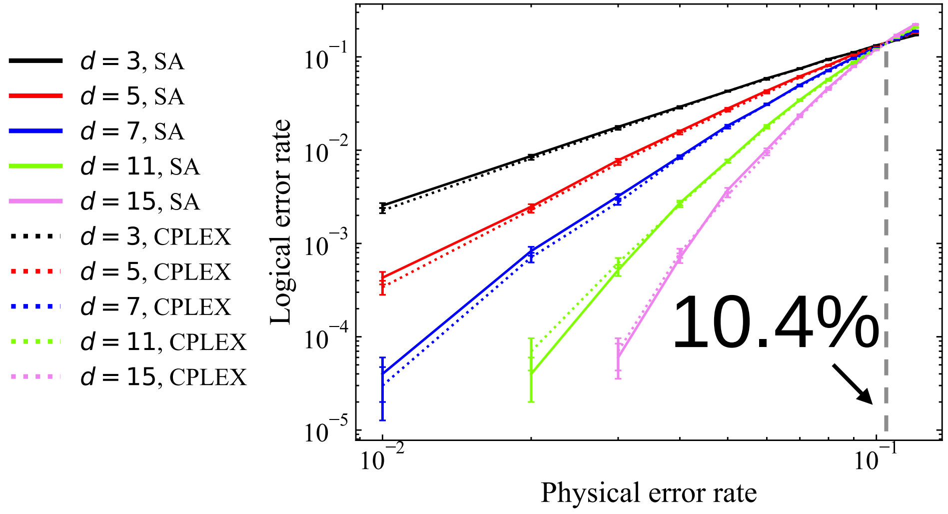
For more detailed information:

Physical Review Research **6**, 013092 (2024).

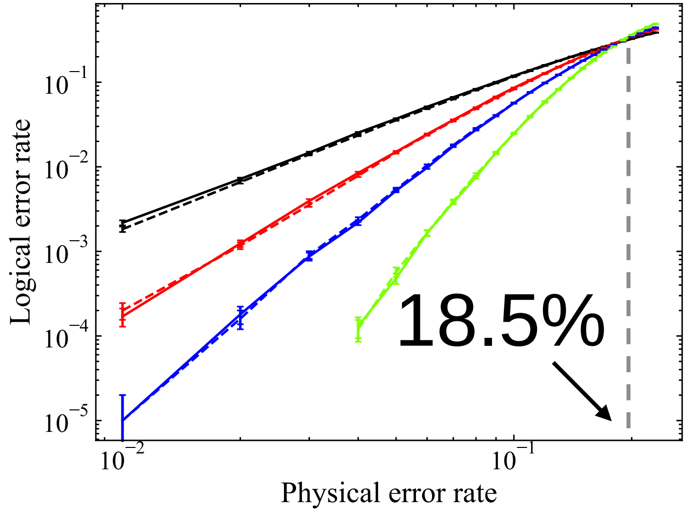
Numerical result — Logical error rate —

- Monte Carlo simulation
- Comparing with the integer programming decoder solved by IBM CPLEX

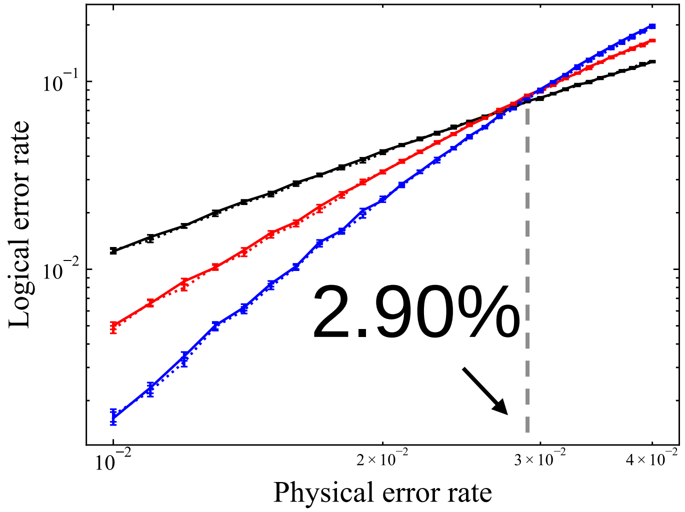
Bit-flip noise



Depolarizing noise



Phenomenological noise



Previous studies (Bit-flip noise)

- 10.6% (CPLEX) J. Landahl et al, arXiv:1108.5738 (2011).
- 10.2% A. Kubica et al, Quantum **7**, 929 (2023).
- 9.8% A. Kubica et al, Quantum **7**, 929 (2023).
- 8.7% H. Bombin et al, New J. Phys, **14**, 073048 (2012).

Previous studies (Depolarizing noise)

- 18.7% (CPLEX)
- 17.5% N. Maskara et al, Phys. Rev. A, **99**, 5(2018).

Previous studies (Phenomenological noise)

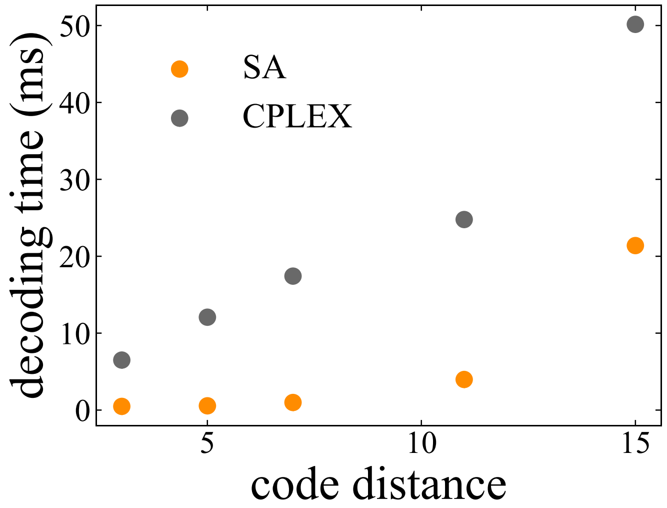
- 3.05% (CPLEX) J. Landahl et al, arXiv:1108.5738 (2011).
- 2.08% M. Stephens, arXiv:1402.3037(2014).

Our thresholds and logical error rates are almost **same** as those obtained by CPLEX

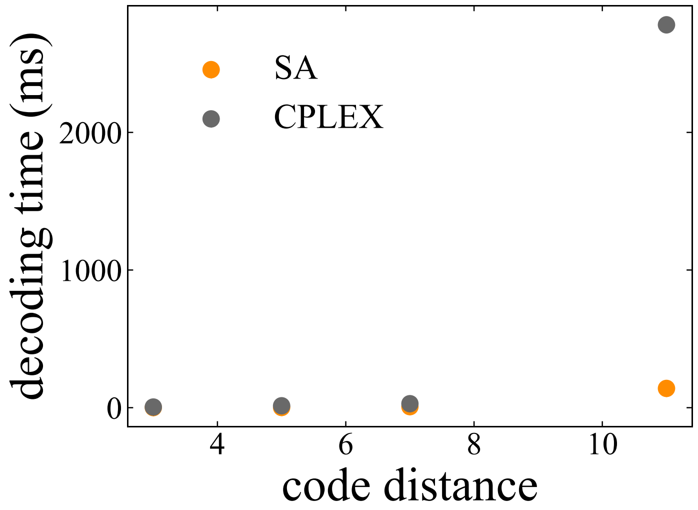
Numerical result — Decoding time —

- Comparing with the integer programming decoder solved by IBM CPLEX
- These figures represent the results when using only single core of a CPU

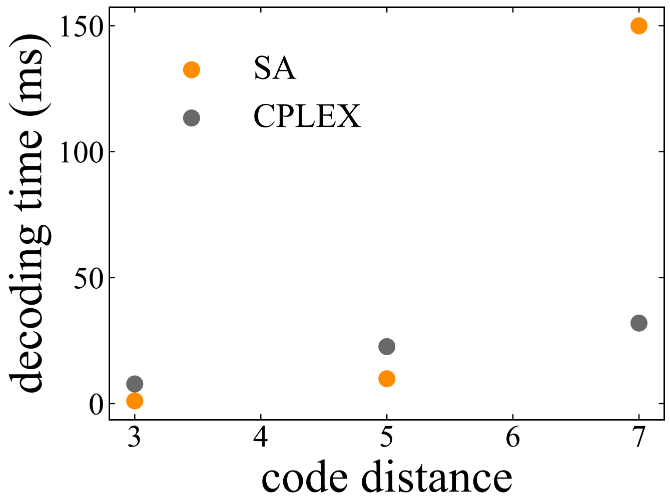
Bit-flip noise



Depolarizing noise



Phenomenological noise



Faster than CPLEX

Faster than CPLEX when d is small

※ We can decode faster by

- parallelizing the iteration of the SA
- using a SA solver that is optimized for the many-body Ising Hamiltonian

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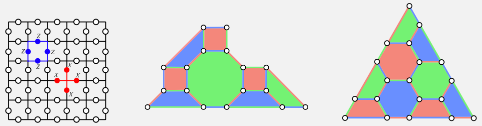
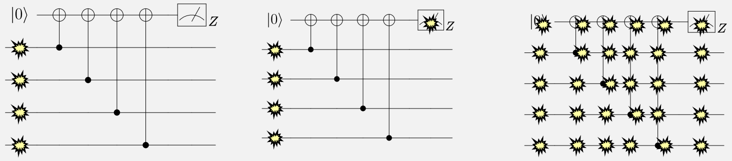
Our focus: syndrome measurement and how to use its information

Factors that determine QEC thresholds

- QEC code

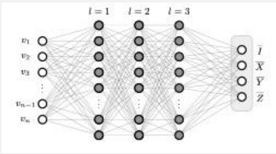


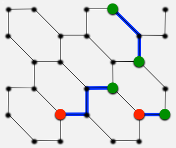
- Noise model



- Decoder (already done!)

$$\begin{cases} \min & \sum_{v} x_v \\ \text{s.t.} & \bigoplus_{v \in f} x_v = s_f \quad \forall f \end{cases}$$





- Syndrome measurement method

syndrome measurement → decoder → recovery

Improve in this research

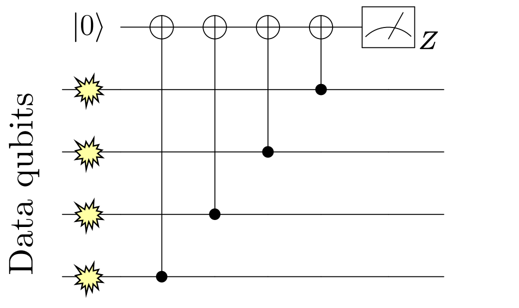
QEC procedure

We use an integer programming decoder, but our method can also be applied to many other decoders.

Why thresholds of color codes under circuit-level noise are low?

Code capacity noise

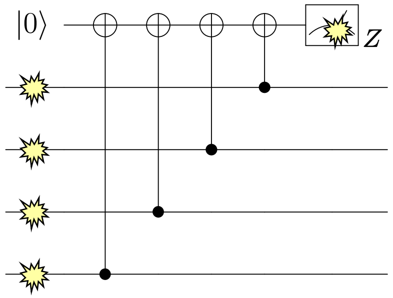
Only data errors



The circuit for measuring $Z^{\otimes 4}$

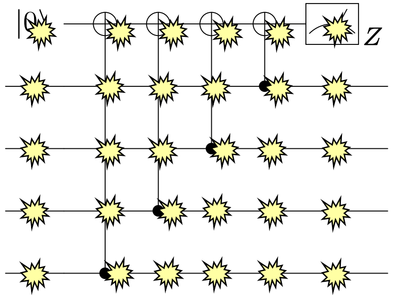
Phenomenological noise

Data errors + measurement errors

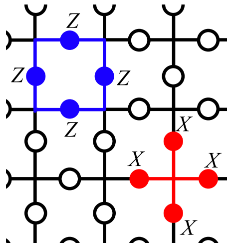


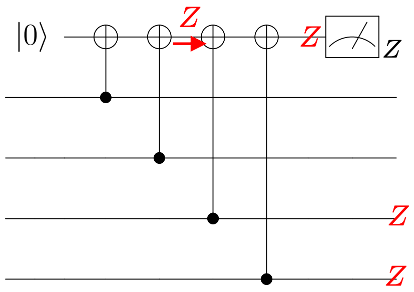
Circuit-level noise

Preparation errors + gate errors
+ measurement errors + idle errors



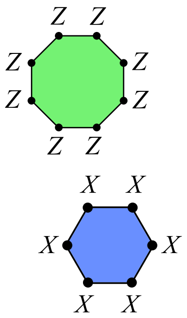
Surface codes

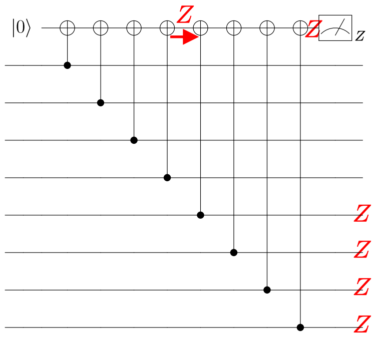




Low-weight stabilizers (weight-**4**)

Color codes





- Higher-depth circuit
- Many error locations
- More error propagation

High-weight stabilizers (weight-**6** or **8**)

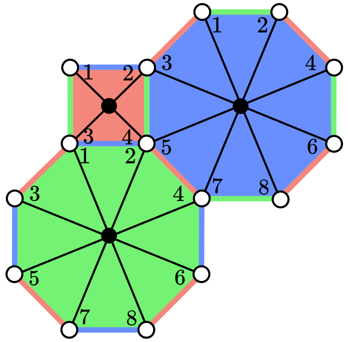
Because of the high-weight stabilizers, color codes are more susceptible to errors

Conventional syndrome measurement circuit

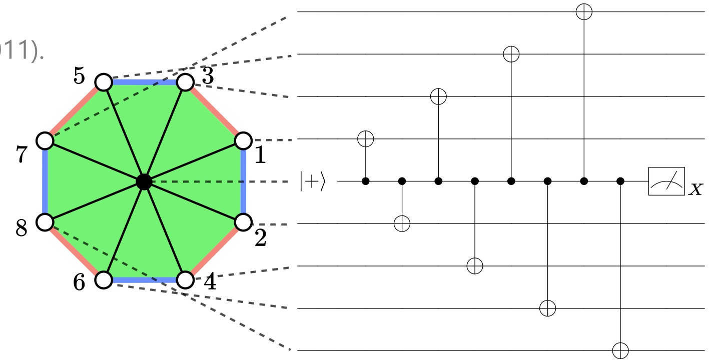
Single ancilla qubit for each face ○ data qubit ● syndrome qubit

(4.8.8) color code

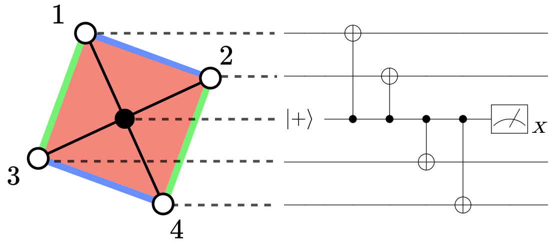
Andrew J. Landahl et al, arXiv:1108.5738 (2011).



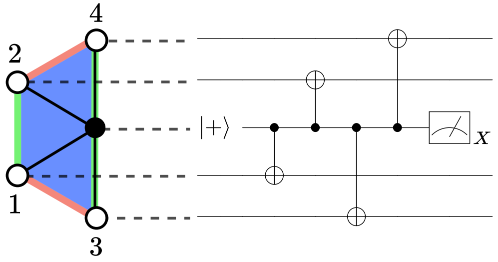
CNOT order (**8** steps)



$X^{\otimes 8}$

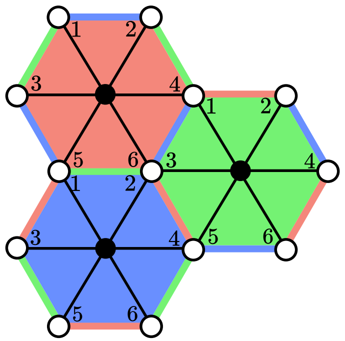


$X^{\otimes 4}$

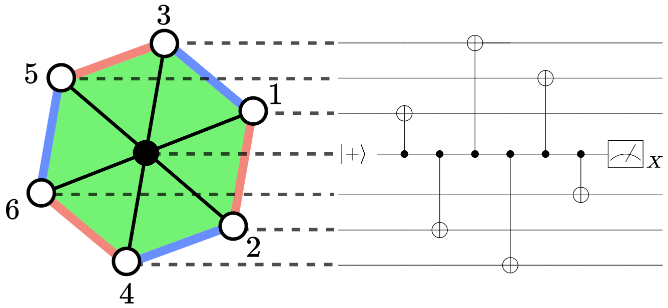


$X^{\otimes 4}$

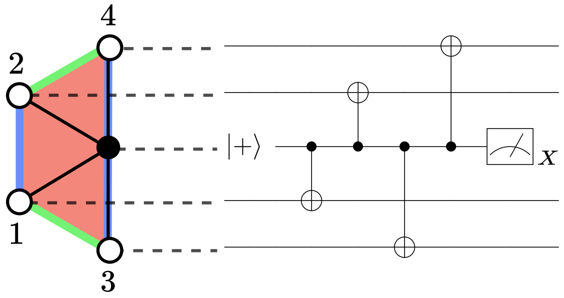
(6.6.6) color code



CNOT order (**6** steps)



$X^{\otimes 6}$

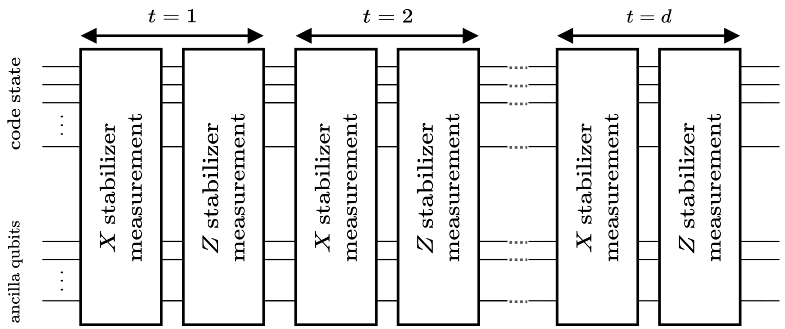


$X^{\otimes 4}$

This approach requires a large number of CNOT steps.

Decoding under phenomenological noise

Probabilities of each data and measurement error occurring are i.i.d.



Repeat syndrome measurement to treat measurement errors

Integer programming decoder

- Given a syndrome, minimize the total number of data errors and measurement errors

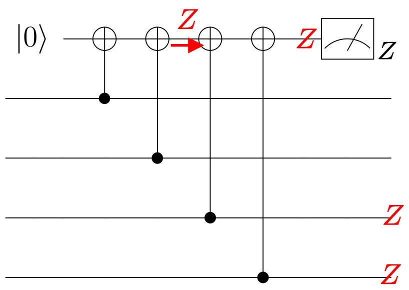
$s_f^{(t)} \in \{0,1\}$ $\qquad$ $x_v^{(t)} \in \{0,1\}$ $\qquad$ $r_f^{(t)} \in \{0,1\}$

$$\min \quad \sum_{v,t} x_v^{(t)} + \sum_{f,t} r_f^{(t)}$$

$$\text{s. t.} \quad \bigoplus_{v \in f} x_v^{(t)} \oplus r_f^{(t)} \oplus r_f^{(t-1)} = s_f^{(t)} \oplus s_f^{(t-1)} \quad \forall f.$$

Decoding under circuit-level noise

Probabilities of each data and measurement error occurring are not independent and not identical.



- There are correlated errors
- The way errors occur and propagate differs for each qubit

⟶ Setting a **weight** of a decoder for each error leads to improved performance.

Conventional weight

- Based on an error probability distribution

weight

$$\min \quad \sum_{v,t} w_v^{(t)} x_v^{(t)} + \sum_{f,t} w_f^{(t)} r_f^{(t)}$$

$$\text{s. t.} \quad \bigoplus_{v\in f} x_v^{(t)} \oplus r_f^{(t)} \oplus r_f^{(t-1)} = s_f^{(t)} \oplus s_f^{(t-1)} \quad \forall f.$$

$$w_v^{(t)} = -\log \frac{p\left(x_v^{(t)} = 1\right)}{1 - p\left(x_v^{(t)} = 1\right)}$$

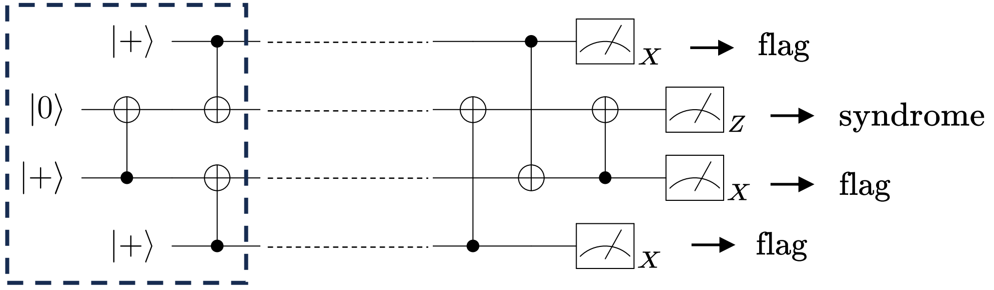
$$w_f^{(t)} = -\log \frac{p(r_f^{(t)} = 1)}{1 - p(r_f^{(t)} = 1)}$$

Conventional weight is not optimal as it fails to account for the impact of correlated errors

Proposed method — Overview —

Flag gadget for each face → Use flag information for weight optimization to decode more accurately

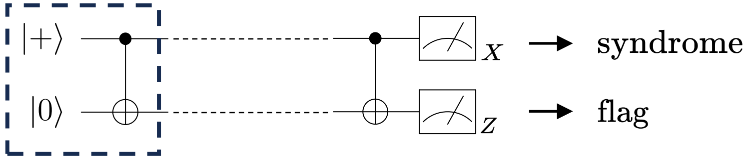
$(|0\rangle^{\otimes 4} + |1\rangle^{\otimes 4})/\sqrt{2}$
cat state



4-qubit flag gadget

David P. DiVincenzo et al, Phys. Rev. Lett. **98**, 020501(2007).

$(|00\rangle + |11\rangle)/\sqrt{2}$
cat state

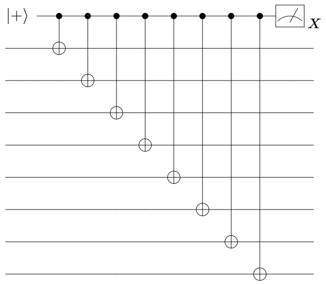


2-qubit flag gadget

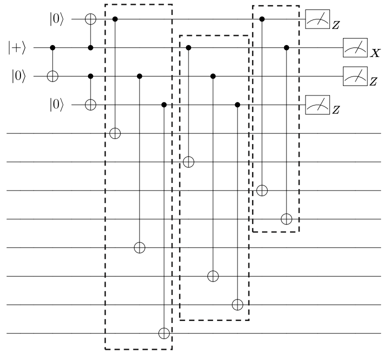
Cat state

- Reduction in the number of CNOT steps to data qubits
  →Reduction in idle noise.
- Reduction in error propagation.

one ancilla (8 steps)

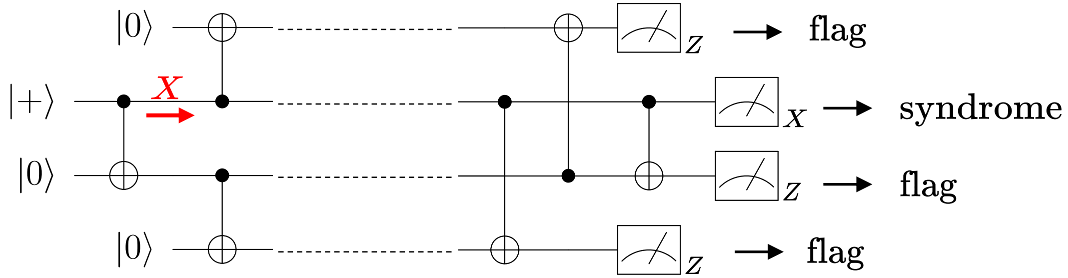


cat state (3 steps)



Flag

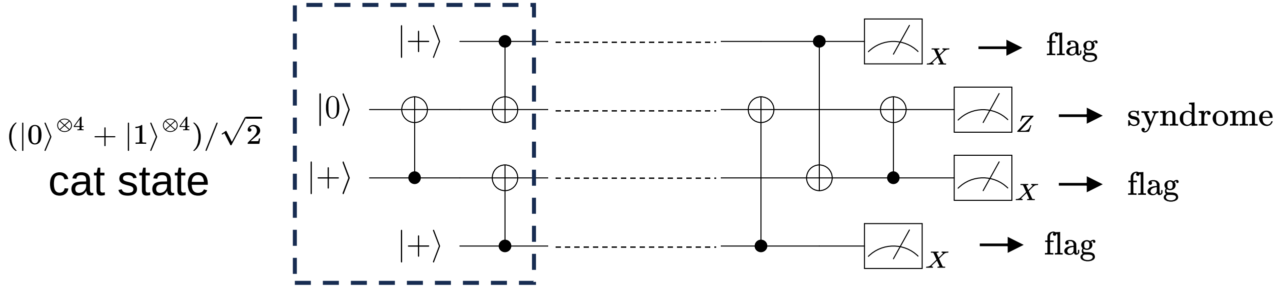
- Providing information about propagated errors.



Flag gadget reduces the occurrence and propagation of errors and allows more accurate error correction.

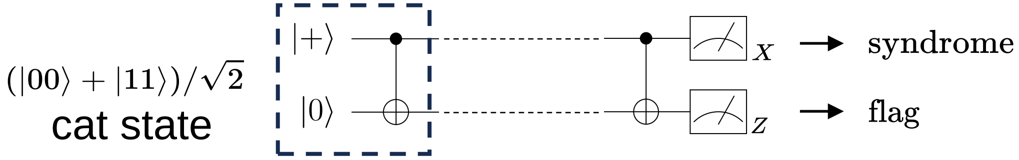
Proposed method — Overview —

Flag gadget for each face → Use flag information for weight optimization to decode more accurately



4-qubit flag gadget

David P. DiVincenzo et al, Phys. Rev. Lett. **98**, 020501(2007).

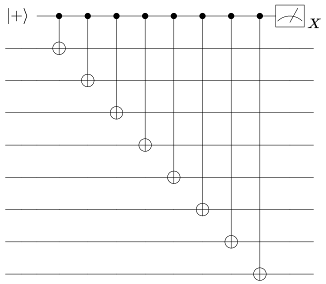


2-qubit flag gadget

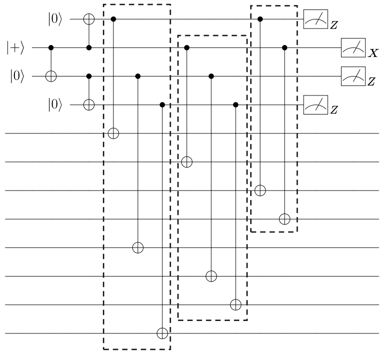
Cat state

- Reduction in the number of CNOT steps to data qubits →Reduction in idle noise.
- Reduction in error propagation.

one ancilla (8 steps)

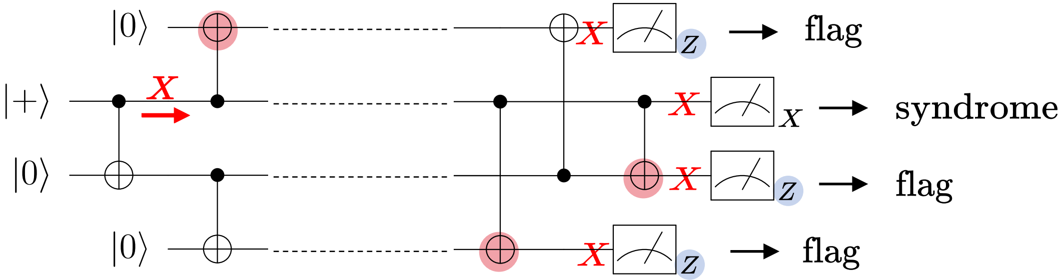


cat state (3 steps)



Flag

- Providing information about propagated errors.



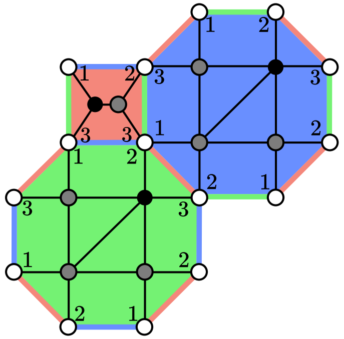
Flag gadget reduces the occurrence and propagation of errors and allows more accurate error correction.

Proposed method — Details of the syndrome measurement schedule—

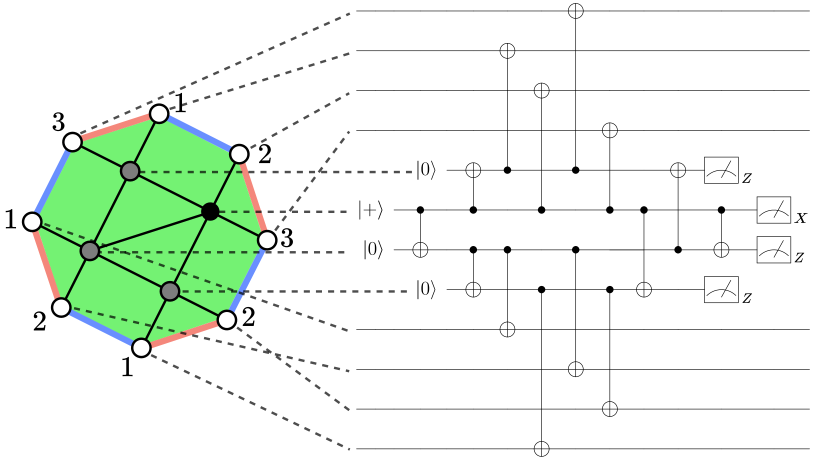
Flag gadget for each face

○ data qubit ● syndrome qubit ● flag qubit

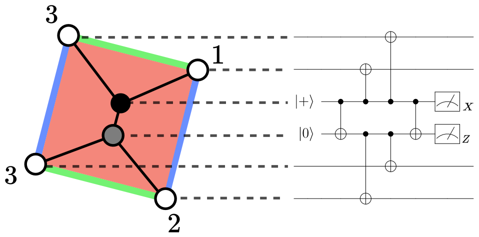
(4.8.8) color code



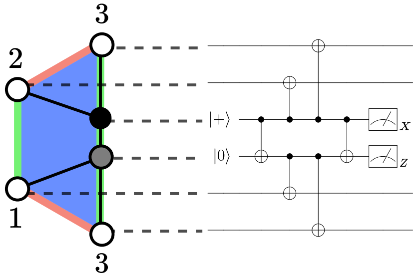
CNOT order (**3** steps)



$X^{\otimes 8}$

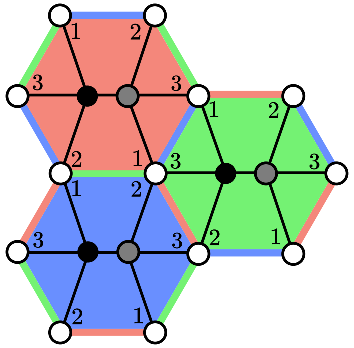


$X^{\otimes 4}$

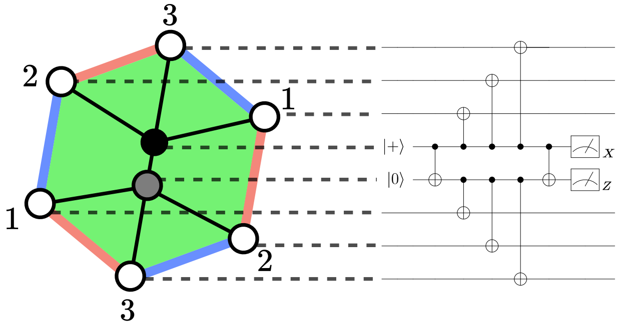


$X^{\otimes 4}$

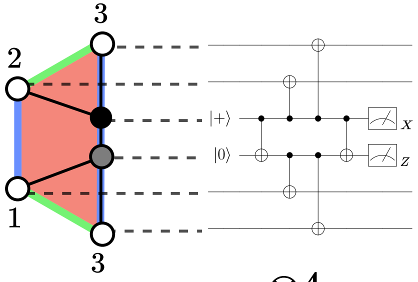
(6.6.6) color code



CNOT order (**3** steps)



$X^{\otimes 6}$



$X^{\otimes 4}$

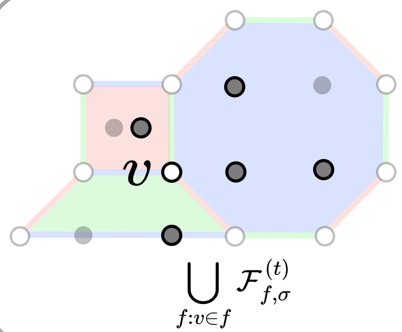
We reduced the number of CNOT steps from 8 (or 6) to **3**

Proposed method — Flagged weight optimization —

We set the weights of a decoder using **conditional error probabilities** conditioned on flag values

- For X error correction

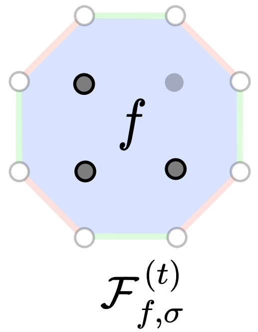
$$w_v^{(t)} = -\log \frac{p\left(x_v^{(t)} = 1 \middle| \bigcup_{f:v\in f} \mathcal{F}_{f,X}^{(t)}\right)}{1 - p\left(x_v^{(t)} = 1 \middle| \bigcup_{f:v\in f} \mathcal{F}_{f,X}^{(t)}\right)}$$



Weights of data errors

● conditioned flag qubits

$$w_f^{(t)} = -\log \frac{p\left(r_f^{(t)} = 1 \middle| \mathcal{F}_{f,Z}^{(t)}\right)}{1 - p\left(r_f^{(t)} = 1 \middle| \mathcal{F}_{f,Z}^{(t)}\right)}$$



Weights of measurement errors

Conventional weight

$$w_v^{(t)} = -\log \frac{p\left(x_v^{(t)} = 1\right)}{1 - p\left(x_v^{(t)} = 1\right)}$$

$$w_f^{(t)} = -\log \frac{p(r_f^{(t)} = 1)}{1 - p(r_f^{(t)} = 1)}$$

$\mathcal{F}_{f,\sigma}^{(t)}$: A set of flag values in the flag gadget used for measuring the $\sigma \in \{X, Z\}$ stabilizers defined on the face f at time t

These weights allow for more accurate decoding than conventional weights

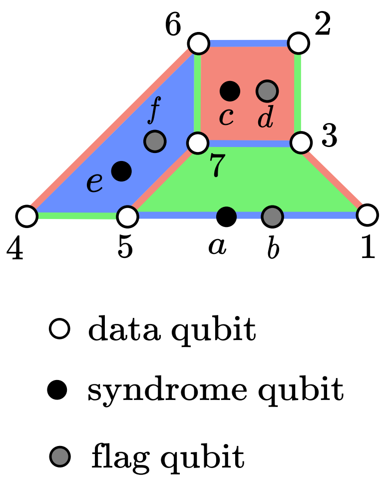
Proposed method — How to estimate the conditional error probabilities —

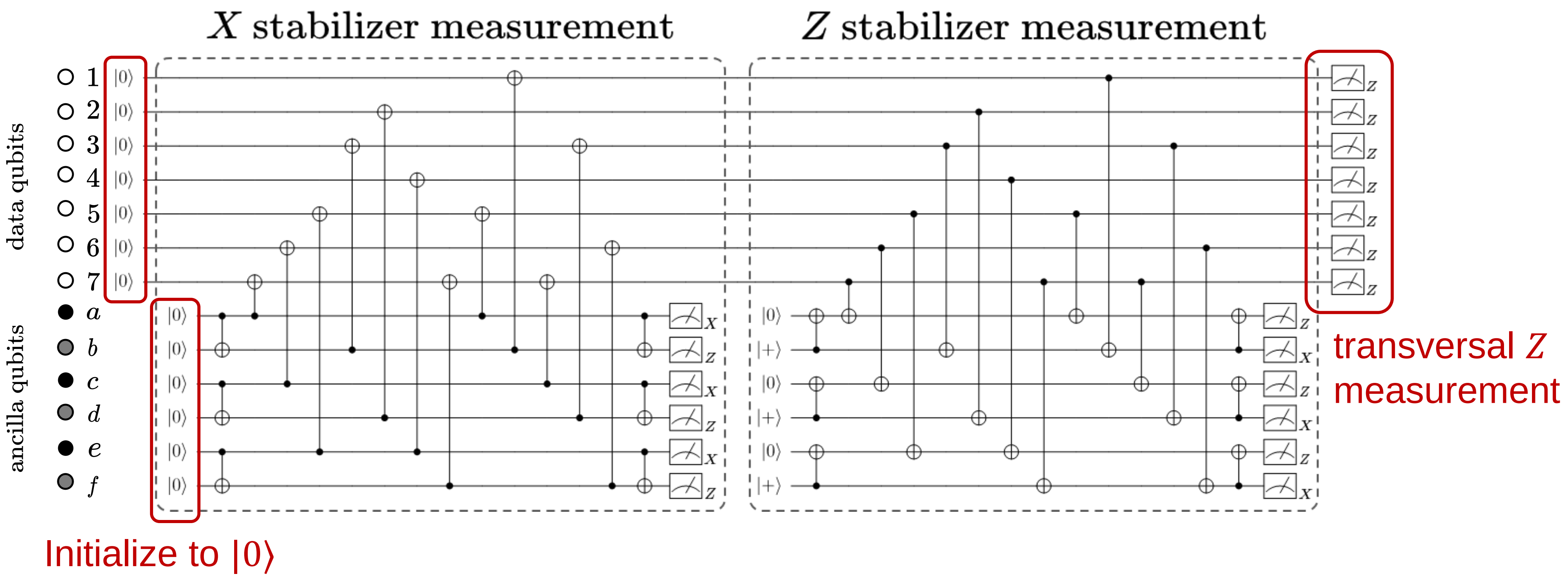
- Repeatedly execute a tailored quantum circuit offline prior to the decoding
- Obtain information about errors from the measurement outcomes

Tailored quantum circuit (for X error correction)

➢ Modified version of a one-cycle syndrome measurement circuit

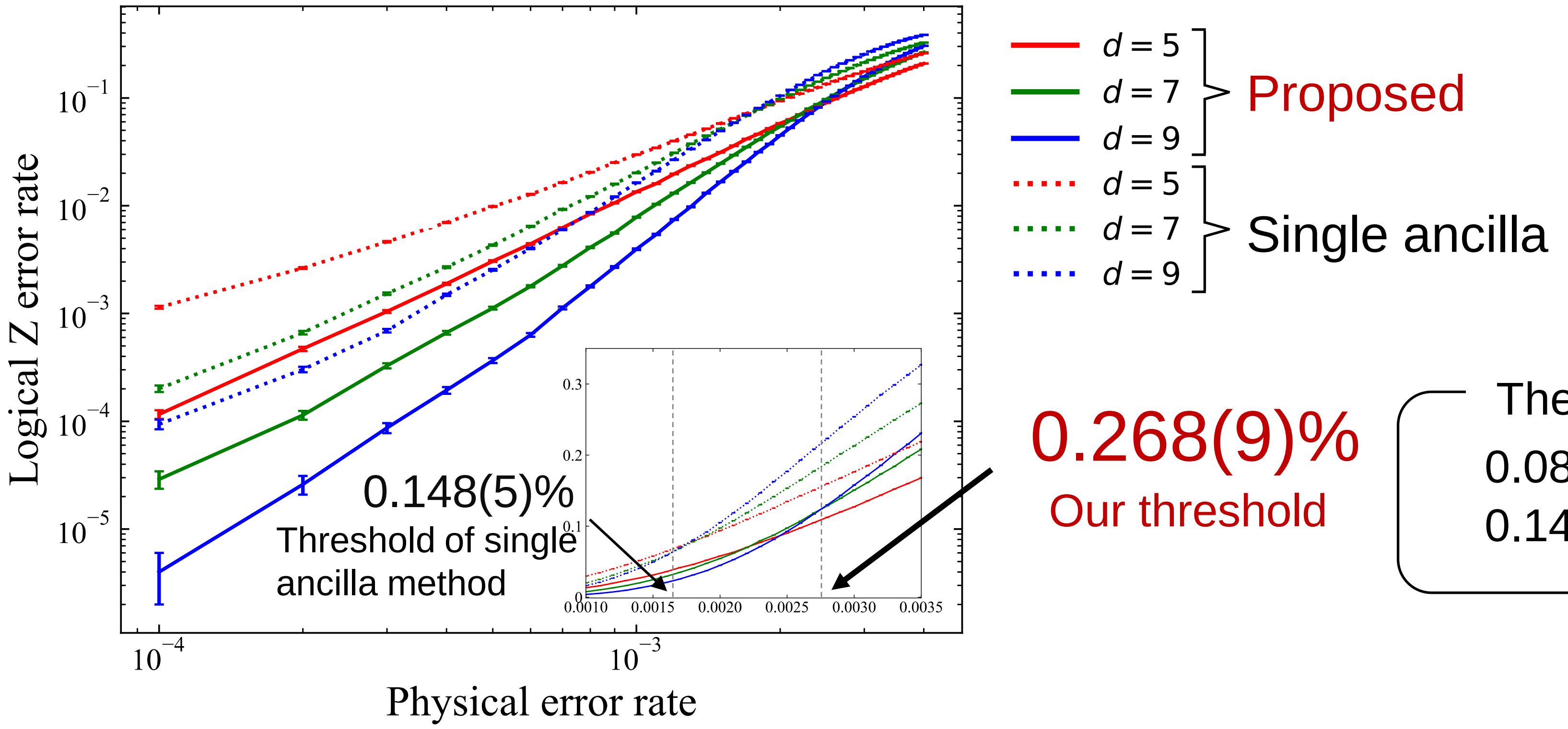
e.g. $d = 3$





Efficient, accurate, and can be used even when the underlying noise is unknown

Numerical result — (4.8.8) color code —



0.268(9)%
Our threshold

The thresholds of previous studies

0.082(3)% Andrew J. Landahl et al, arXiv:1108.5738 (2011).

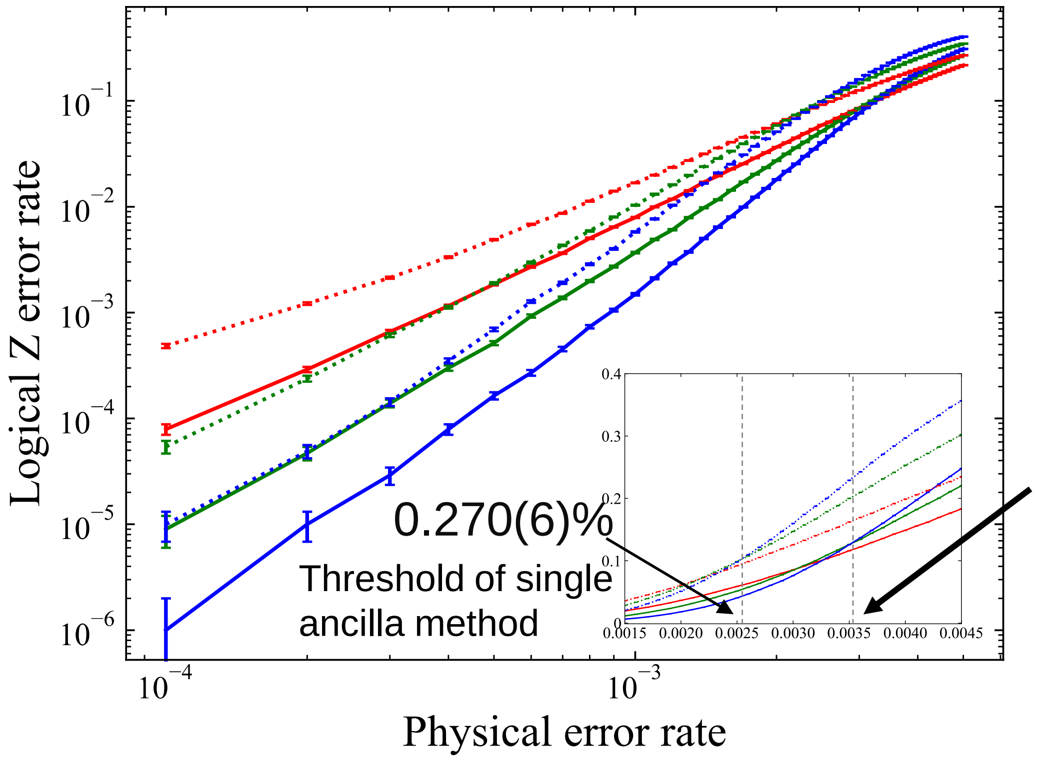
0.143(1)% Ashley M. Stephens, arXiv:1402.3037 (2014).

✓ Compared with the single ancilla method:

- The threshold was improved by about **1.8 times**.
- The logical error rates were improved by almost **one order of magnitude** in the low error rate region.

✓ We achieved the threshold that **surpasses all previous studies**.

Numerical result — (6.6.6) color code —



d = 5
d = 7
d = 9
Proposed

d = 5
d = 7
d = 9
Single ancilla

0.363(9)%
Our threshold

The thresholds of previous studies

- 0.2% C Chamberland et al, New J. Phys. 22, 023019 (2020).
- 0.37(1)% ME Beverland et al, PRX Quantum **2**, 020341 (2021).
- 0.47% Jiaxuan Zhang et al, arXiv:2309.05222 (2023).

Assuming weaker noise compared to our model

✓ Compared with the single ancilla method:

- The threshold was improved by about **1.3 times**.
- The logical error rates were improved by almost **one order of magnitude** in the low error rate region.

✓ We achieved the threshold that is almost same as **the highest value** among the previous studies that employ the same noise model.

1. Background

2. Ising model formulation for highly accurate topological color codes decoding

3. Improving threshold for fault-tolerant color code quantum computing by flagged weight optimization

4. Conclusion

Conclusion

- We proposed an Ising model formulation for decoding color codes.
  - Using simulated annealing as a solver, we achieved almost the same accuracy as the most accurate decoder (CPLEX).
  - The decoding time was shorter than CPLEX in many cases.
  - This method can be applied to wider classes of QEC codes.
- We proposed the method to improve the circuit-level thresholds of color codes by flagged weight optimization.
  - Our result for the (4.8.8) color code surpassed the thresholds of all previous studies. Our result for the (6.6.6) color code is also comparable to the best-known value.
  - This method can also be applied to other weight-based decoders.
  - Another possible future work is to apply our techniques to wider classes of QEC codes, such as high-rate quantum LDPC codes.