Information Perspectives on Turbulent Cascade

Advances in Fluctuating Hydrodynamics: Bridging the Micro and Macro Scales

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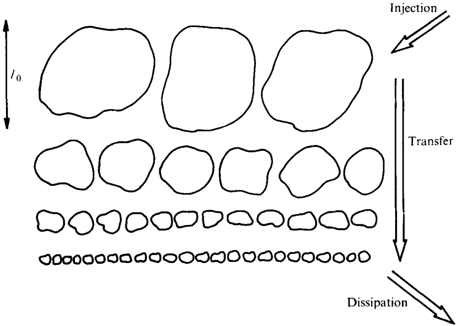
A Paradox of Turbulence from Information Perspectives

Energy Cascade

da Vinci, Royal Collection at Windsor, RCIN 912660v

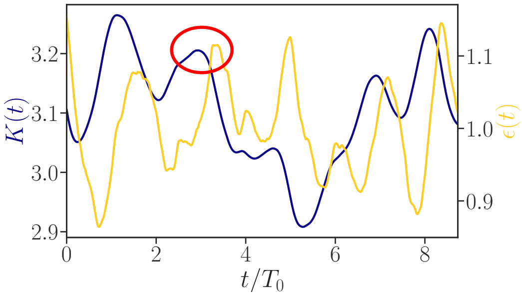


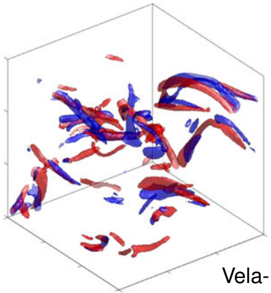
Frisch, Sulem, and Nelkin (1978)

Richardson's poem Richardson (1922, p. 66)

> *[...] big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity [...].*

Causality and "Synchronised" Small Scales





Vela-Martín (2021)

- Time delay between energy

$$K(k,t) := \left\langle \boldsymbol{u}^2 \right\rangle /2$$

  and energy dissipation rate

$$\epsilon(k,t) := \nu(\boldsymbol{\nabla} \boldsymbol{u})^2$$

- "Twin" simulations with synchronised large scales.
- Despite the chaos, small scales are also synchronised.

Statistical Universality and “Forgotten” Large Scales

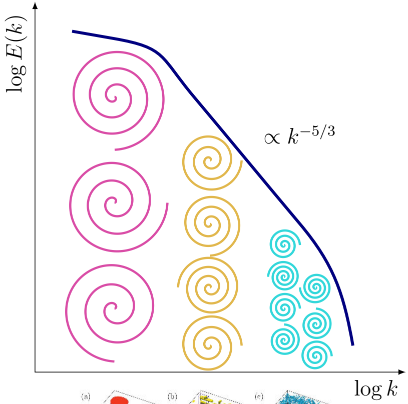
Davidson (2015, § 5.2.1)

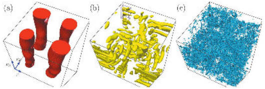
> *[The small scales] do not retain any information which relates to their great-great-great-grand parents.*

G. L. Eyink (2007-2008, § II(E))

> *[...] the small-scales will “forget” about the detailed geometry and statistics of the large-scale flow modes.*

Paradox: Information flows or dams up?





Goto (2018)

Information Flow and Its Bound in Turbulent Cascade

- Tomohiro Tanogami and Ryo Araki (2024). Information-thermodynamic bound on information flow in turbulent cascade. ***Phys. Rev. Res.*** **6** (1), p. 013090
- Remark: Theoretical results are attributed to Tanogami-san.

Information Theory

- "Theory of communication" Shannon (1948)
- Actively applied to cryptography, physics, machine learning, ...

Shannon entropy

- Information $:= -\ln p(x)$ is "amount of surprise" of the event $x \in X$.
  $\therefore$ Unlikely event has large information.
- Shannon entropy $:= H(X)$ is average surprise of possible events.

Information of an observed event

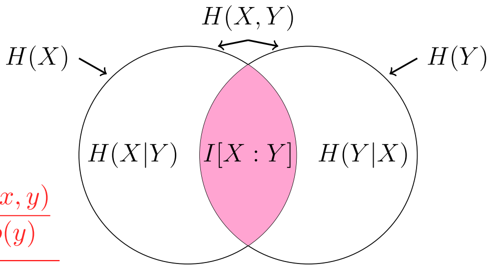
$$H(X) = \sum_{x \in X} p(x) \times [-\ln p(x)]$$

Average of possible events

Mutual information

$$I[X:Y] := \sum_x \sum_y p(x,y) \ln \frac{p(x,y)}{p(x)p(y)}$$
$$= H(X) + H(Y) - H(X,Y)$$
$$= H(X) - H(X|Y)$$
$$\geq 0$$

$$H(X|Y) := -\sum_{x,y} p(x,y) \ln \frac{p(x,y)}{p(y)}$$



- $I[X:Y]$ quantifies
  - Generalised correlation (shared information) between X and Y.
  - Decrease of surprise in Y by knowing X (or vice versa).

Setup: Fluctuating Hydrodynamics

Fluctuating Navier–Stokes equations in Fourier space:

$$\frac{\partial \hat{\boldsymbol{u}}_{\boldsymbol{k}}}{\partial t} = \underbrace{\boldsymbol{B}_{\boldsymbol{k}}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*)}_{\text{Nonlinear}} - \underbrace{\nu k^2 \hat{\boldsymbol{u}}_{\boldsymbol{k}}}_{\text{Viscous}} + \underbrace{\hat{\boldsymbol{f}}_{\boldsymbol{k}}}_{\text{External force}} + \underbrace{\sqrt{\frac{2\nu k^2 k_{\mathrm{B}} T}{\rho}} \hat{\boldsymbol{\xi}}_{\boldsymbol{k}}}_{\text{Thermal agitation}}$$

where noise is delta-correlated

$$\left\langle \hat{\boldsymbol{\xi}}_{\boldsymbol{k}}(t) \hat{\boldsymbol{\xi}}^*_{\boldsymbol{k}'}(t') \right\rangle = \frac{1}{V}\left(\mathsf{I} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right)\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta(t-t').$$

$\hat{\boldsymbol{u}}_{\boldsymbol{k}}$: Fourier-space velocity at wave number $\boldsymbol{k}$ (and time t), ν: kinematic viscosity, ρ: mass density, k_{B}: Boltzmann constant, T: temperature, V: volume, $\cdot^*$: complex conjugate,

Setup: Decomposed Velocity and Mutual Information

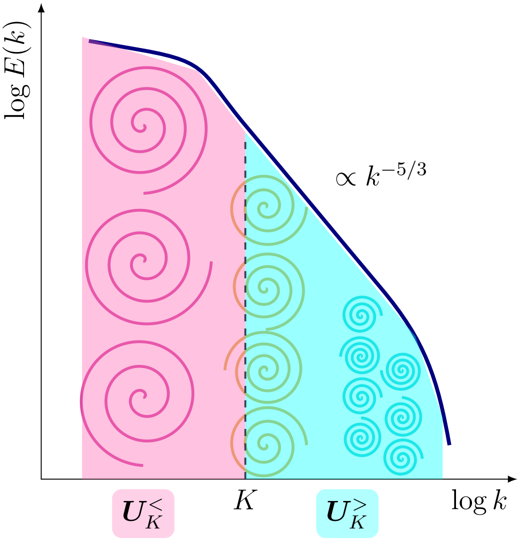
- Decompose the velocity field

$$\{\hat{\boldsymbol{u}}_{\boldsymbol{k}}, \hat{\boldsymbol{u}}^*_{\boldsymbol{k}}\} = \underbrace{\boldsymbol{U}^<_K}_{\substack{\{\hat{\boldsymbol{u}}_{\boldsymbol{k}}, \hat{\boldsymbol{u}}^*_{\boldsymbol{k}} | k \le K\} \\ \text{Large-scale}}} \cup \underbrace{\boldsymbol{U}^>_K}_{\substack{\{\hat{\boldsymbol{u}}_{\boldsymbol{k}}, \hat{\boldsymbol{u}}^*_{\boldsymbol{k}} | k > K\} \\ \text{Small-scale}}}$$

- Define the mutual information

$$I[\boldsymbol{U}^<_K : \boldsymbol{U}^>_K] := \left\langle \ln \frac{p(\boldsymbol{U}^<_K, \boldsymbol{U}^>_K)}{p(\boldsymbol{U}^<_K) p(\boldsymbol{U}^>_K)} \right\rangle,$$

$\rightarrow$ Quantify the correlation between large- and small-scale eddies.



Setup: Information Flow with Learning Rate

"Learning rate" of the large-scale velocity

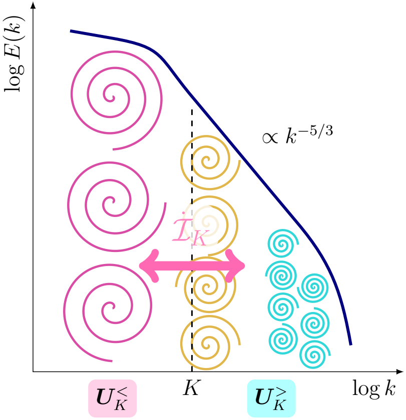
$$\dot{I}_K^{<} := \lim_{\mathrm{d}t \searrow 0} \frac{I[\boldsymbol{U}_K^{<}(t+\mathrm{d}t) : \boldsymbol{U}_K^{>}(t)] - I[\boldsymbol{U}_K^{<}(t) : \boldsymbol{U}_K^{>}(t)]}{\mathrm{d}t}.$$

Learning rate of the small-scale velocity

$$\dot{I}_K^{>} := \lim_{\mathrm{d}t \searrow 0} \frac{I[\boldsymbol{U}_K^{<}(t) : \boldsymbol{U}_K^{>}(t+\mathrm{d}t)] - I[\boldsymbol{U}_K^{<}(t) : \boldsymbol{U}_K^{>}(t)]}{\mathrm{d}t}.$$

At steady state, $\dot{\mathcal{I}}_K := \dot{I}_K^{>} = -\dot{I}_K^{<}$ holds and

- $\dot{\mathcal{I}}_K > 0$: Small scale learns large scale.
- $\dot{\mathcal{I}}_K < 0$: Large scale learns small scale.



Main Result: Information Flow in Turbulence

From the second law of information thermodynamics, we found that at steady state:

1. The following inequality

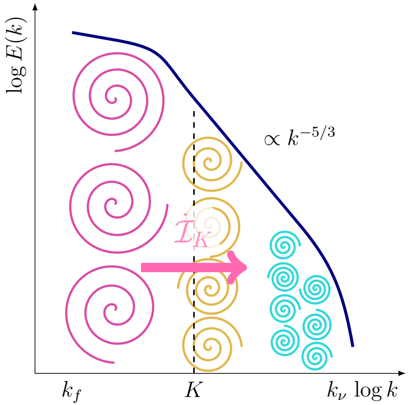
$$\dot{\mathcal{I}}_K \geq 0$$

   holds for wave number K in the inertial range $k_f \ll K \ll k_\nu$.

2. Furthermore,

$$\frac{\rho V \epsilon}{k_{\mathrm{B}} T} \geq \dot{\mathcal{I}}_K$$

   establishes the upper bound of $\dot{\mathcal{I}}_K$.



There is the information flow from large to small scales in turbulence.

Energy Cascade Mechanisms and Information Flow

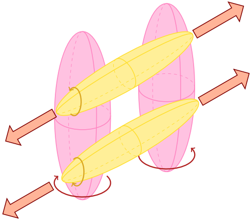
- Ryo Araki, Alberto Vela-Martín, and Adrián Lozano-Durán (2024). Forgetfulness of turbulent energy cascade associated with different mechanisms. ***Journal of Physics: Conference Series*** **2753** (1), p. 012001
- Remark: NO fluctuating hydrodynamics in this section.

Physical Mechanisms of the Energy Cascade

How are energy and information in turbulence transferred
from large to small scales?

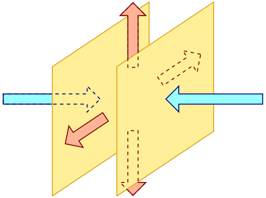
Vortex Stretching (VS)

Coherent hierarchical vortex tubes



Strain Self-Amplification (SSA)

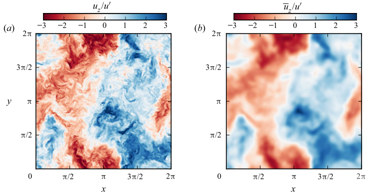
Strong velocity gradient



The physical mechanism of the cascade is still an open question Goto, Saito, and Kawahara (2017), Carbone and Bragg (2020), Johnson (2021), and McKeown et al. (2022).

Coarse-grained velocity field

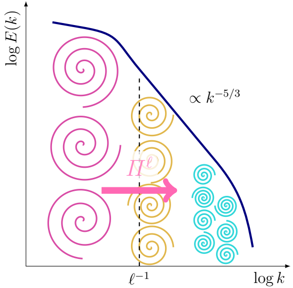
Johnson (2021, Fig. 4)



$$\bar{u}_i^\ell(\boldsymbol{x}) = \iiint_{-\infty}^{\infty} G_\ell(\boldsymbol{r}) u_i(\boldsymbol{x}+\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$$

- Gaussian filter $G_\ell(\boldsymbol{r}) = \mathcal{N}\exp\left(-|\boldsymbol{r}|^2/2\ell^2\right)$

Inter-scale energy flux



$$\Pi^\ell = -\mathring{\tau}_\ell(u_i, u_j)\bar{S}_{ij}^\ell$$

- Reynolds stress[a] $\tau_\ell(u_i, u_j) = \overline{u_i u_j}^\ell - \bar{u}_i^\ell \bar{u}_j^\ell$
- Strain-rate $\bar{S}_{ij}^\ell = \left[\partial \bar{u}_i^\ell/\partial x_j + \partial \bar{u}_j^\ell/\partial x_i\right]/2$

[a] Non-diagonal part $\mathring{v}_{ij} := v_{ij} - v_{kk}\delta_{ij}/3$

Distributions of Decomposed Energy Flux

Johnson (2021, Fig. 8(a))

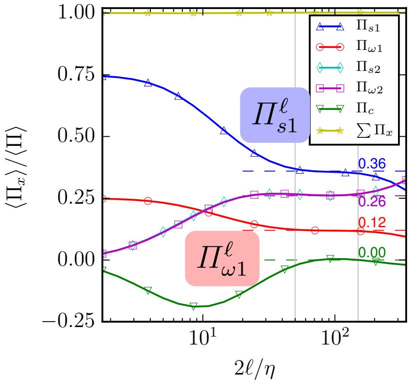
SSA: Strain self-amplification, VS: Vortex stretching

$$\Pi^\ell = -\mathring{\tau}_\ell(u_i, u_j)\bar{S}^\ell_{ij} = \Pi^\ell_{s1} + \Pi^\ell_{\omega 1} + \Pi^\ell_{s2} + \Pi^\ell_{\omega 2} + \Pi^\ell_{c}.$$

- Scale-local SSA Π^ℓ_{s1}
- Scale-local VS $\Pi^\ell_{\omega 1}$
- Scale-nonlocal part: Π^ℓ_{s2}, $\Pi^\ell_{\omega 2}$, Π^ℓ_{c}

In the inertial range:

$$\Pi^\ell_{s1} > \Pi^\ell_{\omega 1}$$



With Johnson (2021), the SSA transfers more energy to smaller scales than the VS.

Quantification of Causality by "Information Flux"

Lozano-Durán and Arranz (2022, § IV)

- Consider N_Y-DoF system: $\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_{N_Y}]$
- Not observe i-th variable: $\boldsymbol{Y}_{\not i} = [Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_{N_Y}]$
- Consider time evolution: $Y_j^{n+1} = f_j(\boldsymbol{Y}^n)$

"Information flux" $T^Y_{i\to j}$ from Y_i^n to Y_j^{n+1}

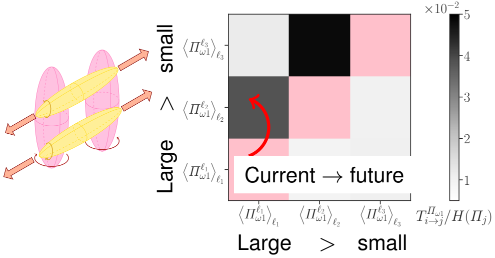
Fully known $\boldsymbol{Y}^n$

$$T^Y_{i\to j} := H\left(Y_j^{n+1} | \boldsymbol{Y}_{\not i}^n\right) - H\left(Y_j^{n+1} | \boldsymbol{Y}^n\right)$$

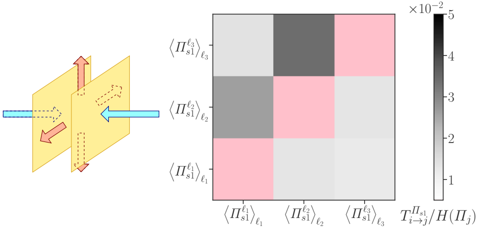
i-th component of $\boldsymbol{Y}^n$ is unknown

Information flux $T^Y_{i\to j}$:= Decrease of "surprise" to observe Y_j^{n+1} when knowing Y_i^n.

Information Flux with Different Cascade Mechanisms



Scale-local VS $T^{\Pi_{\omega 1}}$



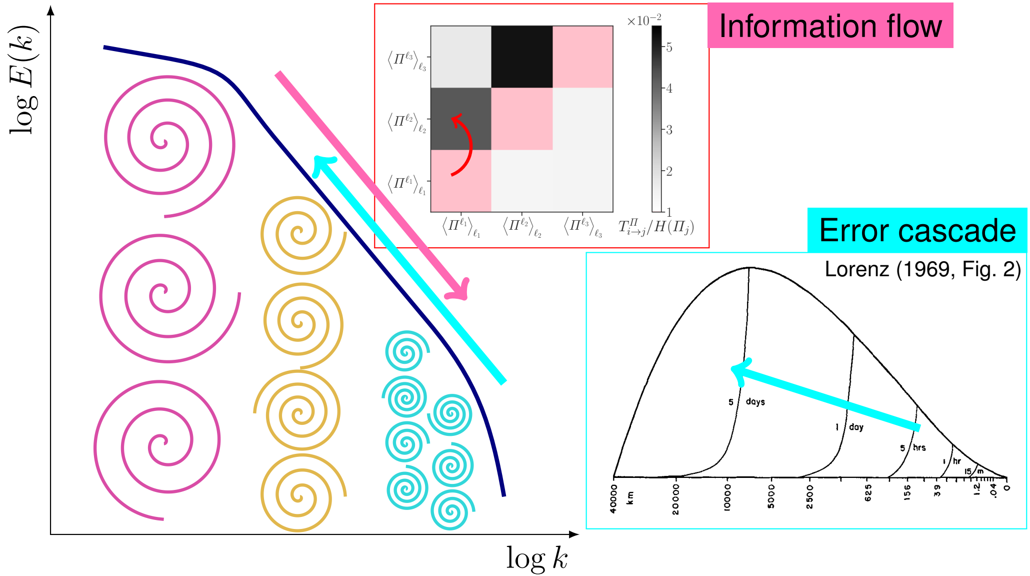
Scale-local SSA $T^{\Pi_{s1}}$

Information flux: $T^{\Pi_{\omega 1}} > T^{\Pi_{s1}}$ $\leftrightarrow$ energy flux: $\Pi_{s1} > \Pi_{\omega 1}$

Efficient ***information*** transfer mechanism $\neq$ efficient ***energy*** transfer mechanism.

Ideas: Turbulence with Fluctuating Hydrodynamics

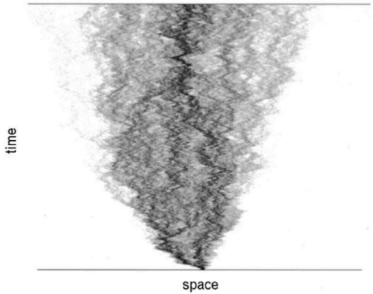
Information Flow and Error Cascade in Turbulence



Spontaneous stochasticity

Tanogami (2023), Palmer (2024), and G. Eyink (2024)

[...] two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, ***which cannot be lengthened by reducing the amplitude of the initial error.*** *Edward N. Lorenz (1969). The predictability of a flow which possesses many scales of motion.* ***Tellus 21*** *(3), pp. 289–307*



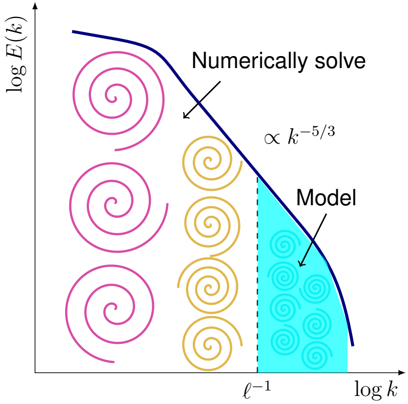
Falkovich, Gawędzki, and Vergassola (2001)

- Finite predictability time horizon
  $\neq$ conventional sensitivity on initial condition
- Uniqueness of solution is violated.
- Affects inertial range dynamics of turbulence
  Thalabard, Bec, and Mailybaev (2020) and Bandak et al. (2024)

Can we establish a unified view of information flow and error cascade in turbulence?

Turbulence Model: Large Eddy Simulation (LES)

- Real-world flows are too demanding.
- Numerically solve the filtered NS equation with modelled small scales.
- Drawback: Unknown initial condition of the modelled scales $\to$ unreliable solution.



Stochastic LES

Pope 2000, § 13.5.6

- Introduce a stochastic term to statistically represent the modelled scales

$$\partial_t \hat{\bar{u}}_i^\ell(\boldsymbol{k}, t) = \cdots + c P_{ij}(\boldsymbol{k}) \underbrace{\mathrm{d}W_j(\boldsymbol{k}, t)}_{\text{Wiener process}} .$$

Can we integrate the fluctuating NS eqs into stochastic turbulence model?

G. Eyink (2024)

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