YITP-05-64

STAR PRODUCT FORMULA OF THETA FUNCTIONS

HIROSHIGE KAJIURA

Abstract. As a noncommutative generalization of the addition formula of theta functions, we construct a class of theta functions which are closed with respect to the Moyal star product of a fixed noncommutative parameter. These theta functions can be regarded as bases of the space of holomorphic homomorphisms between holomorphic line bundles over noncommutative complex tori.

Contents

1. Introduction

Theta functions are associated with various algebraic relations. One of them is the addition formula, which also appears in the context of the homological mirror symmetry [14] for elliptic curves [14, 20], abelian varieties [4] and noncommutative real two tori with complex structures [9, 19, 12, 10]. It is known that the bases of the space of sections of a holomorphic line bundle on an abelian variety are described by theta functions. However, in the context of homological mirror symmetry, theta functions are regarded rather as the bases of the space of holomorphic homomorphisms between two holomorphic line bundles. The composition of two holomorphic homomorphisms is just the product of two theta functions, which by the addition formula turns out to be a linear combination of theta functions. Homological mirror symmetry then asserts that such formulas can be reproduced in a geometric way by the mirror dual symplectic torus (see subsection 3.3).

A noncommutative extension of these stories is given in the case of elliptic curves [9, 19, 12, 10] based on A. Schwarz's framework of noncommutative complex tori [21, 3]. However, the conclusion is that the structure constants of the product are independent of the noncommutative parameter θ,

Date: November 8, 2005.

which implies that the derived category of holomorphic vector bundles on a noncommutative real two-torus is independent of θ [19].

Thus, in order to obtain noncommutative deformations of the structure constants, one should discuss higher dimensional complex tori. In this case, again, an extension of the framework of A. Schwarz's noncommutative complex tori gives various explicit noncommutative deformations [11], which include the deformations described in more familiar terminologies by the Moyal star product of theta functions. In this paper, we present the noncommutative deformation of the addition formula of theta functions for higher dimensional tori (Theorem 4.1). For a more categorical set-up describing these phenomena, see [11]. To explore geometric interpretations of this theorem from the mirror dual side should be especially interesting. We hope to discuss on it elsewhere.

In section 2, we start from the commutative case; we present explicitly the addition formula of theta functions corresponding to the composition of holomorphic homomorphisms between holomorphic line bundles on the n-dimensional complex torus $T^{2n} := \mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n)$. In section 3, we explain various aspects of the addition formula. Though the readers can move ahead to section 4 directly, this section provides us with interesting and pedagogical backgrounds on the product of these theta functions, together with an introduction to the approach by noncommutative complex tori. In subsection 3.1, we explain the relation of these theta functions with the *theta vectors* introduced by A. Schwarz [21, 3] (see also [2]) in the framework of (non)commutative complex tori. The framework of the theta vectors provides us with an underlying key structure in the addition formula in the commutative case and its noncommutative generalization in section 4. In subsection 3.2, these theta functions or the theta vectors are interpreted in terms of holomorphic line bundles on complex tori. In subsection 3.3, we give explicitly a geometric realization of the addition formula in the commutative case by the mirror dual symplectic torus based on the homological mirror symmetry [14]. This result can be regarded as a consequence of [4], but still it should be valuable enough to give such a correspondence in our situation together with the addition formula explicitly as we do. In section 4, we give a noncommutative generalization of this addition formula, the main theorem of this paper (Theorem 4.1). Of course, we can replace the product of the addition formula in the commutative case by the Moyal star product. However, the result is no longer described by any linear combination of the theta functions. The important point is that we should and in fact can find a class of theta functions which are closed with respect to the Moyal star product. Finally, an example of these noncommutative theta functions in the case of complex two-tori is presented in section 5.

Throughout this paper, any (graded) vector space stands for the one over the field $k = \mathbb{C}$.

Acknowledgments: I would like to thank A, Kato, T. Kawai and K. Saito for valuable discussions and useful comments. The author is supported by JSPS Research Fellowships for Young Scientists.

2. Commutative theta functions

The theta function $\vartheta : (\mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n/\mathbb{Z}^n) \times \mathfrak{H} \times \mathbb{C}^n \to \mathbb{C}$ is defined by

$$\vartheta[c_1, c_2](\Omega, z) := \sum_{m \in \mathbb{Z}^n} \exp(\pi\sqrt{-1}(m + c_1)^t\Omega(m + c_1) + 2\pi\sqrt{-1}(m + c_1)^t \cdot (z + c_2)) \ , \tag{2.1}$$

where $c_1, c_2 \in \mathbb{R}^n/\mathbb{Z}^n$ and $\mathfrak{H}$ is the Siegel upper half plane, that is, the space of $\mathbb{C}$ valued n by n symmetric matrices whose imaginary parts are positive definite. Here, for two symmetric matrices $A_a, A_b \in \mathrm{Mat}_n(\mathbb{Z})$ such that $A_{ab} := A_b - A_a$ is positive definite, we define

$$\mathfrak{e}^{\mu}_{ab}(z) = \frac{1}{\sqrt{\det(A_{ab})}} \vartheta[0, -A_{ab}^{-1}\mu](\sqrt{-1}A_{ab}^{-1}, z) \ , \qquad \mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n \ , \tag{2.2}$$

where $\sharp(\mathbb{Z}^n/A_{ab}\mathbb{Z}^n) = \det(A_{ab})$. One obtains the following addition formula:

Theorem 2.1. *Given three symmetric matrices $A_a, A_b, A_c \in \mathrm{Mat}_n(\mathbb{Z})$ such that A_{ab}, A_{bc} are positive definite, the following product formula holds:*

$$\left(\mathfrak{e}^{\mu}_{ab} \cdot \mathfrak{e}^{\nu}_{bc}\right)(z) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n} C^{\mu\nu}_{abc,\rho}\, \mathfrak{e}^{\rho}_{ac}(z) \ ,$$

where the structure constant $C^{\mu\nu}_{abc,\rho} \in \mathbb{C}$ is given by

$$C^{\mu\nu}_{abc,\rho} = \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]}{}^{\mu}_{-u+\rho}\, \delta_{[A_{bc}]}{}^{\nu}_{u} \exp\left(-\pi(u - A_{bc}A_{ac}^{-1}\rho)^t (A_{ab}^{-1} + A_{bc}^{-1})(u - A_{bc}A_{ac}^{-1}\rho)\right) \ . \tag{2.3}$$

As explained in the next section, in particular, in subsection 3.2, the collection of these theta functions $\{\mathfrak{e}^{\mu}_{ab}\}_{\mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n}$ can be interpreted as the basis of holomorphic homomorphisms between a holomorphic line bundle specified by A_a and the one specified by A_b on the n-dimensional complex torus $T^{2n} = \mathbb{C}^n/(\mathbb{Z}^n + \sqrt{-1}\mathbb{Z}^n)$. The addition formula above is then interpreted as the composition of the holomorphic homomorphisms.

Let $\mathrm{Ob} := \{a, b, \cdots\}$ be a finite collection of labels, where any $a \in \mathrm{Ob}$ is associated with a nondegenerate symmetric matrix $A_a \in \mathrm{Mat}_n(\mathbb{Z})$ such that, for any $a, b \in \mathrm{Ob}$, A_{ab} is nondegenerate if $a \neq b$. For any $a, b \in \mathrm{Ob}$, define a vector space $H^0(a, b)$ over $\mathbb{C}$ as follows:

- If A_{ab} is positive definite, $H^0(a, b)$ is the $\det(A_{ab})$-dimensional vector space spanned by the theta functions $\{\mathfrak{e}^{\mu}_{ab}\}$.
- If $a = b$, then $H^0(a, b) := \mathbb{C}$.
- If otherwise, then we set $H^0(a, b) = 0$.

For any $a, b \in \mathrm{Ob}$, $\mathrm{Hom}(a, a)$ and $\mathrm{Hom}(b, b)$ act on $\mathrm{Hom}(a, b)$ from the left and the right, respectively, as the trivial multiplication by complex numbers. Then, the product formula in Theorem 2.1 defines an algebraic structure on $\oplus_{a,b \in \mathrm{Ob}} H^0(a, b)$. This can be in fact described by the zero-th cohomology of an appropriate differential graded category (see [11]).

The main result of this paper is a noncommutative generalization of Theorem 2.1 by the Moyal star product (Theorem 4.1).

For the proof of Theorem 2.1, it is convenient to prepare the following notion.

Definition 2.2. Given two symmetric matrices $A_a, A_b \in \mathrm{Mat}_n(\mathbb{Z})$ such that A_{ab} is nondegenerate, let μ be an element in $\mathbb{Z}^n/A_{ab}\mathbb{Z}^n$ and we define a linear map $T^{\mu}_{A_{ab}} : \mathcal{S}(\mathbb{R}^n) \to C^{\infty}(T^n)$ by

$$(T^{\mu}_{A_{ab}}\xi)(x) = \sum_{w \in \mathbb{Z}^n} \xi(x + w - A_{ab}^{-1}\mu) \ , \qquad x \in \mathbb{R}^n \ .$$

Here, $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, that is, the space of smooth functions on $\mathbb{R}^n$ whose derivatives tend to zero faster than any polynomial on $\mathbb{R}^n$ (see [8], p.40).

Lemma 2.3. *Let $A_a, A_b, A_c \in \mathrm{Mat}_n(\mathbb{Z})$ be symmetric matrices such that A_{ab}, A_{bc} and A_{ac} are nondegenerate. For $\xi_{ab}, \xi_{bc} \in \mathcal{S}(\mathbb{R}^n)$, the following formula holds:*

$$(T^\mu_{A_{ab}}\xi_{ab})\cdot(T^\nu_{A_{bc}}\xi_{bc}) = \sum_{\rho\in\mathbb{Z}^n/A_{ac}\mathbb{Z}^n} (T^\rho_{A_{ac}}\xi^\rho_{ac}) \ ,$$

where $\xi^\rho_{ac} \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\xi^\rho_{ac}(x) := \sum_{u\in\mathbb{Z}^n} \delta_{[A_{ab}]}{}^{\mu}_{-u+\rho}\delta_{[A_{bc}]}{}^{\nu}_{u}\xi_{ab}(x + A_{ab}^{-1}(u - A_{bc}A_{ac}^{-1}\rho)) \cdot \xi_{bc}(x - A_{bc}^{-1}(u - A_{bc}A_{ac}^{-1}\rho)) \ . \tag{2.4}$$

Here, $\delta_{[A_{ab}]}{}^{\mu}_{\rho}$ is the Kronecker's delta mod $\mathbb{Z}^n/A_{ab}\mathbb{Z}^n$, that is,

$$\delta_{[A_{ab}]}{}^{\mu}_{\rho} = \begin{cases} 1 & \rho - \mu \in A_{ab}\mathbb{Z}^n \ , \\ 0 & \textit{otherwise} \ . \end{cases}$$

Proof. By direct calculation, the left hand side is

$$(T^\mu_{A_{ab}}\xi_{ab})\cdot(T^\nu_{A_{bc}}\xi_{bc})(x) = \sum_{v\in\mathbb{Z}^n} \delta_{[A_{ab}]}{}^{\mu}_{-v}\xi_{ab}(x + A_{ab}^{-1}v) \sum_{v'\in\mathbb{Z}^n} \delta_{[A_{bc}]}{}^{\nu}_{-v'}\xi_{bc}(x + A_{bc}^{-1}v') \ .$$

By the transformation

$$\begin{pmatrix} v \\ v' \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & A_{ab} \\ -\mathbf{1}_n & A_{bc} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} - \begin{pmatrix} \rho \\ \mathbf{0}_n \end{pmatrix} \ ,$$

the equation above is rewritten as

$$(T^\mu_{A_{ab}}\xi_{ab})\cdot(T^\nu_{A_{bc}}\xi_{bc})(x) = \sum_{\rho\in\mathbb{Z}^n/A_{ac}\mathbb{Z}^n} \sum_{u,w\in\mathbb{Z}^n} \delta_{[A_{ab}]}{}^{\mu}_{-u+\rho}\delta_{[A_{bc}]}{}^{\nu}_{u}\xi_{ab}(x+w+A_{ab}^{-1}(u-\rho))\xi_{bc}(x+w-A_{bc}^{-1}u) \ .$$

On the other hand, the right hand side can be computed directly as

$$\begin{aligned} (T^\rho_{A_{ac}}\xi^\rho_{ac})(x) = &\sum_{\rho\in\mathbb{Z}^n/A_{ac}\mathbb{Z}^n} \sum_{u,w\in\mathbb{Z}^n} \delta_{[A_{ab}]}{}^{\mu}_{-u+\rho}\delta_{[A_{bc}]}{}^{\nu}_{u} \\ &\qquad \xi_{ab}(x + w - A_{ac}^{-1}\rho + A_{ab}^{-1}(u - A_{bc}A_{ac}^{-1}\rho)) \cdot \xi_{bc}(x + w - A_{ac}^{-1}\rho - A_{bc}^{-1}(u - A_{bc}A_{ac}^{-1}\rho)) \\ = &\sum_{u\in\mathbb{Z}^n} \delta_{[A_{ab}]}{}^{\mu}_{-u+\rho}\delta_{[A_{bc}]}{}^{\nu}_{u}\xi_{ab}(x + w + A_{ab}^{-1}(u - \rho)) \cdot \xi_{bc}(x + w - A_{bc}^{-1}u) \ . \end{aligned}$$

Thus, the left hand side coincides with the right hand side. □

For A_a, A_b such that $A_{ab} \in \mathrm{Mat}_n(\mathbb{Z})$ is positive definite, define a function $e_{ab} \in \mathcal{S}(\mathbb{R}^n)$ by

$$e_{ab}(x) = \exp\left(-\pi x^t A_{ab} x\right) \ . \tag{2.5}$$

Then, by the Poisson resummation formula (see [17], p.195-197), one can rewrite the theta functions $\{\mathfrak{e}^\mu_{ab}\}$ as

$$\mathfrak{e}^\mu_{ab}(z) := T^\mu_{A_{ab}}(e_{ab})(z) \ , \qquad \mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n \ , \tag{2.6}$$

where, for $T^\mu_{A_{ab}}(e_{ab}) \in \mathcal{S}(\mathbb{R}^n)$, $T^\mu_{A_{ab}}(e_{ab})(z)$ stands for the holomorphic extension.

Thus, for symmetric matrices $A_a, A_b, A_c \in \mathrm{Mat}_n(\mathbb{Z})$ such that A_{ab} and A_{bc} are positive definite, apply Lemma 2.3 with $\xi_{ab} = e_{ab}$, $\xi_{bc} = e_{bc}$, and the holomorphic extension leads to Theorem 2.1.

3. Various interpretations of the product formula

In this section, we give various interpretations of Theorem 2.1.

3.1. **The tensor product of Heisenberg modules.** Theorem 2.1 can be understood directly in A. Schwarz's framework of noncommutative complex tori [21, 3]. A *noncommutative torus* $\mathcal{A}_\theta^d$ is an algebra defined by unitary generators $U_1, \cdots, U_d$ with relations

$$U_i U_j = e^{-2\pi\sqrt{-1}\theta_{ij}} U_j U_i \ , \qquad \theta_{ij} = -\theta_{ji} \in \mathbb{R} \tag{3.1}$$

for $i, j = 1, \cdots, d$. Now, we shall consider $2n$-dimensional commutative torus $\mathcal{A}^{2n} := \mathcal{A}^{2n}_{\theta=0}$. Namely, $\mathcal{A}^{2n}$ is thought of as the space of functions on a $2n$-dimensional commutative torus T^{2n}. Thus, the generators $U_1, \cdots, U_{2n}$ now commute with each other.

A pair $E_a := (E_{A_a}, \nabla_a)$ of a finitely generated projective module E_{A_a}, called a *Heisenberg module* (see [13]), with a constant curvature connection ∇_a is constructed as follows. The Heisenberg module is defined by

$$E_{A_a} := \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n / A_a \mathbb{Z}^n))$$

for a fixed nondegenerate symmetric matrix $A_a \in \mathrm{Mat}_n(\mathbb{Z})$. The left action of $\mathcal{A}^{2n}$ on E_{A_a} is defined by specifying the left action of each generator; for $\xi_a \in E_{A_a}$, it is given by

$$\begin{aligned} (U_i \xi_a)(x;\mu) &= e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))} \xi_a(x;\mu) \ , \\ (U_{n+i}\xi_a)(x;\mu) &= \xi_a(x + A_a^{-1} t_i; \mu - t_i) \ , \qquad i = 1, \cdots, n \ , \end{aligned} \tag{3.2}$$

where $x := (x_1 \cdots x_n)^t \in \mathbb{R}^n$ (^{t} indicates the transpose), $\mu \in \mathbb{Z}^n / A_a \mathbb{Z}^n$ and $t_i \in \mathbb{R}^n$ is defined by $(t_1 \cdots t_n) = \mathbf{1}_n$. A constant curvature connection $\nabla_{a,i} : E_{A_a} \to E_{A_a}$, $i = 1, \cdots, 2n$, is given by

$$(\nabla_{a,1} \cdots \nabla_{a,2n})^t = \begin{pmatrix} \mathbf{1}_n & \\ & -A_a \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix} \ , \tag{3.3}$$

where $\partial_x := \begin{pmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{pmatrix}^t$, whose curvature $F_a := \{ \frac{\sqrt{-1}}{2\pi} [\nabla_{a,i}, \nabla_{a,j}] \}_{i,j=1,\cdots,2n}$ is

$$F_a := \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix} \ .$$

The generators of the endomorphism algebra is the same as U_i, $i = 1, \cdots, 2n$:

$$\begin{aligned} (\xi_a Z_i)(x;\mu) &= \xi_a(x;\mu)\, e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))} \ , \\ (\xi_a Z_{n+i})(x;\mu) &= \xi_a(x + A_a^{-1} t_i; \mu - t_i) \ , \qquad i = 1, \cdots, n \ . \end{aligned}$$

Namely, the endomorphism algebra also forms a commutative torus $\mathcal{A}^{2n}$.

Given E_a and E_b such that A_{ab} is nondegenerate, the space $\mathrm{Hom}(E_a, E_b)$ is defined again as the Schwartz space $\mathrm{Hom}(E_a, E_b) := \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n / A_{ab}\mathbb{Z}^n))$. For $\xi_{ab} \in \mathrm{Hom}(E_a, E_b)$, the left action of $\mathcal{A}^{2n}$, generated by U_i, $i = 1, \cdots, 2n$, and the right action of $\mathcal{A}^{2n}$, generated by Z_i, $i = 1, \cdots, 2n$, are defined by

$$\begin{aligned} (U_i \xi_{ab})(x;\mu) &= e^{2\pi\sqrt{-1}(x_i + (A_{ab}^{-1}\mu)_i))}\, \xi_{ab}(x;\mu) \ , & (U_{n+i}\xi_{ab})(x;\mu) &= \xi_{ab}(x + A_{ab}^{-1} t_i; \mu - t_i) \ , \\ (\xi_{ab} Z_i)(x;\mu) &= \xi_{ab}(x;\mu)\, e^{2\pi\sqrt{-1}(x_i + (A_{ab}^{-1}\mu)_i))} \ , & (\xi_{ab} Z_{n+i})(x;\mu) &= \xi_{ab}(x + A_{ab}^{-1} t_i; \mu - t_i) \ , \end{aligned}$$

where $\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n$. In fact, all these generators U_i and Z_i, $i=1,\cdots,2n$, commute with each other. The constant curvature connection $\nabla_i:\mathrm{Hom}(E_a,E_b)\to\mathrm{Hom}(E_a,E_b)$, $i=1,\cdots,2n$, is given by

$$(\nabla_1\cdots\nabla_{2n})^t:=\begin{pmatrix}\mathbf{1}_n & \\ & -A_{ab}\end{pmatrix}\begin{pmatrix}\partial_x\\ 2\pi\sqrt{-1}x\end{pmatrix}\ .$$

For $\xi_{ab}\in\mathrm{Hom}(E_a,E_b)$ and $\xi_{bc}\in\mathrm{Hom}(E_b,E_c)$, the tensor product $m:\mathrm{Hom}(E_a,E_b)\otimes\mathrm{Hom}(E_b,E_c)\to\mathrm{Hom}(E_a,E_c)$ is defined by

$$m(\xi_{ab},\xi_{bc})(x,\rho)=\sum_{u\in\mathbb{Z}^n}\xi_{ab}(x+A_{ab}^{-1}(u-A_{bc}A_{ac}^{-1}\rho),-u+\rho)\cdot\xi_{bc}(x-A_{bc}^{-1}(u-A_{bc}A_{ac}^{-1}\rho),u)\ . \quad (3.4)$$

One can see that this tensor product formula is just the definition of ξ_{ac}^{ρ} in eq.(2.4). This tensor product is in fact associative and the connection $\nabla_i:\mathrm{Hom}(E_a,E_b)\to\mathrm{Hom}(E_a,E_b)$ satisfies the Leibniz rule with respect to this product (see [11]). [1]

Now suppose we consider a n-dimensional *complex* torus $T^{2n}:=\mathbb{C}^n/(\mathbb{Z}^n\oplus\sqrt{-1}\mathbb{Z}^n)$. For $E_a=(E_{A_a},\nabla_a)$ a Heisenberg module with the constant curvature connection, the holomorphic structure $\bar\nabla_{a,i}:E_{A_a}\to E_{A_a}$, $i=1,\cdots,n$, is defined by

$$\bar\nabla_{a,i}=\nabla_{a,i}+\sqrt{-1}\nabla_{a,n+i}\ .$$

Also, for given E_a,E_b, the holomorphic structure $\bar\nabla_i:\mathrm{Hom}(E_a,E_b)\to\mathrm{Hom}(E_a,E_b)$, $i=1,\cdots,n$, is defined in the same way:

$$\bar\nabla_i:=\nabla_i+\sqrt{-1}\nabla_{n+i}\ ,\qquad i=1,\cdots,n\ .$$

When A_{ab} is positive definite, the space $H^0(E_a,E_b):=\cap_{i=1}^n\mathrm{Ker}(\bar\nabla_i:\mathrm{Hom}(E_a,E_b)\to\mathrm{Hom}(E_a,E_b))$ forms a $\det(A_{ab})$-dimensional vector space. The bases e_{ab}^{μ}, $\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n$, are called A. Schwarz's *theta vectors* [21] (see also [2]), which are just the function $e_{ab}\in\mathcal{S}(\mathbb{R}^n)$ defined in eq.(2.5):

$$e_{ab}^{\mu}(x,\rho)=\delta_{[A_{ab}]}{}^{\mu}_{\rho}\exp\left(-\pi x^tA_{ab}x\right)\ . \quad (3.5)$$

The Leibniz rule of $\bar\nabla$ then guarantees that the tensor product $m(e_{ab}^{\mu},e_{bc}^{\nu})$ turns out to be the linear combination of e_{ac}^{ρ}, $\rho\in\mathbb{Z}^n/A_{ac}\mathbb{Z}^n$.

This approach by Heisenberg modules allows us various noncommutative deformations of these structures (see [11]), but some of such deformations can be lifted to theta functions as the Moyal star product; the consequence is the one presented in section 4.

3.2. **Holomorphic line bundles on tori.** In this subsection, the theta functions $\{\mathfrak{e}_{ab}^{\mu}\}$ in eq.(2.6), or equivalently, the theta vectors $\{e_{ab}^{\mu}\}$ in eq.(3.5), are interpreted in terms of holomorphic line bundles on complex tori.

[1] In [11], left modules in this paper is flipped to be right modules. The relation of the conventions between this paper and [11] is as follows. First, consider a bimodule $\mathrm{Hom}(E_a,E_b)$ in this paper. Replace A_a by $-A_b$ and A_b by $-A_a$. Then, one gets a bimodule in [11]. In both cases, a left/right module E_{A_b} is obtained by setting $A_a=0$.

Given a d-dimensional-torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$, let $\pi : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$ be the projection. The space $\tilde{E}$ of sections of a vector bundle of rank $q \in \mathbb{Z}_{>0}$ is described by the space of q copies of functions on the covering space $\mathbb{R}_d$ equipped with a $\mathbb{Z}^d \ni \lambda$ action

$$\tilde{\xi}(x+\lambda) := c_\lambda(x)\xi(x) \ , \qquad \tilde{\xi} \in \tilde{E} \subset (C^\infty(\mathbb{R}^d))^{\oplus q} \ , \qquad c_\lambda \in U(q; C^\infty(\mathbb{R}^d)) \tag{3.6}$$

satisfying the following condition:

$$c_{\lambda'}(x+\lambda)c_\lambda(x) = c_{\lambda+\lambda'}(x) \ .$$

Thus, c_γ is regarded as a transition function of the vector bundle. A connection $\nabla_i : \tilde{E} \to \tilde{E}$, $i = 1, \cdots, d$, is defined so that the following compatibility conditions hold:

$$(\nabla_i)(x+\lambda) = c_\lambda(x)(\nabla_i)(x)c_\lambda^{-1}(x) \ , \tag{3.7}$$

where the *curvature* is defined by

$$F = \{F_{ij}\}_{i,j=1,\cdots,d} \ , \qquad F_{ij} := \frac{\sqrt{-1}}{2\pi}[\nabla_i, \nabla_j] \ .$$

Now, consider a complex torus $T^{2n} := \mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n)$, where the coordinates of the covering space $\mathbb{C}^n$ is denoted $z := (z_1 \cdots z_n)^t$, $z_i := x_i + \sqrt{-1}y_i$, $i = 1, \cdots, n$. For a nondegenerate symmetric matrix $A_a \in \mathrm{Mat}_n(\mathbb{Z})$, the space of sections $\tilde{E}_{A_a}$ of a line bundle ($q = 1$ case) on T^{2n} is constructed by setting

$$c_{(\lambda_x,0)}(x,y) = 1 \ , \qquad c_{(0,\lambda_y)}(x,y) = e^{-2\pi\sqrt{-1}x^t A_a \lambda_y} \cdot 1 \ ,$$

where $x := (x_1 \cdots x_n)^t$, $y := (y_1 \cdots y_n)^t$ and $\lambda_x, \lambda_y \in \mathbb{Z}^n$ such that $\lambda = (\lambda_x, \lambda_y) \in \mathbb{Z}^{d=2n}$. The general form of sections in $\tilde{E}_{A_a}$ is given by

$$\tilde{\xi}_a(x,y) = \sum_{w\in\mathbb{Z}^n}\sum_{\mu\in\mathbb{Z}^n/A\mathbb{Z}^n} \exp\left(2\pi\sqrt{-1}y^t\left(-A_a\left(x+w\right)+\mu\right)\right)\xi_a^\mu\left(x+w-A_a^{-1}\mu\right) \ , \quad \xi_a^\mu \in \mathcal{S}(\mathbb{R}^n) \ ,$$

as a natural extension of the two dimensional case ([5, 7, 15] and see [13], the vector bundles constructed there are called *twisted bundles*). For $\xi_a^\mu(x) =: \xi_a(x,\mu)$, $\xi_a \in \mathcal{S}(\mathbb{R}^n \otimes (\mathbb{Z}^n/A_a\mathbb{Z}^n)) = E_{A_a}$, we regard $\tilde{}$ in the formula above as the isomorphism from E_{A_a} to $\tilde{E}_{A_a}$ which sends ξ_a to $\tilde{\xi}_a$. This line bundle can be equipped with the following constant curvature connection $\{\nabla_{a,i} : \tilde{E}_{A_a} \to \tilde{E}_{A_a}\}_{i=1,\cdots,2n}$ with its curvature F_a:

$$(\nabla_{a,1}, \cdots, \nabla_{a,n})^t = \partial_x + 2\pi\sqrt{-1}Ay \ , \quad (\nabla_{a,n+1}, \cdots, \nabla_{a,2n})^t = \partial_y \ , \qquad F_a = \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix} \ ,$$

where $\partial_x := (\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n})^t$, $\partial_y := (\frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_n})^t$. Let us define the generators of the space $C^\infty(T^{2n})$ of functions by

$$\tilde{U}_i = e^{\pi\sqrt{-1}x_i} \ , \quad \tilde{U}_{n+i} = e^{\pi\sqrt{-1}y_i} \ , \qquad i = 1, \cdots, n \ .$$

Then, the relationship of $\tilde{E}_a := (\tilde{E}_{A_a}, \nabla_a)$ with $E_a = (E_{A_a}, \nabla_a)$ in the previous subsection can be summarized as follows: for $\xi_a \in E_{A_a}$,

$$\tilde{U}_i\tilde{\xi}_a = \widetilde{U_i\xi_a} \ , \quad \tilde{U}_{n+i}\tilde{\xi}_a = \widetilde{U_{n+i}\xi_a} \ , \quad \nabla_{a,i}\tilde{\xi}_a = \widetilde{\nabla_{a,i}\xi} \ , \quad \nabla_{a,n+i}\tilde{\xi}_a = \widetilde{\nabla_{a,n+i}\xi_a} \ , \qquad i = 1, \cdots, n \ .$$

In a similar way, for given $\tilde{E}_a$ and $\tilde{E}_b$ such that A_{ab} is nondegenerate, the space $\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)$ of homomorphisms from $\tilde{E}_a$ from $\tilde{E}_b$ is the space whose elements are described of the form:

$$\tilde{\xi}_{ab}(x,y)=\sum_{w\in\mathbb{Z}^n}\sum_{\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n}\exp\left(2\pi\sqrt{-1}y^t\left(-A_{ab}\left(x+w\right)+\mu\right)\right)\xi_{ab}^{\mu}\left(x+w-A_{ab}^{-1}\mu\right)\ , \qquad (3.8)$$

for $\xi_{ab}^{\mu}\in\mathcal{S}(\mathbb{R}^n)$, where the compatible constant curvature connection $\nabla_i:\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)\to\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)$, $i=1,\cdots,n$, is given by

$$(\nabla_1,\cdots,\nabla_n)^t:=\partial_x+2\pi\sqrt{-1}A_{ab}y\ ,\quad (\nabla_{n+1},\cdots,\nabla_{2n})^t:=\partial_y\ ,\qquad F_{ab}=\begin{pmatrix}\mathbf{0}_n & A_{ab}\\ -A_{ab} & \mathbf{0}_n\end{pmatrix}\ .$$

Again, for $\xi_{ab}^{\mu}(x)=:\xi_{ab}(x,\mu)$, $\xi_{ab}\in\mathcal{S}(\mathbb{R}^n\otimes(\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))=\mathrm{Hom}(E_a,E_b)$, $\tilde{}$ in eq.(3.8) is regarded as the isomorphism from $\mathrm{Hom}(E_a,E_b)$ to $\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)$ which sends ξ_{ab} to $\tilde{\xi}_{ab}$.

Actually, for E_a,E_b,E_c, $\xi_{ab}\in\mathrm{Hom}(E_a,E_b),\xi_{bc}\in\mathrm{Hom}(E_b,E_c)$ and the corresponding elements $\tilde{\xi}_{ab}\in\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b),\tilde{\xi}_{bc}\in\mathrm{Hom}(\tilde{E}_b,\tilde{E}_c)$, the pointwise product $\tilde{\xi}_{ab}\cdot\tilde{\xi}_{bc}$ turns out to be

$$\tilde{\xi}_{ab}\cdot\tilde{\xi}_{bc}=m\widetilde{(\xi_{ab},\xi_{bc})}\ ,$$

where m is the tensor product of the Heisenberg modules defined in eq.(3.4). The proof is essentially the same as that of Lemma 2.3.

Now, for T^{2n} as a *complex* torus, the holomorphic structure $\{\bar{\nabla}_{a,i}:\tilde{E}_a\to\tilde{E}_a\}_{i=1,\cdots,n}$ is defined by $\bar{\nabla}_{a,i}:=\nabla_{a,i}+\sqrt{-1}\nabla_{a,n+i}$. Similarly, given $\tilde{E}_a$ and $\tilde{E}_b$, the holomorphic structure $\{\bar{\nabla}_i:\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)\to\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)\}_{n=1,\cdots,n}$ is defined by $\bar{\nabla}_i:=\nabla_i+\sqrt{-1}\nabla_{n+i}$. The space of holomorphic sections in $\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)$ is then defined by $H^0(\tilde{E}_a,\tilde{E}_b):=\cap_{i=1}^n\mathrm{Ker}(\bar{\nabla}_i:\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b)\to\mathrm{Hom}(\tilde{E}_a,\tilde{E}_b))$. This space $H^0(\tilde{E}_a,\tilde{E}_b)$ forms a $\det(A_{ab})$-dimensional vector space spanned by $\{\tilde{e}_{ab}^{\mu}\}$, the extension of the theta vectors $\{e_{ab}^{\mu}\}_{\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n}$ in (3.5) by eq.(3.8). Also, the explicit relation of these $\tilde{e}_{ab}^{\mu}$ with the theta functions $\mathfrak{e}_{ab}^{\mu}$ (2.6) is given by

$$\mathfrak{e}_{ab}^{\mu}(z)=\exp\left(\pi y^tA_{ab}y\right)\cdot\tilde{e}_{ab}^{\mu}(x,y)\ .$$

3.3. **Lagrangian submanifolds and triangles.** The homological mirror symmetry [14] asserts that the product $m(\mathfrak{e}_{ab}^{\mu},\mathfrak{e}_{bc}^{\nu})$ can also be derived from geometry of the mirror dual torus $\hat{T}^{2n}$, a symplectic $2n$-dimensional torus with the symplectic structure

$$\omega=\begin{pmatrix}\mathbf{0}_n & -\mathbf{1}_n\\ \mathbf{1}_n & \mathbf{0}_n\end{pmatrix}\ . \qquad (3.9)$$

For the covering space $\mathbb{R}^{2n}$ of $\hat{T}^{2n}$, let $\pi:\mathbb{R}^{2n}\to\hat{T}^{2n}$ be the natural projection. The coordinates for $\mathbb{R}^{2n}$ is denoted $(x_1,\cdots,x_n,\hat{y}_1,\cdots,\hat{y}_n)$.

The affine lagrangian submanifold mirror dual to $E_a=(E_{A_a},\nabla_a)$ over $\mathcal{A}^{2n}$, the space of functions on T^{2n}, is defined by the image of the affine subspace in $\mathbb{R}^{2n}$

$$L_a:\hat{y}=A_ax$$

by the projection $\pi:\mathbb{R}^{2n}\to\hat{T}^{2n}$. Thus, we have

$$\pi^{-1}\pi(L_a)=\{\hat{y}=A_ax+c_a,\ c_a\in\mathbb{Z}\}\ .$$

Let us define the space of morphisms $\mathrm{Hom}(L_a, L_b)$ which is isomorphic to the $\det(A_{ab})$-dimensional vector space $H^0(E_a, E_b)$ in subsection 3.1. Denote the basis of $\mathrm{Hom}(L_a, L_b)$ by v_{ab}^{μ}, $\mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n$, to which is associated the image of the intersection point of $\hat{y} = A_b x + \mu$ with $\hat{y} = A_a x$ in $\mathbb{C}^n$ by $\pi : \mathbb{C}^n \to \hat{T}^{2n}$. One can see that actually the number of the intersection points of $\pi(L_a)$ and $\pi(L_b)$ in $\hat{T}^{2n}$ is $\det(A_{ab})$. For a base v_{ab} of $\mathrm{Hom}(L_a, L_b)$, we denote the corresponding point in $\hat{T}^{2n}$ also by v_{ab}, which defines the set $\tilde{V}_{ab} := \pi^{-1}(v_{ab})$ of points in the covering space $\mathbb{R}^{2n}$.

The structure constant $C_{abc,\rho}^{\mu\nu} \in \mathbb{C}$ (2.3) can be identified with the sum of the exponentials of the symplectic areas of the triangles $\tilde{v}_{ab}\tilde{v}_{bc}\tilde{v}_{ac}$ for any $\tilde{v}_{ab} \in \tilde{V}_{ab}$, $\tilde{v}_{bc} \in \tilde{V}_{bc}$ and $\tilde{v}_{ac} \in \tilde{V}_{ac}$ with respect to the symplectic structure ω in eq.(3.9), where the triangles related by parallel transformations on the covering space $\mathbb{R}^{2n}$ are identified with each other.

It is calculated as follows. Consider three affine subspaces L'_a, L'_b, L'_c in $\mathbb{R}^2$ as follows:

$$L'_a : \hat{y} = A_a x + c_a \ , \quad L'_b : \hat{y} = A_b x + c_b \ , \quad L'_c : \hat{y} = A_c x + c_c \ .$$

If A_{ab} is nondegenerate, the intersection of L'_a and L'_b is a point v_{ab}; the coordinates $\left(\begin{smallmatrix} x \\ \hat{y} \end{smallmatrix}\right)$ are:

$$v_{ab} = \begin{pmatrix} -(A_{ab})^{-1}(c_b - c_a) \\ -A_a A_{ab}^{-1} c_b + A_b A_{ab}^{-1} c_a \end{pmatrix} .$$

Now, assume that A_{ab} and A_{bc} are positive definite. Then, A_{ac} is also positive definite. The three intersection points v_{ab}, v_{bc}, v_{ac} form a triangle, where the edges $(v_{ab}v_{bc})$, $(v_{bc}v_{ac})$, $(v_{ac}v_{ab})$ belong to L'_b, L'_c and L'_a, respectively. The symplectic area of the triangle is given by

$$(v_{ab} - v_{ac})^t \omega (v_{bc} - v_{ac}) = \begin{pmatrix} (c_c - c_a)^t & (c_b - c_a)^t \end{pmatrix} \begin{pmatrix} A_{bc}^{-1} & A_{ac}^{-1} \\ A_{ab}^{-1} A_{ac} A_{bc}^{-1} & A_{ab}^{-1} \end{pmatrix} \begin{pmatrix} c_b - c_c \\ c_a - c_c \end{pmatrix} .$$

Let us put $c_a = 0$, $c_b = u'$ and $c_c = -\rho$ so that $\pi(v_{ac}) = v_{ac}^{\rho}$. Then, consider

$$\sum_{u'} \delta_{[A_{ab}]}{}^{\mu}_{-u'} \delta_{[A_{bc}]}{}^{\nu}_{u'+\rho} \exp\left(\begin{pmatrix} -\rho^t & u'^t \end{pmatrix} \begin{pmatrix} A_{bc}^{-1} & A_{ac}^{-1} \\ A_{ab}^{-1} A_{ac} A_{bc}^{-1} & A_{ab}^{-1} \end{pmatrix} \begin{pmatrix} u'+\rho \\ \rho \end{pmatrix} \right) ,$$

where $\delta_{[A_{ab}]}{}^{\mu}_{-u'}$ and $\delta_{[A_{bc}]}{}^{\nu}_{u'+\rho}$ correspond to the condition of $\pi(v_{ab}) = v_{ab}^{\mu}$ and $\pi(v_{bc}) = v_{bc}^{\nu}$, respectively. One can see that, by the replacement $u' + \rho =: u$, this coincides with the structure constant $C_{abc,\rho}^{\mu\nu}$ of the product of the theta functions in eq.(2.3).

4. Noncommutative theta functions

The Moyal star product [16] is an associative noncommutative product on the space of functions on a flat space. It gives the first example of deformation quantization [1] and is also used as a building block of deformation quantization on arbitrary symplectic manifolds (see [18, 6]). A Moyal star product for functions on $\mathbb{C}^n$ is defined by

$$(f * g)(z) = f(z) e^{-\frac{\sqrt{-1}}{4\pi} \overleftarrow{\partial}_z \theta \overrightarrow{\partial}_z} g(z) \ ,$$

where $\overleftarrow{\partial}_z \theta \overrightarrow{\partial}_z := \sum_{i,j=1}^{n} \frac{\overleftarrow{\partial}}{\partial z^i} \theta^{ij} \frac{\overrightarrow{\partial}}{\partial z^j}$. Note that this skewsymmetric matrix $\theta \in \mathrm{Mat}_n(\mathbb{R})$ can be thought of as the restriction of $\theta = \{\theta_{ij}\}_{i,j=1,\cdots,2n}$ in eq.(3.1) to $\theta = \{\theta_{ij}\}_{i,j=1,\cdots,n}$. [2]

[2] This skewsymmetric matrix $\theta \in \mathrm{Mat}_n(\mathbb{R})$ corresponds to θ_1 in [11].

Now, for two symmetric matrices $A_a, A_b \in \mathrm{Mat}_n(\mathbb{Z})$ such that A_{ab} is nondegenerate, the following matrix $M_{ab} \in \mathrm{Mat}_n(\mathbb{C})$,

$$M_{ab} := \left(\mathbf{1}_n + \frac{\sqrt{-1}}{2}A^+_{ab}\theta\right)^{-1} A_{ab}\ , \qquad A^+_{ab} := A_a + A_b\ ,$$

is symmetric if and only if the the following condition holds:

$$A_a\theta A_a = A_b\theta A_b\ . \tag{4.1}$$

For two symmetric matrices $A_a, A_b \in \mathrm{Mat}_n(\mathbb{Z})$ satisfying the condition (4.1), the real part of M_{ab} is positive definite if and only if A_{ab} is positive definite (see [8], p.5). For two symmetric matrices $A_a, A_b \in \mathrm{Mat}_n(\mathbb{Z})$ such that A_{ab} is positive definite, define theta functions $\mathfrak{e}^\mu_{ab}$, $\mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n$, by

$$\mathfrak{e}^\mu_{ab}(z) = \frac{\det(\mathbf{1}_n + \sqrt{-1}A_a\theta)^{\frac14}\det(\mathbf{1}_n + \sqrt{-1}A_b\theta)^{\frac14}}{\det(A_{ab})^{\frac12}}\,\vartheta[0, -A_{ab}\mu](\sqrt{-1}M^{-1}_{ab}, z)\ . \tag{4.2}$$

It is clear that these theta functions actually coincide with those in eq.(2.2) if $\theta = 0$.

Then, we get the $*$ product formula of these noncommutative theta functions.

Theorem 4.1. *For a fixed skewsymmetric matrix $\theta \in \mathrm{Mat}_n(\mathbb{R})$, consider a set of symmetric matrices $A_a, A_b, A_c \in \mathrm{Mat}_n(\mathbb{Z})$ such that $A_a\theta A_a = A_b\theta A_b = A_c\theta A_c$ and $A_{ab}, A_{bc} \in \mathrm{Mat}_n(\mathbb{Z})$ are positive definite. Then, the following product formula holds:*

$$\left(\mathfrak{e}^\mu_{ab} * \mathfrak{e}^\nu_{bc}\right)(z) = \sum_{\rho\in\mathbb{Z}^n/A_{ac}\mathbb{Z}^n} C^{\mu\nu}_{abc,\rho}\,\mathfrak{e}^\rho_{ac}(z)\ ,$$

$$C^{\mu\nu}_{abc,\rho} := \sum_{u\in\mathbb{Z}^n}\delta_{[A_{ab}]}{}^{\mu}_{-u+\rho}\,\delta_{[A_{bc}]}{}^{\nu}_{u}\exp\left(-\pi(u - A_{bc}A^{-1}_{ac}\rho)^t\left((A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1} + \sqrt{-1}A_b\theta)^{-1}\right)(u - A_{bc}A^{-1}_{ac}\rho)\right)\ .$$

Note that the matrix $(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1} + \sqrt{-1}A_b\theta)^{-1} \in \mathrm{Mat}_n(\mathbb{C})$ is symmetric due to the condition (4.1).

Proof. Again, by the Poisson resummation formula, the theta functions $\{\mathfrak{e}^\mu_{ab}\}$ in eq.(4.2) can be rewritten as $\mathfrak{e}^\mu_{ab}(z) = T^\mu_{A_{ab}}(e_{ab})(z)$, where

$$e_{ab}(x) := C_{ab}\cdot e^{-\pi x^t M_{ab} x}\ , \qquad C_{ab} := \frac{\det(\mathbf{1}_n + \sqrt{-1}A_a\theta)^{\frac14}\det(\mathbf{1}_n + \sqrt{-1}A_b\theta)^{\frac14}}{\det(\mathbf{1}_n + \frac{\sqrt{-1}}{2}A^+_{ab}\theta)^{\frac12}} \in \mathbb{C}\ .$$

As in the commutative case in subsection 3.1, one can consider the corresponding Heisenberg modules with a constant curvature connection ∇, where the tensor product is given just by replacing the product $\cdot$ in the right hand side of eq.(2.4) by the star product, the constant curvature connection ∇ satisfies the Leibniz rule with respect to the tensor product, and the the theta vectors are obtained just as the function e_{ab} above [11]. The Leibniz rule of ∇ then guarantees that the tensor product $m(e^\mu_{ab}, e^\nu_{bc})$ is a linear combination of e^ρ_{ac}. The appropriate coefficients $C_{ab} \in \mathbb{C}$ and the structure constant $C^{\mu\nu}_{abc,\rho} \in \mathbb{C}$ are obtained by direct calculations. □

In the same way as in the commutative ($\theta = 0$) case, the product formula above leads to the following. Let $\mathrm{Ob} := \{a, b, \cdots\}$ be a finite collection of labels, where any $a \in \mathrm{Ob}$ is associated with a nondegenerate symmetric matrix $A_a \in \mathrm{Mat}_n(\mathbb{Z})$ such that for any $a, b \in \mathrm{Ob}$ the condition (4.1) holds and A_{ab} is nondegenerate if $a \neq b$. For any $a, b \in \mathrm{Ob}$, define a vector space $H^0(a, b)$ as follows:

- If A_{ab} is positive definite, $H^0(a,b)$ is the $\det(A_{ab})$-dimensional vector space spanned by the theta functions $\{\mathfrak{e}^{\mu}_{ab}\}$.
- If $a=b$, then $H^0(a,b) := \mathbb{C}$.
- If otherwise, then we set $H^0(a,b)=0$.

Then, the product formula in Theorem 4.1 defines an algebraic structure on $\oplus_{a,b\in \mathrm{Ob}} H^0(a,b)$. The condition (4.1) has an interpretation in a categorical setting of these structures (see [11]).

5. An example

We end with showing an example for the case of noncommutative complex two-torus ($n=2$). In this case, for any fixed nonzero θ, the condition $A_a\theta A_a = A_b\theta A_b$ reduces to

$$\det(A_a) = \det(A_b)\ .$$

In general there exist infinitely many symmetric matrices $A \in \mathrm{Mat}_2(\mathbb{Z})$ for a fixed $\det(A)$. For instance, let us consider symmetric matrices $A \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det(A) = -4$. If we concentrate on diagonal matrices $A \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det(A)=-4$, all such matrices are given by

$$A_{\mathbf{1}} = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}\ ,\quad A_{\mathbf{2}} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\ ,\quad A_{\mathbf{3}} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\ ,$$

and $A_{\mathbf{1}'} := -A_{\mathbf{1}}$, $A_{\mathbf{2}'} := -A_{\mathbf{2}}$, $A_{\mathbf{3}'} := -A_{\mathbf{3}}$. Since $H^0(i,j') = H^0(i',j) = 0$ for any $i,j = \mathbf{1},\mathbf{2},\mathbf{3}$, let us concentrate on the one side $\{\mathbf{1},\mathbf{2},\mathbf{3}\}$. Then, one obtains $H^0(i,j)\neq 0$ if and only if $i \le j$ and in particular

$$\dim(H^0(\mathbf{1},\mathbf{2})) = 2\ ,\qquad \dim(H^0(\mathbf{2},\mathbf{3})) = 2\ ,\qquad \dim(H^0(\mathbf{1},\mathbf{3})) = 9\ .$$

Thus, one obtains the following quiver:

$$\begin{tikzcd} \mathbf{1} \arrow[rr, "9"] \arrow[dr, "2"'] & & \mathbf{3} \\ & \mathbf{2} \arrow[ur, "2"'] & \end{tikzcd}\ .$$

However, if we allow symmetric matrices with nonzero off-diagonal elements, there exist infinitely many symmetric matrices $A \in \mathrm{Mat}_n(\mathbb{Z})$ with $\det(A) = -4$, since the matrix g^tAg has $\det(A) = -4$ for any $SL(2,\mathbb{Z})$ element g. For instance, for $g_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, $\alpha\in\mathbb{Z}$, one has $A_{\mathbf{1},\alpha} := g^t_\alpha A_{\mathbf{1}} g_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ \alpha & \alpha^2-4\end{smallmatrix}\right)$. Clearly, $A_{\mathbf{1},\alpha} \neq A_{\mathbf{1},\alpha}$ if $\alpha\neq\alpha'$. Similarly, $g^t_\alpha A_{\mathbf{2}} g_\alpha$ and $g^t_\alpha A_{\mathbf{3}} g_\alpha$ define new symmetric matrices for each $\alpha \in \mathbb{Z}$. Then, these infinitely many symmetric matrices together with the vector space $H^0(*,*)$ in fact define a connected quiver.

The fact that one can still consider a connected quiver of infinite type, as in the commutative case ($\theta = 0$), might imply that our approach gives an interesting model of noncommutative deformations in particular from a viewpoint of homological mirror symmetry.

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Yukawa Institute for Theoretical Physcics, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `kajiura@yukawa.kyoto-u.ac.jp`