SIMULATION OF THE IMPACT OF TWO-DIMENSIONAL ELASTIC DISKS [1]

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9th. Nisshin Engineering Particle Technology International

Symposium at Kyoto, January 9, 2001

[1] The paper is accesible at cond-mat/0011294 and cond-mat/0011375.

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Contents

- Introduction - What is the problem on impacts? -
- Quasi-Static Theory - What can we deduce from Hertzian contact theory? -
- The model of our simulation
  - Model A - Model consist of particles and springs -
  - Model B - Mode analysis of an elastic disk (Gerl and Zippelius) -
  - Parameters in our simulation
- Results
  - Simulation at $T = 0$ - (1)COR decreas with the impact velocity. (2) Our result is not consistent with quasi-static theory-
  - Simulation at finite T - Abrupt drop of COR (Plastic deformation) -
- Discussion - What is the actual effect of temperature? -
- Conclusion

1 Introduction

- The inelastic collision of particles : $\leftarrow$ the internal degrees of freedom
  - a subject of high school physics
  - micromechanisim of granular dynamics
- Interaction
  - Hertzian contact mechanics : static interaction between granular (elastic) particles
  - Dynamic Interaction : Unknown or not confirmed
- Charcteristic Quantity
- The coefficient of restitution (COR)

$$e = -v_r/v_i, \tag{1}$$

  where v_i and v_r are the relative velocities of incoming and outgoing particles respectively.

- COR e had been believed to be a material constant

  $\rightarrow$

  Not constant. It depends on the impact velocity.

- The origin of the dissipation

  $\leftarrow$ The transfer of the kinetic energy of the center of mass into the internal degrees of freedom

  - Quasi-static theory by Kuwabara and Kono(87), Morgado and Oppenheim (97), Brilliantov et al (96).
  - Microscopic simulation by Gerl and Zippelius (99) for a Hamilton system
  - The relation between two approaches is not clear.

2 Quasi-static theory

Quasi-static theory predicts

$$\ddot{h} = -kh^{3/2} - \gamma\sqrt{h}\dot{h} \tag{2}$$

in a collision of two spheres. where $h = R_1 + R_2 - |\mathbf{r}_1 - \mathbf{r}_2|$ with the radius $R_i (i = 1, 2)$.

- Method of projection operator by Morgado and Oppenheim (97)
  - They introduce the thermal deformation.
  - They assume quick relaxation of the temperature field. (Uniform T)
- The simplest derivation by Brilliantov *et al.* (96).
  - Assumption on the stress tensor

    $$\sigma_{ij} = \sigma^{(el)}{}_{ij} + \sigma^{(vis)}{}_{ij}$$

  - The first term $\rightarrow$ the static stress tensor
  - The second term $\rightarrow$ the viscous stress tensor.

    They assume that $\sigma^{(vis)}{}_{ij}$ has the same form as that for $\sigma^{(el)}{}_{ij}$ replacing the deformation field by the local velocity.

– 2D Hertzian contact law

$$h \simeq -\frac{F_{el}}{\pi Y}\{\ln\left(\frac{4\pi Y R}{F_{el}\,(1-\sigma^2)}\right) - 1\}, \tag{3}$$

F_{el} : elastic force, Y : Young modulus, σ Poisson's ratio and R : the radius of the disk without deformation

– Brilliantov *el al.*(96) assumed

$$\dot{u}_i \simeq \dot{h}(\partial u_i/\partial h)$$

With $F_{el} \simeq -\pi Y h/\ln(4R/h) \to$ 2D quasi-static theory as

$$F_{tot} \simeq -\frac{\pi Y h}{\ln(4R/h)} - A\frac{\pi Y \dot{h}}{\ln(4R/h)}, \tag{4}$$

3 Our models

Wall exists at $y = 0$, and the center of mass keeps the position at $x = 0$. The disk approaches from $y > 0$ region and is rebounded by the wall.

3.1 Model A

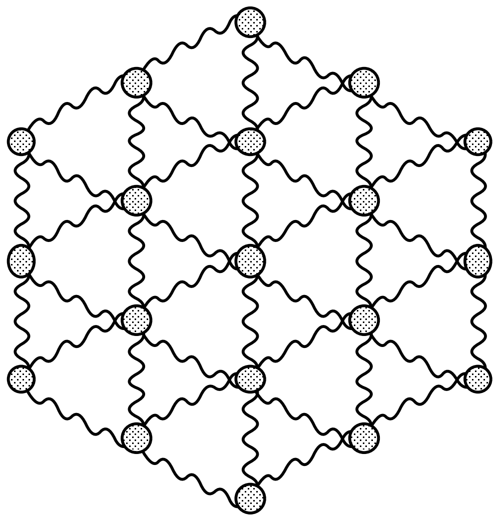

Figure 1: A schematic figure of a disk used in model A.

- The disk in model A: mass points (with the mass m) & linear springs κ.
- Continuum limit: $Y = 2\kappa/\sqrt{3}$ and $\sigma = 1/3$.

- Equation of motion

$$m\frac{d^2\mathbf{r}_p}{dt^2} = -\kappa\sum_{i=1}^{6}(d_0 - |\mathbf{r}_p - \mathbf{r}_i|)\frac{\mathbf{r}_p - \mathbf{r}_i}{|\mathbf{r}_p - \mathbf{r}_i|} + \hat{y}aV_0e^{-ay_p} \quad (5)$$

- d_0 : the lattice constant, $\mathbf{r}_i$: the position of the nearest neighbor mass points of $\mathbf{r}_p$, y_p is the y coordinate of $\mathbf{r}_p$, and $\hat{y}$ is the unit vector of y direction.

- The wall potential:

$$V_0e^{-ay}$$

$V_0 = a/2$ and $a = 100/d_0$. The number of the mass points is fixed to 1459

3.2 Model B

- Model B = model by Gerl and Zippelius (99).

- They analyze Hamilton's equation ;

$$\dot{P}_{n,l} = -\frac{\partial H}{\partial Q_{n,l}}; \quad \dot{Q}_{n,l} = \frac{\partial H}{\partial P_{n,l}} \quad (6)$$

$$H = \frac{p_0^2}{2M} + \sum_{n,l}^{N}\left(\frac{P_{n,l}^2}{2M} + \frac{1}{2}M\omega_{n,l}^2Q_{n,l}^2\right) + V_0\int_{-\pi/2}^{\pi/2} d\phi e^{-ay(\phi,t)}. \quad (7)$$

- $Q_{n,l}$: the expansion coefficient of $\mathbf{u} = (u_r, u_\phi)$

$$(u_r(r,\phi), u_\phi(r,\phi)) = \sum_{n,l} Q_{n,l}(u_r^{n,l}(r)\cos n\phi, u_\phi^{n,l}(r)\sin n\phi), \tag{8}$$

u_r and u_ϕ can be determined by the wave equations.

- $k'_{n,l} = k_{n,l}\sqrt{2(1+\sigma)/(1-\sigma^2)}$ and $k_{n,l}$ is the solution of an eigenvalue equation. $\Leftarrow$ force free boundary conditions

$$\sigma_{rr}(R,\phi) = 0, \quad \sigma_{r\phi}(R,\phi) = 0$$

- For fixed n: Many solutions exist. $\omega_{n,l} = k_{n,l}\sqrt{Y/\{\rho(1-\sigma^2)\}}$ numbered by $l = 0, 1, \cdots, \infty$.

- $P_{n,l}$: the canonical momentum.

- $y(\phi, t)$: the shape of the elastic disk

$$y(\phi,t) = y_0(t) + \sum_{n,l} Q_{n,l}(C_{n,l}\cos(n\phi)\cos\phi - S_{n,l}\sin(n\phi)\sin\phi) \tag{9}$$

- $y_0(t)$ the position of the center of mass

- $C_{n,l}$ and $S_{n,l}$ determined by the maximal radial and tangential displacement at the edge of the disk as $C_{n,l} =$

$u_r^{n,l}(R)$ and $S_{n,l} = u_\phi^{n,l}(R)$.[3]

- M : the mass of the disk,
- $\dot{p}_0 = -(\partial H/\partial y_0)$
- System size: $N = 1189$ ($n \leq 50$ and $\kappa_n \leq 50$)or $N = 437$ ($n \leq 30$ and $\kappa_n \leq 30$)
- $V_0 = a/2$ and $a = 500/R$ with the radius of the disk R.

3.3 Parameters in both models

- Poisson's ratio $\sigma = 1/3$.
- The numerical scheme
  - model A : fourth order Runge-Kutta method & $\Delta t = 1.6 \times 10^{-3}\sqrt{m/\kappa}$.
  - model B: symplectic integrator & $\Delta t = 5.0 \times 10^{-3} R/c$ with $c = \sqrt{Y/\rho}$
- Temperature.

[3] $u_r^{n,l}(r)$ and $u_\phi^{n,l}$ are the radial dimensionless deformation fields.

- model A: Maxwellian (velocity) and equilbrium positions
- model B: Gibbs state for harmonic oscillations $\leftarrow$ Normal random number

- Differences between two models

1. Model A: all of the mass points interact with the wall. Model B: exterior boundary has the influence of the potential.
2. Model A can have nonlinear deformations. Model B is based on the theory of linear elasticity.
3. Model A can express some plastic deformations, but model B cannot.
4. Model A has the six folds symmetry but model B has the rotational symmetry.

4 Results

4.1 Simulation at $T = 0$: Macroscopic disks

- COR decreases with the impact velocity.
- Gap between the results of two models for low v_i
- Force acting on the center of the mass
  - The hysteresis curve above $v_i = 0.01c$ ($c = \sqrt{Y/\rho}$)
  - The hysteresis loop disappears at $v_i = 0.001c \Rightarrow$ Deviation from quasi-static theory
  - The turning point at $\dot{F} = 0$ is deviated from the Hertzian curve. $\rightarrow$ No hope to recover quasi-static theory $\Leftarrow$ No thermal deformation in our model
- Contact time can be evaluated by the theory of elasticity

$$\tau \simeq \frac{\pi R}{c}\sqrt{\ln(4c/v_i)}$$

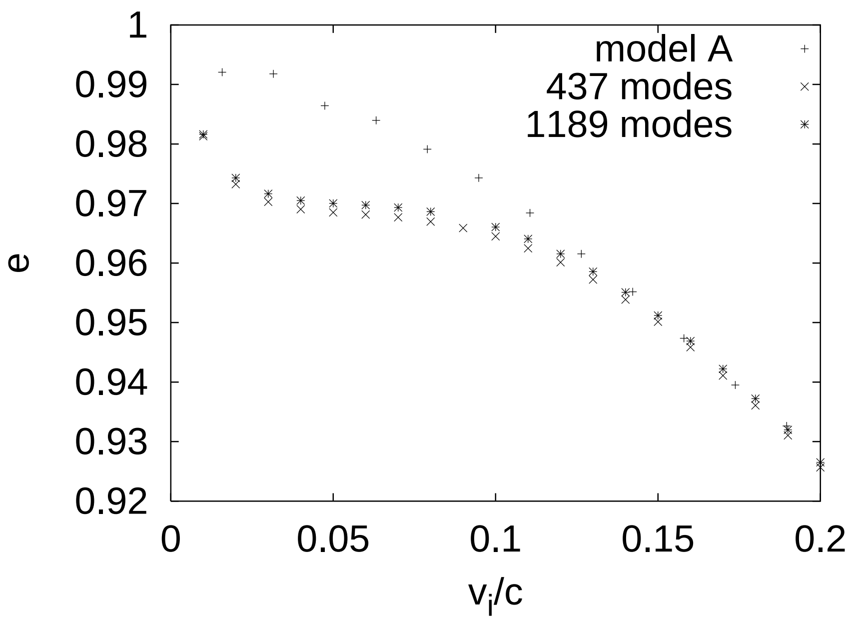

Figure 2: Coefficient of restitution for normal collision of the Model A and Model B as a function of impact velocity.

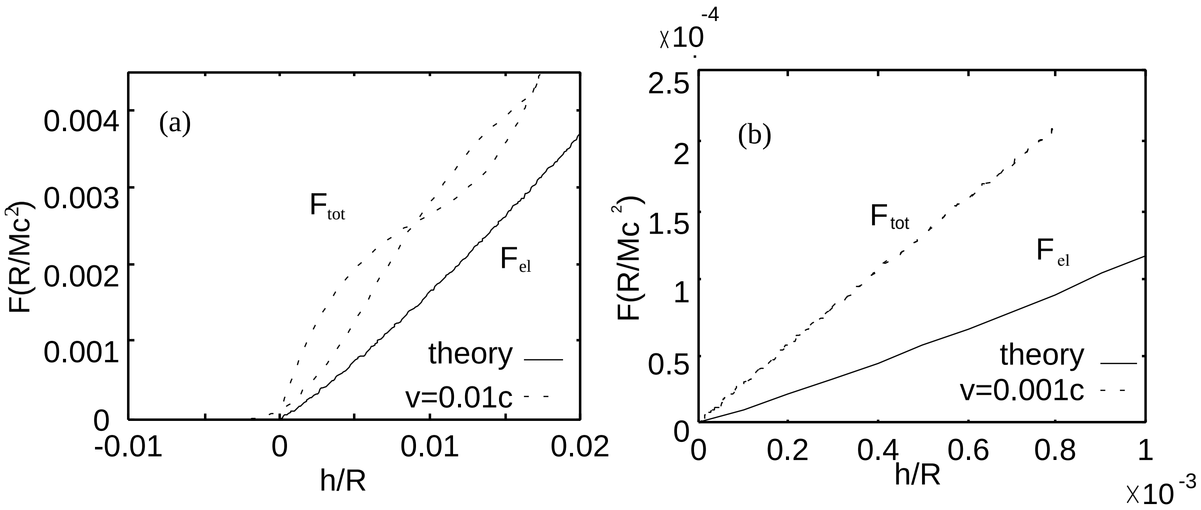

Figure 3: The comparison of the Hertzian force in eq.(3) with our simulation at $v_i = 0.01c$ (a) and $v_i = 0.001c$(b) at $T = 0$ in model B.

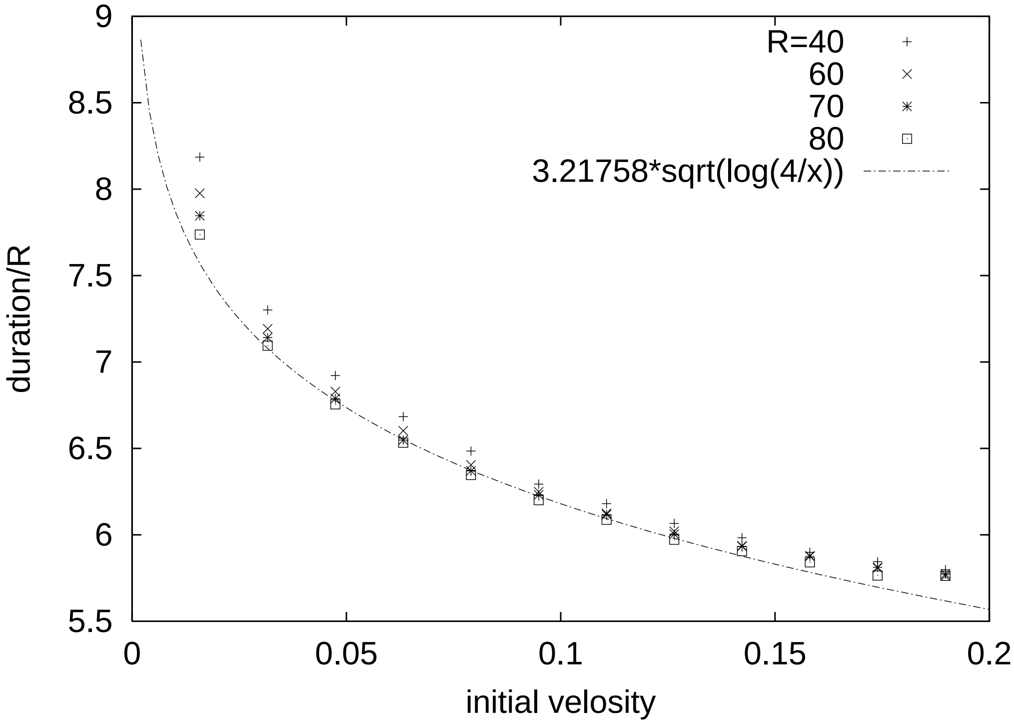

Figure 4: The plot of contact time versus the impact velocity. R represents the radius of the disk, in which $R = 40$,60,70 and 80 correspond to the number of mass points 5815, 13057, 17761 and 23233, respectively. The dotted line is fitting curve based on the quasi-static theory.

4.2 Simulation at finite T: Mesoscopic disks

- For small v_i: $\Rightarrow$ See Discussion.
- For large v_i :
  - Model B : No effects
  - Model A : Abrupt drop of COR
    - Drop of COR may be universal. (Fig.5)
    - $v_{cr}/c \simeq aT/T_0 + b$ ($T_0 = mc^2/k_B$)
    - Plastic deformation
    - Plastic deformation predicts $e \sim v_i^{-1/3}$ but this study shows $e \sim v_i^{-1.2}$

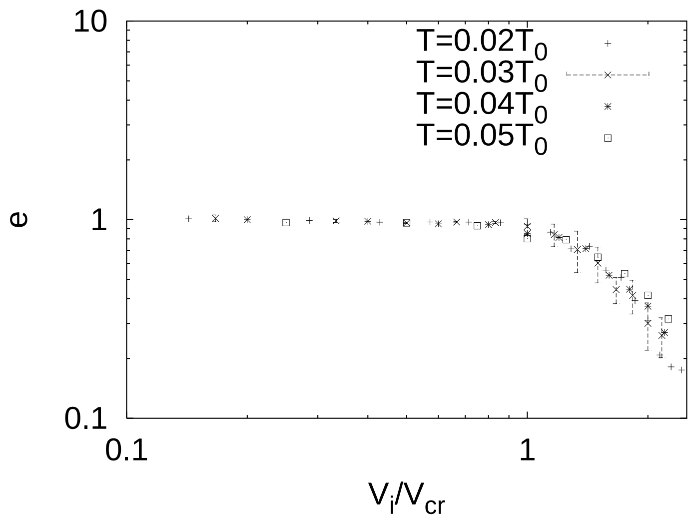

Figure 5: The relation the coefficient of restitution and the impact velocity rescaled by the critical velocity for each temperature. Curves are plotted in the log-log scale. The temperature is scaled by $T_0 = mc^2/k_B$ with the mass of the mass points m and Boltzmann constant k_B. Note that the error bars are plotted only in the case $T/T_0 = 0.03$ but is the same order even at other T.

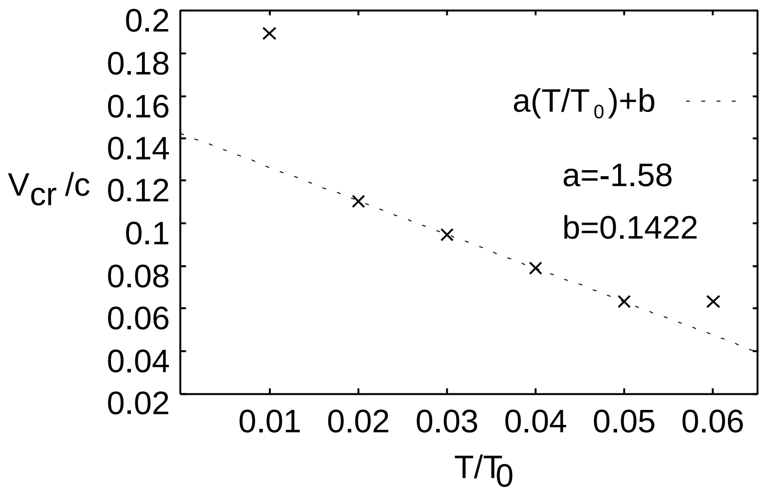

Figure 6: The plot of the initial temperature and the critical velocity causing the plastic deformation. $v_{cr}/c = a(T/T_0) + b$ is the fitting curve line from the data between $T/T_0 = 0.02$ and 0.05.

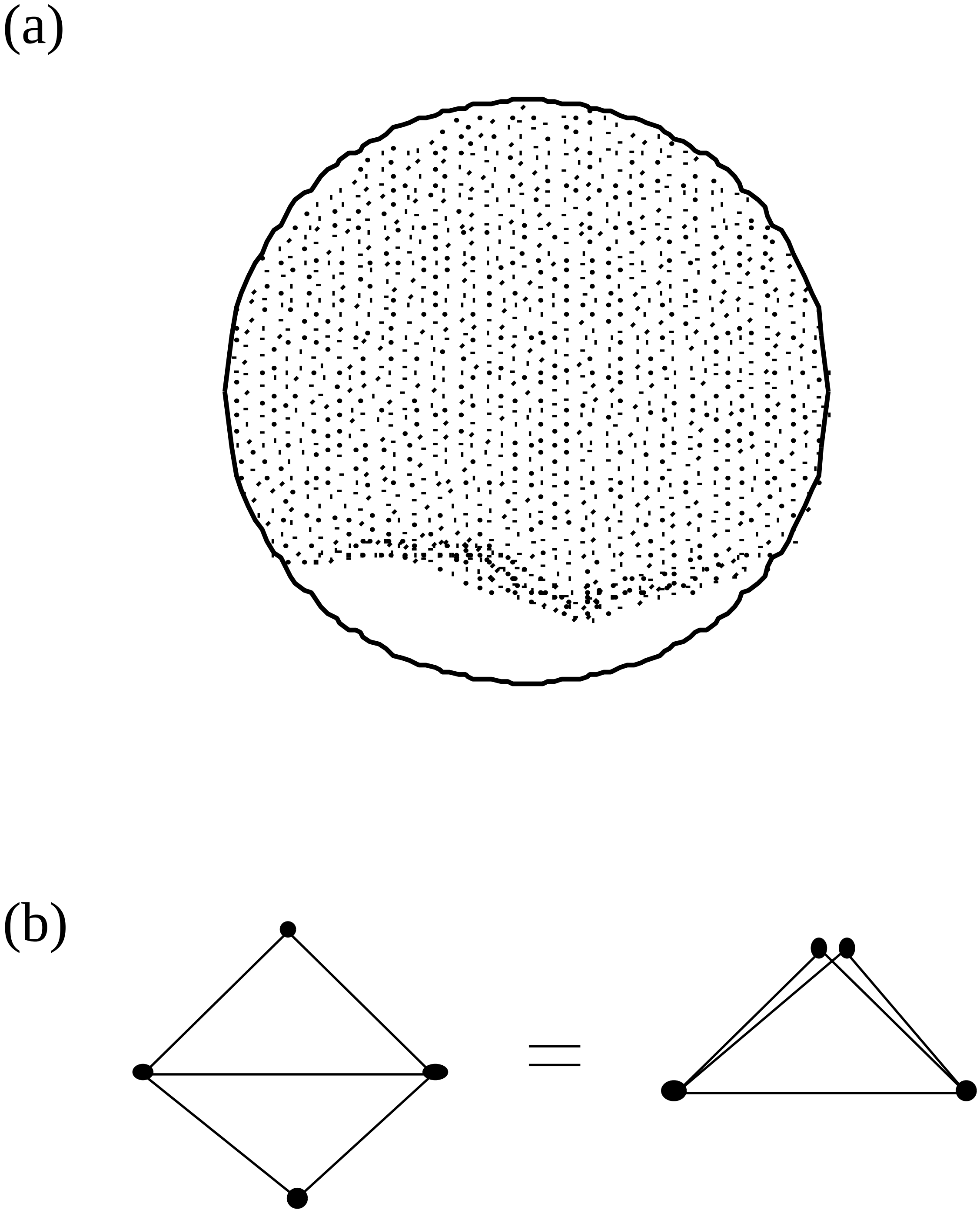

Figure 7: (a) Plastic deformation of model A with $v_i = 0.22$ at $T = 0.03$. The solid circle represents the initial circle. The cross points are positions of the mass points after the collision. (b) Two configurations are energetically equivalent.

5 discussion

- Let us discuss the effect of temperature of COR for small v_i, where the thermal velocity $v_{th} = \sqrt{k_B T/M}$ is not negligible.
- At low v_i and fixed T, the fluctuation of COR becomes very large. $\leftarrow v_{th}/v_i$ becomes large. However, the average of COR does not have any systematic change.
- At fixed v_i and change of T: The average of COR decreases with T but the flutuation increases with T.
- The distribution of COR is currently unknown.
- $\Rightarrow$ We need systematic investigation for low v_i.
- We also need a 'thermal' deformed model.

6 Conclusion

6.1 Summary of our results

1. We have numerically studied the impact of a two dimensional elastic disk. with
2. The coefficient of restitution (COR) decreases with the impact velocity.
3. The result of our simulation is not consistent with the result of the two-dimensional quasi-static theory.
   - For large impact velocity, there is hysteresis in the deformation of the center of mass.
   - For small velocity, there remains the inelastic force even at $\dot{h} = 0$.
4. There are drastic effects of temperature in both small and large impact velocity.
5. In particular, for large impact velocity of model A, we have found the abrupt drop of COR above the critical impact velocity by the plastic deformation.

6. The critical velocity obey a simple linear function of temperature.

6.2 Perspective

- Rigorious application of thermodynamic argument to evaluate COR (Tasaki, Sasa).
- Study of three dimensional impact (Coming soon)
- Consideration of the degree of freedom of the wall.
- Tangential effects including contact friction