Hydrodynamic fluctuations and its kinetic descriptions

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Advances in Fluctuating Hydrodynamics:
Bridging the Micro and Macro Scales

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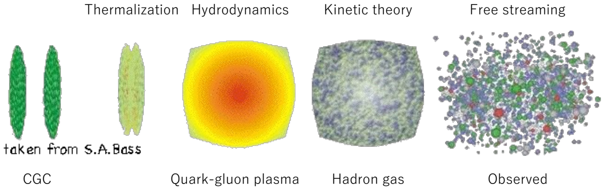
Contents

Contents

Relativistic heavy-ion collisions

Key achievements

- Formation of Quark-Gluon Plasma
- Nearly perfect fluid $\eta/s \sim 1/4\pi \rightarrow$ Strongly coupled plasma



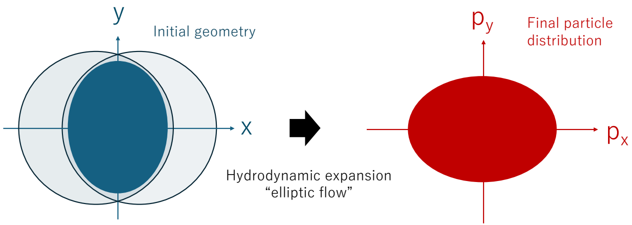
Challenges

- Thermalization and hydrodynamization
- Hydrodynamics with $N_{\text{particle}} \sim 10^{3-4} \rightarrow$ Fluctuating hydrodynamics?

How to measure hydrodynamic properties?

Response to initial transverse geometry

- Almond shape on average $\rightarrow$ elliptic flow "v_2"
- Elliptic flow data fitted well by $\eta/s \sim 1/4\pi$

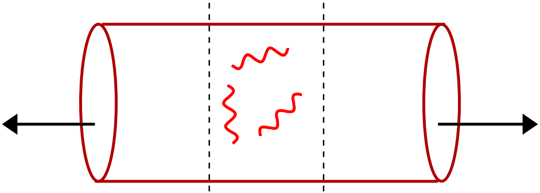


- Event-by-event shape $\rightarrow$ higher harmonics "v_n"

Bjorken expansion

Longitudinal expansion along the beam direction

- Equilibration in a Bjorken expansion is still a challenging problem



I assume local equilibration and discuss
How hydrodynamic fluctuations evolve on a Bjorken background?

Contents

Relativistic fluctuating hydrodynamics

- Conservation law

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}}(e, v) + T^{\mu\nu}_{\text{viscous}}(e, v, \partial) + T^{\mu\nu}_{\text{noise}}$$

- Assume conformal fluid $T^\mu_\mu = -e + 3p = 0$

sound velocity $c_s^2 = dp/de = 1/3$, bulk viscosity $\zeta = 0$

- Linearized hydrodynamic fluctuations $\phi = (c_s \delta e, \boldsymbol{g} = w\boldsymbol{v})^T$

$$\frac{\partial}{\partial t}\phi_a(t, \boldsymbol{k}) = \underbrace{-i\mathcal{L}_{ab}\phi_b}_{\text{ideal}} - \underbrace{\mathcal{D}_{ab}\phi_b}_{\text{viscous}} - \underbrace{\xi_a}_{\text{noise}},$$

$$\mathcal{L} = \begin{pmatrix} 0 & c_s\boldsymbol{k} \\ c_s\boldsymbol{k} & 0 \end{pmatrix}, \quad \mathcal{D} = \gamma_\eta \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & k^2\delta_{ij} + \frac{1}{3}k_ik_j \end{pmatrix},$$

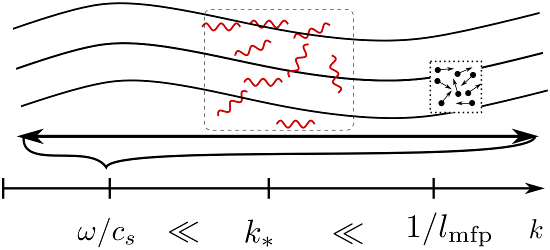
$$\langle \xi_a(t, \boldsymbol{k})\xi_b(t', -\boldsymbol{k}')\rangle = 2\,Tw\mathcal{D}_{ab}(2\pi)^3\delta(\boldsymbol{k} - \boldsymbol{k}')\delta(t - t')$$

Kinetic regime for a non-static and non-uniform background

Relaxation on a background flow of time scale ω^{-1}

$$\underbrace{\gamma_\eta k_*^2 = \omega}_{\text{balance}} \quad \to \quad \underbrace{\frac{\omega}{c_s} \ll k_* = \sqrt{\frac{\omega}{\gamma_\eta}} \ll \frac{c_s}{\gamma_\eta}}_{\text{short wavelength}} \quad \because \underbrace{\epsilon \equiv \frac{\gamma_\eta \omega}{c_s^2} \ll 1}_{\text{gradient expansion}}$$

- Kinetic regime k_*: non-equilibrium scale, wavepacket on a background



Counting on a background

1. Expand up to linear fluctuations ϕ on background Φ

$$\partial_\mu T^{\mu\nu}_{\rm id} = \partial_\mu \left[(e+p)u^\mu u^\nu + p g^{\mu\nu}\right] \sim \partial \left[\Phi^{(0)}_{\rm id} + \Phi^{(1)}_{\rm id}\phi + \cdots\right]$$

$$\partial_\mu T^{\mu\nu}_{\rm vis} = \partial_\mu \left[\eta \partial u + \cdots\right] \sim \partial \left[\partial\Phi^{(0)}_{\rm vis} + \Phi^{(1)}_{\rm vis}\partial\phi + \partial\Phi^{(1)}_{\rm vis}\cdot\phi + \cdots\right]$$

2. Background solution

$$\underbrace{\partial\Phi^{(0)}_{\rm id}}_{\omega\sim 1} + \underbrace{\partial^2\Phi^{(0)}_{\rm vis}}_{\gamma_\eta\omega^2/c_s^2\sim\epsilon} = 0$$

3. Langevin equation for fluctuation

$$\underbrace{\Phi^{(1)}_{\rm id}\partial\phi}_{c_s k_*\sim 1} + \underbrace{\phi\partial\Phi^{(1)}_{\rm id}}_{\omega\sim\epsilon^{1/2}} + \underbrace{\Phi^{(1)}_{\rm vis}\partial^2\phi}_{\gamma_\eta k_*^2\sim\epsilon^{1/2}} + \underbrace{\cancel{\partial\Phi^{(1)}_{\rm vis}\cdot\partial\phi}}_{\gamma_\eta\omega k_*/c_s\sim\epsilon} + \underbrace{\cancel{\partial^2\Phi^{(1)}_{\rm vis}\cdot\phi}}_{\gamma_\eta\omega^2/c_s^2\sim\epsilon^{3/2}} + \text{noise} = 0$$

4. Gapped modes do not mix at time scales ω^{-1}

$$\underbrace{e^{ic_s k_*/\omega} \sim e^{i\epsilon^{-1/2}} \sim 0}_{\text{rotating wave approximation}}$$

Equal-time two-point functions

Wigner function

$$N_{ab}(t,\boldsymbol{x},\boldsymbol{k}) \equiv \int d^3r e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \langle \phi_a(t,\boldsymbol{x}+\boldsymbol{r}/2)\phi_b(t,\boldsymbol{x}-\boldsymbol{r}/2)\rangle$$

Special cases: we can take a uniform comoving frame where $\boldsymbol{v}=0$

- Bjorken expansion

$$g_{\mu\nu} = \mathrm{diag}(-1,1,1,\tau^2)$$

- weak metric perturbation

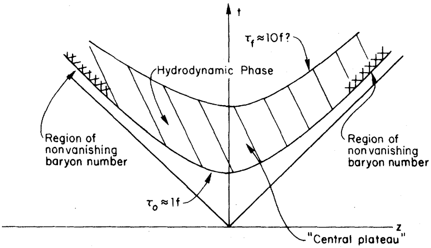
$$g_{\mu\nu} = \mathrm{diag}(-1,1+h(t),1+h(t),1-2h(t))$$

Uniform comoving frame: $N_{ab}(t,\boldsymbol{x},\boldsymbol{k}) \to N_{ab}(t,\boldsymbol{k})$

Contents

Milne coordinates for Bjorken expansion

Approximate boost invariance along the beam in ultrarelativistic collisions



[Bjorken 1983]

- Coordinate transformation

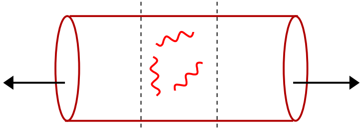
$$\tau = \sqrt{t^2 - z^2}, \quad \eta = \ln \frac{t+z}{t-z}, \quad ds^2 = -d\tau^2 + dx_\perp^2 + \tau^2 d\eta^2$$

- Fourier transformation

$$\phi(\tau, \boldsymbol{x}_\perp, \eta) = \int d^2 k_\perp d\kappa \, e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + i\kappa\eta} \phi(\tau, \boldsymbol{k}_\perp, \kappa)$$

$$\boldsymbol{k} = (\boldsymbol{k}_\perp, \kappa), \quad \boldsymbol{K}(\tau) = (\boldsymbol{k}_\perp, k_z(\tau) \equiv \kappa/\tau)$$

Hydrodynamic fluctuations on a Bjorken expansion



1. Background ideal fluid (viscous corrections are subleading for ϕ)

$$\underbrace{u_\eta(\tau) = 0 \leftrightarrow v_z(t, \boldsymbol{x}) = z/t}_{\text{1-dim. Hubble expansion}}, \quad \underbrace{s(\tau)\tau = s(\tau_0)\tau_0}_{\text{entropy conservation}}$$

$$\rightarrow \quad \text{“}\omega\text{”} \sim \frac{1}{\tau}, \quad k_* = \sqrt{\frac{1}{\gamma_\eta(\tau)\tau}}$$

2. Langevin equation for fluctuations $\phi = (c_s\delta e, \boldsymbol{G}) = (c_s\delta e, \boldsymbol{g}_\perp, \tau g^\eta)$

$$\frac{\partial}{\partial\tau}\phi_a(\tau, \boldsymbol{k}) = \underbrace{-i\mathcal{L}_{ab}\phi_b}_{\text{ideal}} - \underbrace{\mathcal{D}_{ab}\phi_b}_{\text{viscous}} - \underbrace{\mathcal{P}_{ab}\phi_b}_{\text{expansion}} - \underbrace{\xi_a}_{\text{noise}},$$

$$\mathcal{P} = \frac{1}{\tau}\text{diag}(1 + c_s^2, 1, 1, 2)$$

Kinetic equations for the two-point functions

1. Two-point function

$$\langle \phi_a(\tau, \boldsymbol{k}) \phi_b(\tau, -\boldsymbol{k}') \rangle = N_{ab}(\tau, \boldsymbol{k})(2\pi)^3 \delta(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp)\delta(\kappa - \kappa')$$

2. Evolution equation

$$\frac{\partial}{\partial \tau} N(\tau, \boldsymbol{k}) = -i[\mathcal{L}, N] - \{\mathcal{D}, N\} - \{\mathcal{P}, N\} + \frac{2Tw}{\tau}\mathcal{D}$$

3. Eigen modes of $\mathcal{L}$: $\phi_\alpha = (\phi_+, \phi_-, \phi_{T_1}, \phi_{T_2})$

$$\underbrace{\text{left moving sound}}_{\lambda_- = -c_s|\boldsymbol{K}|}, \quad \underbrace{\text{right moving sound}}_{\lambda_+ = c_s|\boldsymbol{K}|}, \quad \underbrace{\text{transverse modes}}_{\lambda_T = 0}$$

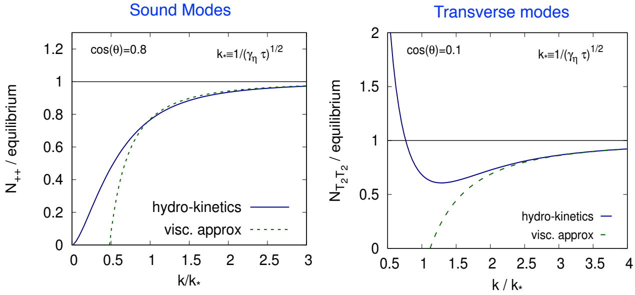
4. Rotating wave approximation with eigenmodes

$$\frac{\partial}{\partial \tau} N_{\alpha\alpha}(\tau, \boldsymbol{k}) = \underbrace{-2\mathcal{D}_{\alpha\alpha}\left[N_{\alpha\alpha} - \frac{Tw}{\tau}\right]}_{\text{relaxation} \sim \gamma_\eta k_*^2} - \underbrace{2\mathcal{P}_{\alpha\alpha}N_{\alpha\alpha}}_{\text{expansion} \sim 1/\tau}$$

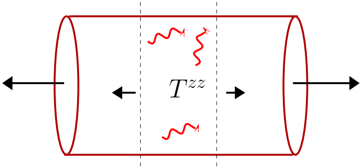
Large wavenumber asymptotics

$$\frac{\partial}{\partial \tau} N_{\pm\pm}(\tau, \boldsymbol{k}) = -\frac{4}{3}\gamma_\eta K^2 \left[N_{\pm\pm} - \frac{T(e+p)}{\tau} \right] - \frac{1}{\tau}(2 + c_s^2 + \cos^2\theta_K) N_{\pm\pm}$$

$$N_{\pm\pm}(\tau, \boldsymbol{k}) = \frac{Tw}{\tau}\Big[\underbrace{1 + \frac{c_s^2 - \cos^2\theta_K}{\frac{4}{3}\gamma_\eta K^2 \tau}}_{\text{visc. approx}} + \underbrace{\cdots}_{\text{time integral}} \Big] \quad \text{for } K \gg k_*$$



Evolution of the background



Hydrodynamic equation in Bjorken expansion

$$\frac{d}{d\tau}\langle \tau\, T^{\tau\tau}\rangle = -\langle T^{zz}\rangle$$

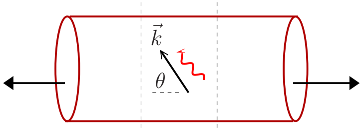
- Without hydrodynamic fluctuations

$$T^{zz} = \underbrace{p}_{\text{ideal}} - \underbrace{\frac{4\eta}{3\tau}}_{\text{1st visc.}} + \underbrace{(\lambda_1 - \eta\tau_\pi)\frac{8}{9\tau^2}}_{\text{2nd visc.}}$$

- Hydrodynamic fluctuations give another contribution

$$\langle T^{zz}_{\text{fluct}}(\tau)\rangle = \frac{\langle G_z^2\rangle}{w}$$

Nonlinear contribution from $K \sim k_*$ to the background



Nonlinear contribution to T^{zz}

$$\langle G_z^2 \rangle = \tau \int_K \underbrace{\frac{N_{++} + N_{--}}{2} \cos^2\theta_K + N_{T_2 T_2} \sin^2\theta_K}_{\propto 1 + \#/\gamma_\eta K^2 \tau + \cdots \text{ for } K \gg k_*}$$

- Regularize cubic and linear UV divergences by a cutoff Λ

Renormalize the divergences [Kovtun-Yaffe (03), Kovtun-Moore-Romatschke (11)]

$$\langle T^{zz} \rangle = \underbrace{p(\Lambda) + \frac{\Lambda^3 T}{6\pi^2}}_{\equiv p} - \frac{4}{3\tau} \underbrace{\left[\eta(\Lambda) + \frac{17 \Lambda T}{120\pi^2} \frac{1}{\gamma_\eta(\Lambda)} \right]}_{\equiv \eta} + \text{finite} + \cdots$$

Finite contributions: Long-time tails

Evaluate the finite parts after renormalization

$$\langle T^{zz} \rangle = \underbrace{p}_{\text{ideal}} - \underbrace{\frac{4\eta}{3\tau}}_{\text{1st visc.}} + \underbrace{1.08318 \;\; T \left(\frac{1}{4\pi\gamma_\eta\tau}\right)^{3/2}}_{\text{long-time tail}} + \cdots$$

Simple understanding of the scaling

$$\langle T^{zz}_{\rm fluct} \rangle \sim T \underbrace{\int d^3K}_{\#\text{ of modes}} \sim T k_*^3 \sim T\left(\frac{1}{\gamma_\eta \tau}\right)^{3/2}$$

Order counting

$$\frac{\langle T^{zz}(\tau) \rangle}{w} = \underbrace{\frac{p}{w}}_{\sim 1} - \underbrace{\frac{4\gamma_\eta}{3\tau}}_{\sim \epsilon} + \underbrace{\frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}}}_{\sim 1/s\ell_*^3 = 1/N_*},$$

The finite contribution from k_* gives the long-time tails

Contents

Effect of the initial conditions

Longitudinal pressure: initial time can be taken to 0

$$\frac{\langle T^{zz}(\tau)\rangle}{w} = \frac{p}{w} - \frac{4\gamma_\eta}{3\tau} + \frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}},$$

Transverse pressure: initial time should be finite (regulator)

$$\frac{\langle T^{xx,yy}(\tau)\rangle}{w} = \frac{p}{w} + \frac{2\gamma_\eta}{3\tau} + \underbrace{\left[\frac{\chi_{\tau_0} + \delta\chi_{\tau_0}}{\tau^2 w^2}\right]\frac{1}{(12\pi\gamma_\eta\tau)}}_{\text{what's this?}} - \frac{0.273836}{s(4\pi\gamma_\eta\tau)^{3/2}}$$

Time dependence of each term

$$\frac{p}{w} = \frac{1}{4}, \quad \frac{\gamma_\eta}{\tau} \propto \frac{1}{\tau^{2/3}}, \quad \underbrace{\frac{\delta\chi_{\tau_0}}{\gamma_\eta w^2 \tau^3} \propto \frac{1}{\tau}\left(\frac{\tau}{\tau_0}\right)^{1/3}}_{\text{initial sensitivity}}, \quad \frac{1}{s(\gamma_\eta\tau)^{3/2}} \propto \frac{1}{\tau}$$

Initial transverse momentum fluctuations

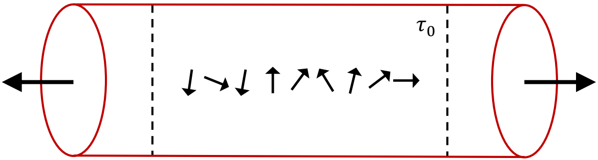
$N_{T_1 T_1}$ at initial moment gives divergent contribution in $\tau_0 \to 0$

- Particular kinematic region contributes ($k_*(\tau) \propto \tau^{-2/3}$)

$$\underbrace{k_\perp \sim k_*(\tau)}_{\text{kinetic regime at } \tau} \ll k_*(\tau_0), \qquad \underbrace{\kappa \sim k_*(\tau_0)\tau_0}_{\text{kinetic regime at } \tau_0} \ll k_*(\tau)\tau$$

  Almost uniform in $\boldsymbol{x}_\perp$ at initial times, and in η at later times

- Initial transverse momentum distribution



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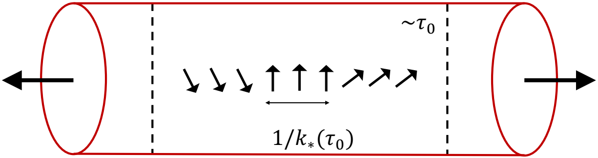
  Almost uniform in $\boldsymbol{x}_\perp$ at initial times, and in η at later times

- Diffusion in z direction at the initial moment



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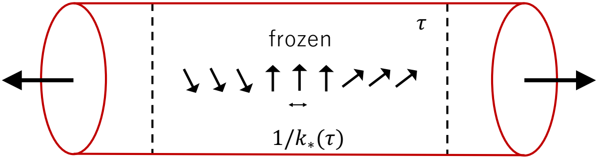
Almost uniform in $\boldsymbol{x}_\perp$ at initial times, and in η at later times

- Diffusion in z direction at later time is ineffective



Initial time effects on transverse pressure

1. Initial momentum fluctuations of modes $\boldsymbol{k} \sim (k_*(\tau), k_*(\tau_0))$

$$\langle (\boldsymbol{g}_\perp(\tau_0, \boldsymbol{x}))^2 \rangle \sim T(\tau_0) w(\tau_0) \underbrace{k_*(\tau)^2 k_*(\tau_0)}_{\#\text{ of modes}}$$

2. Diluted momentum fluctuations at τ

$$\langle (\boldsymbol{g}_\perp(\tau, \boldsymbol{x}))^2 \rangle \sim \left(\frac{\tau_0}{\tau}\right)^2 \langle (\boldsymbol{g}_\perp(\tau_0, \boldsymbol{x}))^2 \rangle$$

3. Transverse pressure

$$\langle T^{xx}(\tau) \rangle = \frac{1}{2} \frac{\langle (\boldsymbol{g}_\perp(\tau, \boldsymbol{x}))^2 \rangle}{w(\tau)} \sim \left(\frac{\tau_0}{\tau}\right)^2 \frac{w(\tau_0)}{w(\tau)} T(\tau_0) k_*(\tau)^2 k_*(\tau_0)$$

$$\sim \frac{k_*(\tau)^2}{\tau^2 w(\tau)} \underbrace{\tau_0^2 T(\tau_0) w(\tau_0) k_*(\tau_0)}_{= \delta\chi_{\tau_0}}$$

Contents

Energy density for non-conformal fluid

Using the hydro-kinetic equation, we get the energy density as

$$\langle T^{\tau\tau}\rangle = \underbrace{e(\Lambda) + \frac{T\Lambda^3}{2\pi^2}}_{\text{renormalize}} - \frac{T\Lambda}{6\pi^2\tau}\left[\left(1+\frac{3\,T}{2}\frac{dc_s^2}{dT} - 3\,c_s^2\right)\frac{1}{\gamma_\zeta} + 4\left(1-3c_s^2\right)\frac{1}{2\gamma_\eta}\right] \underbrace{+\mathcal{O}(\Lambda^0)}_{\text{long-time tail}}\,,$$

- Λ^3 term can be renormalized into energy density
- How to renormalize Λ/τ term?
- What is background temperature?

Cutoff dependent temperature

Landau condition $T^{\mu\nu}u_\nu = -eu^\mu$

- On a rest frame, energy density and temperature are related by

$$e(T) = \langle T^{\tau\tau}\rangle = \underbrace{\langle T^{\tau\tau}\rangle_{k>\Lambda}}_{= e(T(\Lambda),\Lambda)} + \underbrace{\langle T^{\tau\tau}\rangle_{k<\Lambda}}_{\text{fluctuations}}$$

- $T(\Lambda)$ depends on Λ only when the system is out of equilibrium

$$e(T(\Lambda),\Lambda) = e(T+\Delta T(\Lambda),\Lambda) \simeq e(T,\Lambda) + \frac{de(T)}{dT}\Delta T(\Lambda)$$

- Renormalization and temperature shift

$$\langle T^{\tau\tau}\rangle = \underbrace{e(T,\Lambda) + \frac{T\Lambda^3}{2\pi^2}}_{\text{renormalize}} + \frac{de(T)}{dT}\Delta T(\Lambda)$$

$$\underbrace{-\frac{T\Lambda}{6\pi^2\tau}\left[\left(1+\frac{3T}{2}\frac{dc_s^2}{dT}-3c_s^2\right)\frac{1}{\gamma_\zeta} + 4\left(1-3c_s^2\right)\frac{1}{2\gamma_\eta}\right] + \cdots}_{\text{cancelled by temperature shift } \Delta T}$$

Renormalization of bulk viscosity

Temperature shift affects renormalization of bulk viscosity

- Order counting $1+\epsilon+1/N_*+\cancel{\epsilon/N_*} \to \zeta(T+\Delta T,\Lambda) \simeq \zeta(T,\Lambda)$

$$\frac{1}{4}\langle T^{xx}+T^{yy}-2T^{zz}\rangle = \frac{1}{\tau}\underbrace{[\eta(\Lambda)+\#\Lambda]}_{=\eta}+\cdots$$

$$\begin{aligned}\frac{1}{3}\langle T^{xx}+T^{yy}+T^{zz}\rangle &= p(T+\Delta T,\Lambda)-\frac{\zeta(\Lambda)}{\tau}+\#\Lambda^3+\frac{\#\Lambda}{\tau}+\cdots \\ &= \underbrace{p(T,\Lambda)+\#\Lambda^3}_{=p(T)}\underbrace{-\frac{\zeta(\Lambda)}{\tau}+\frac{\#\Lambda}{\tau}+s(T)\Delta T}_{=-\zeta/\tau}+\cdots\end{aligned}$$

Renormalization of bulk viscosity [Kovtun-Yaffe (03)]

$$\zeta=\zeta(\Lambda)+\frac{T\Lambda}{18\pi^2}\left[\left(1+\frac{3T}{2}\frac{dc_s^2}{dT}-3c_s^2\right)^2\frac{1}{\gamma_\zeta+\frac{4}{3}\gamma_\eta}+\left(1-3c_s^2\right)^2\frac{2}{\gamma_\eta}\right]$$

Contents

Summary

What I talked:

- Kinetic regime k_* is the non-equilibrium scale
- Nonlinear fluctuations from k_* gives long-time tail
- Some details (initial fluctuations, temperature shift) are discussed

What I promised to talk but I did not: [An et al (2019)]

- Hydro-kinetic theory on general background flow
- Affine connection between local rest frame $\rightarrow$ confluent derivative
- Kinetic theory on local flows (shear, bulk, rotation, acceleration)

Future direction:

- Critical fluctuations of model H near the QCD critical point
- Kibble-Zurek scaling by mode coupling treatment? [YA et al, in progress]
- Photon and dilepton spectrum [YA et al, in prep.]

Backup slides

Time dependence of transverse pressure

Transverse pressure

$$\frac{\langle T^{xx,yy}(\tau)\rangle}{w} = \frac{p}{w} + \frac{2\gamma_\eta}{3\tau} + \left[\frac{\chi_{\tau_0} + \delta\chi_{\tau_0}}{\tau^2 w^2}\right]\frac{1}{(12\pi\gamma_\eta\tau)} - \frac{0.273836}{s(4\pi\gamma_\eta\tau)^{3/2}}$$

Time dependence of each term

$$\frac{p}{w} = \frac{1}{4}, \quad \frac{\gamma_\eta}{\tau} = \underbrace{\frac{\eta}{s}\cdot\frac{1}{T\tau^{1/3}}}_{\text{const}}\cdot\frac{1}{\tau^{2/3}}, \quad \frac{1}{\gamma_\eta w^2\tau^3} = \underbrace{\frac{s}{\eta}\cdot\frac{T\tau^{1/3}}{(w\tau^{4/3})^2}}_{\text{const}}\cdot\frac{1}{\tau^{2/3}},$$

$$\frac{1}{s(\gamma_\eta\tau)^{3/2}} = \underbrace{\frac{1}{s\tau}\cdot\left(\frac{s}{\eta}\cdot T\tau^{1/3}\right)^{3/2}}_{\text{const}}\cdot\frac{1}{\tau}$$

$$\chi_{\tau_0} \sim \left[\frac{\tau_0^2\,Tw}{\sqrt{\gamma_\eta\tau_0}}\right]_{\tau_0} \sim \underbrace{\left[Tw\tau_0^{5/3}\right]_{\tau_0}\cdot\left[\frac{\tau_0^{1/6}}{\gamma_\eta^{1/2}}\right]_{\tau_0}}_{\text{const}}\cdot\frac{1}{\tau_0^{1/3}}$$

Particle correlations

Fundamental tool for data analysis

- Anisotropic flows (elliptic v_2, triangular v_3, $\cdots$) from particle correlations

$$v_n\{2\} = \sqrt{\langle e^{in(\phi_1 - \phi_2)} \rangle}, \quad \cdots$$

Correlation via hydrodynamic fluctuations

- Boltzmann-Langevin equation

$$f(x,p) = \underbrace{f_{\text{ideal}}(x,p) + f_{\text{visc.}}(x,p)}_{\text{viscous hydro}} + \underbrace{\delta f_{\text{fluct.}}(x,p)}_{\text{noise}}$$

$$\langle f^{(2)}(x_1, p_1, x_2, p_2) \rangle \neq \langle f(x_1, p_1) \rangle \langle f(x_2, p_2) \rangle$$

- Hydro fluctuation should influence particle correlation

$$f_{\text{ideal}}(x,p) = [\exp(\beta_\mu(x) p^\mu) \mp 1]^{-1}$$

Nonlocal correlation from $\langle \beta_\mu(x) \beta_\nu(y) \rangle$ out of equilibrium