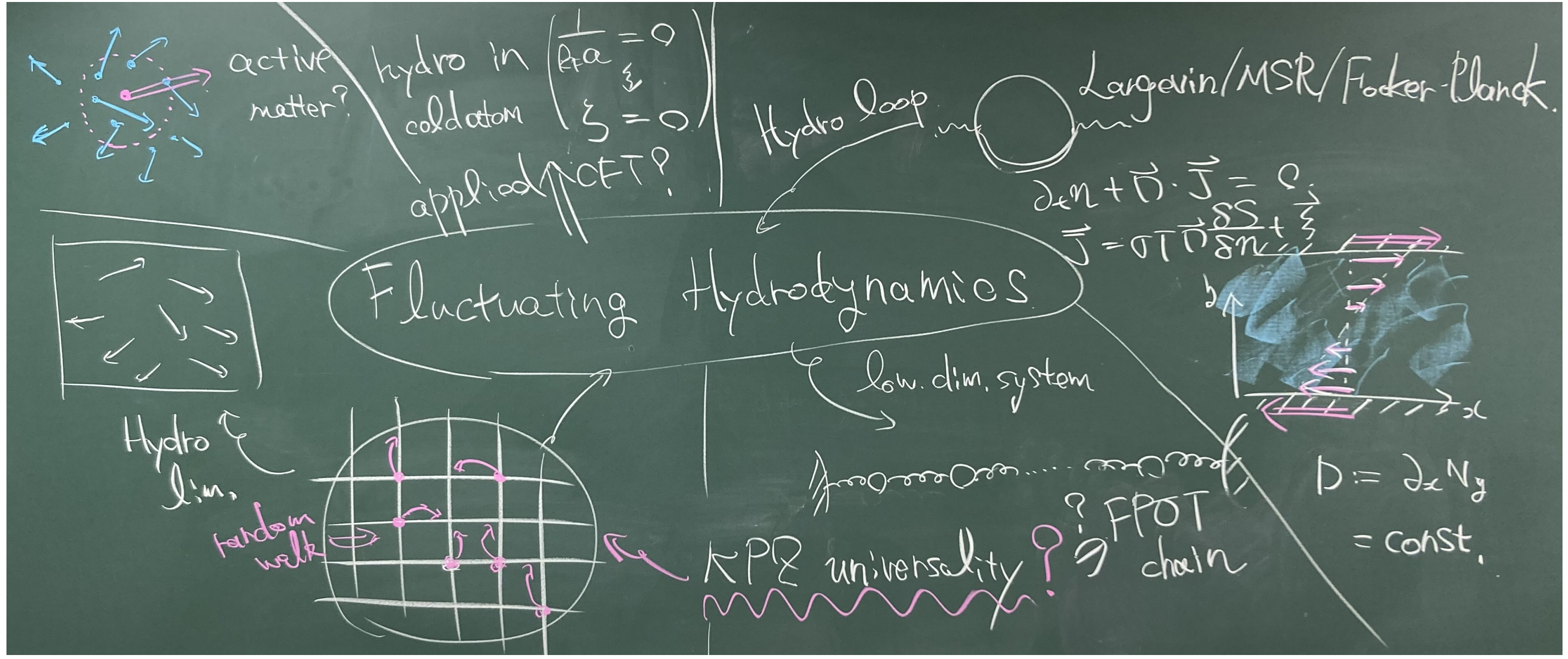


# Minimum scaling model and exact exponents for the Nambu-Goldstone modes in the Vicsek Model

(arXiv:2401.01603)

Harukuni Ikeda
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# Overview

- **Introduction**
- Previous research: Toner-Tu theory
- Model
- Theory
- Comparison with numerical simulations
- Summary and discussions

# Introduction

## Continuous symmetry breaking in equilibirum

**O(n) model** (n=2→ XY, n=3 → Heisenberg)

$$F[\boldsymbol{\phi}] = \int dx \left[ \frac{k}{2}(\nabla \boldsymbol{\phi})^2 + f(\boldsymbol{\phi}) \right]$$

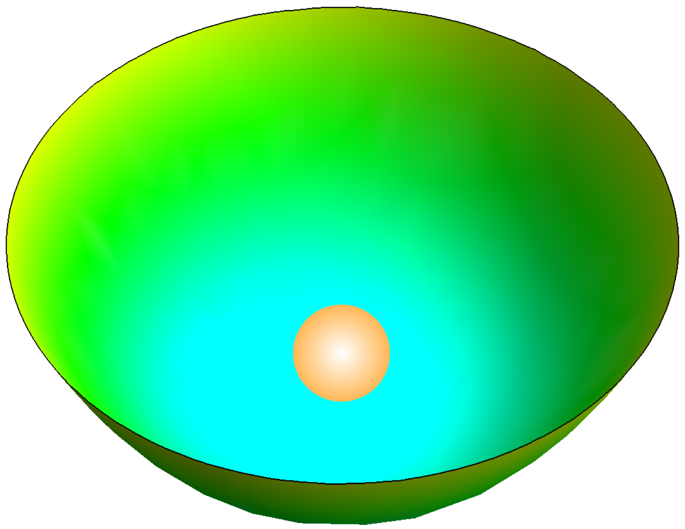
$$f(\boldsymbol{\phi}) = a\frac{|\boldsymbol{\phi}|^2}{2} + b\frac{|\boldsymbol{\phi}|^4}{4}$$

Order parameter: $\boldsymbol{\phi} = \{\phi_1, \cdots, \phi_n\}$

**Schematic pictures of $f(\boldsymbol{\phi})$ for n=2**

**Disordered phae**

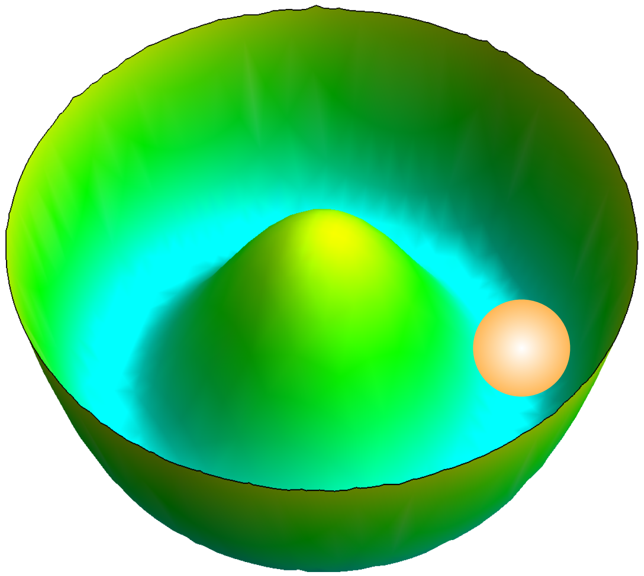
$a > 0$



$\langle \boldsymbol{\phi} \rangle = 0$

**Ordered phae**

$a < 0$



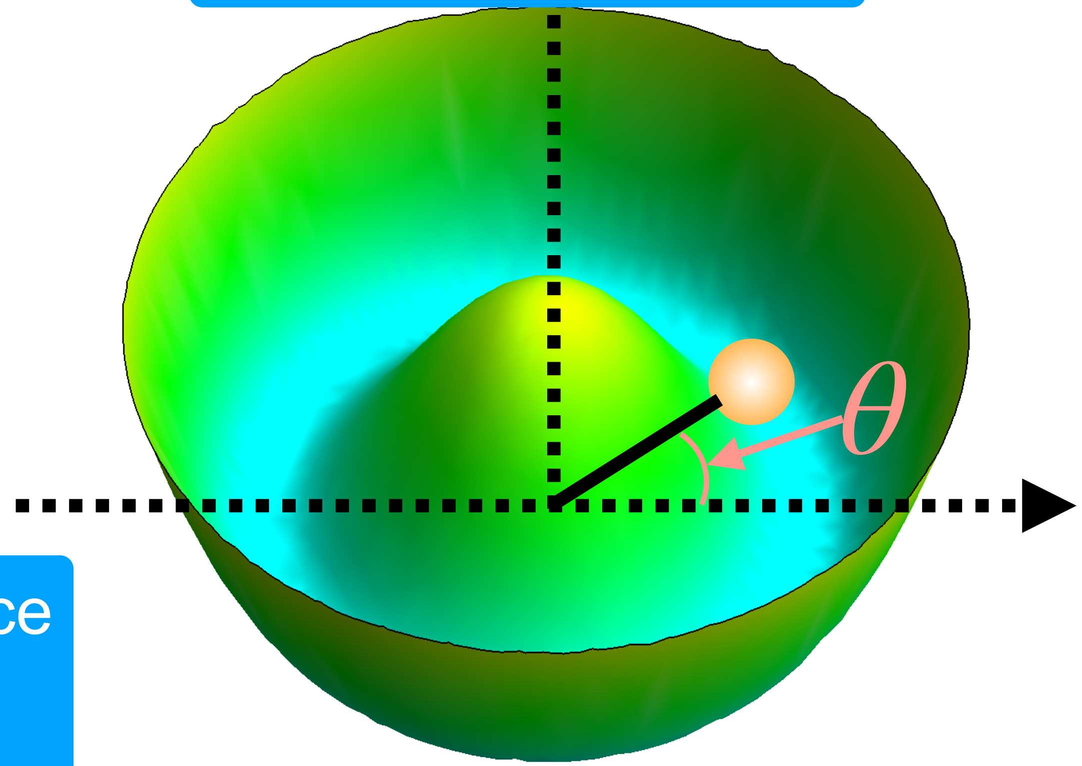
$\langle \boldsymbol{\phi} \rangle \neq 0$

# Introduction

## Hohenberg-Merming-Wagner theorem

$$F[\boldsymbol{\phi}] = \int dx \left[\frac{k}{2}(\nabla\boldsymbol{\phi})^2 + f(\boldsymbol{\phi})\right]$$

$$\boldsymbol{\phi} = \begin{pmatrix}\phi\cos\theta \\ \phi\sin\theta\end{pmatrix}$$




$f(\boldsymbol{\phi})$ does not depend on $\theta$

➡ **$\theta$ is a NG mode!**

Fluctuation in momentum space
$$\langle|\delta\theta_q|^2\rangle = \frac{1}{kq^2}$$

Fluctuation in real space

$$\langle\delta\theta(\boldsymbol{x})^2\rangle \propto \int d\boldsymbol{q}\langle|\delta\theta_q|^2\rangle \sim \int_q \frac{d\boldsymbol{q}}{kq^2} \sim \begin{cases}\infty & (d \le 2) \\ const & (d > 2)\end{cases}$$

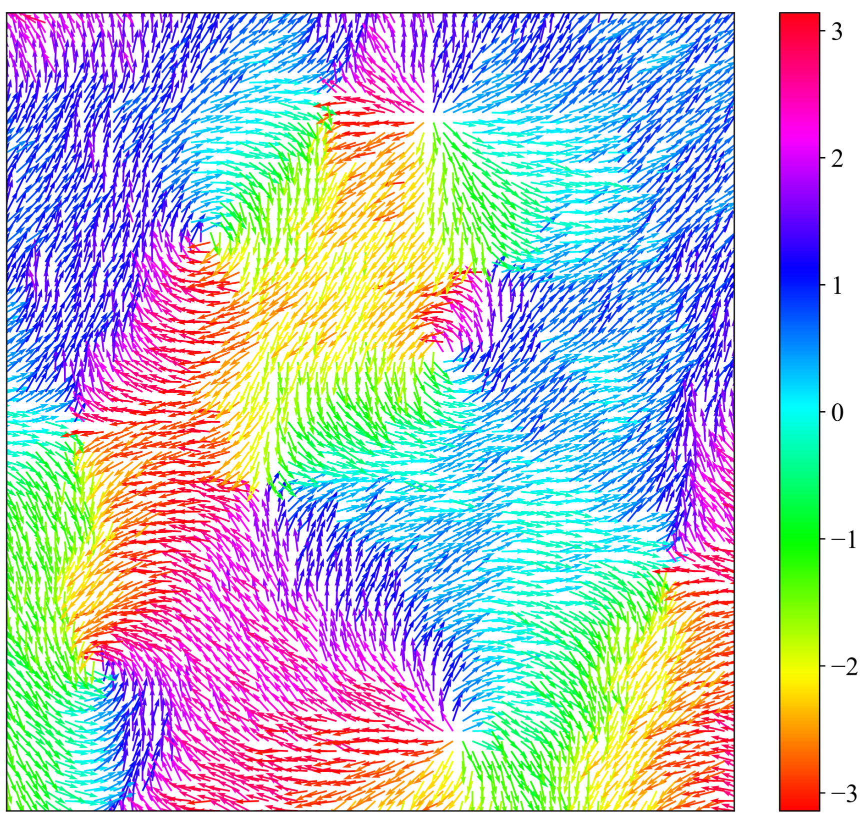
**Fluctuations of the NG mode diverge in two dimensions, which destroy the long-range order.**

# Introduction

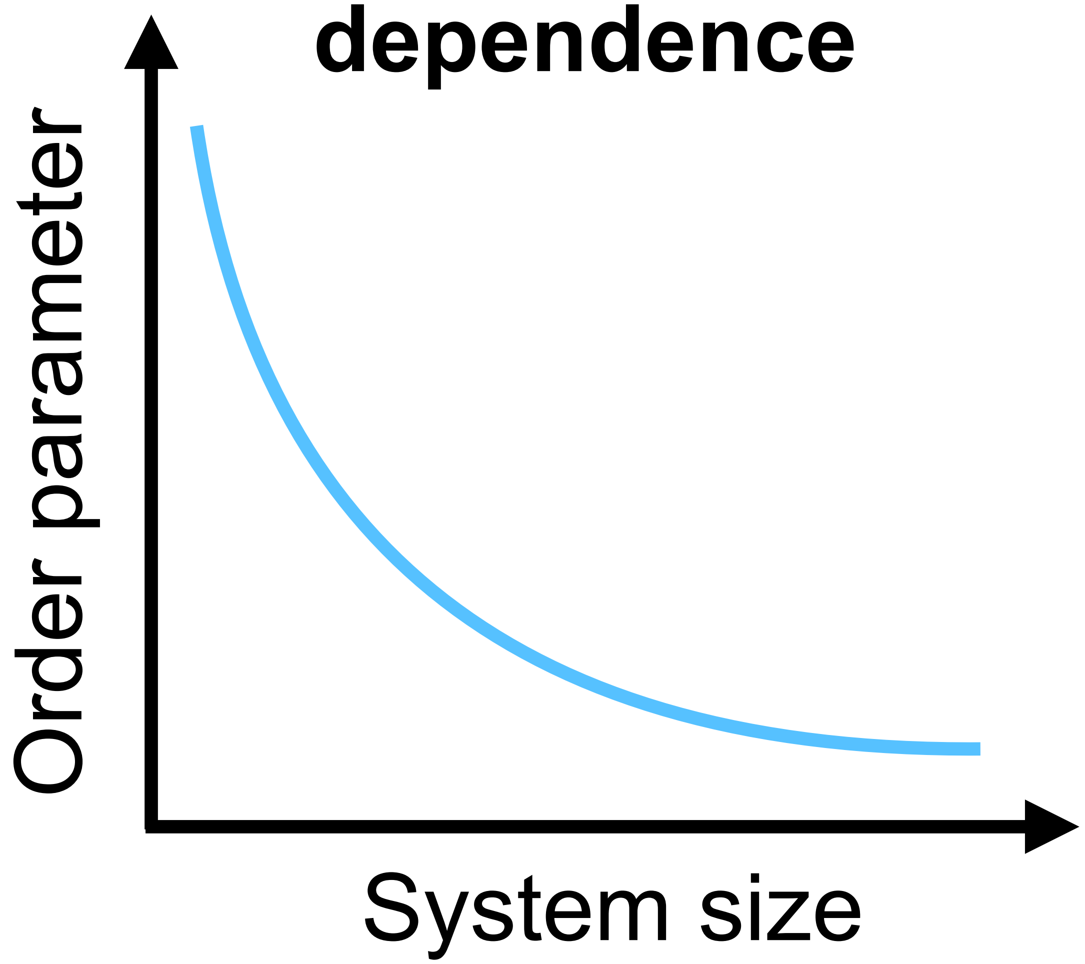
## Hohenberg-Merming-Wagner theorem

In $d \leq 2$, systems with short range interactions do not show continuous symmetry breaking in equilibirum.

**XY model in 2d**




**System size dependence**




Continous symmetry breakings do not occur in d=2 in equilibirum.
**How about systems far from equilibirum?**

# Introduction

## Continuous symmetry breaking far from equilibirum

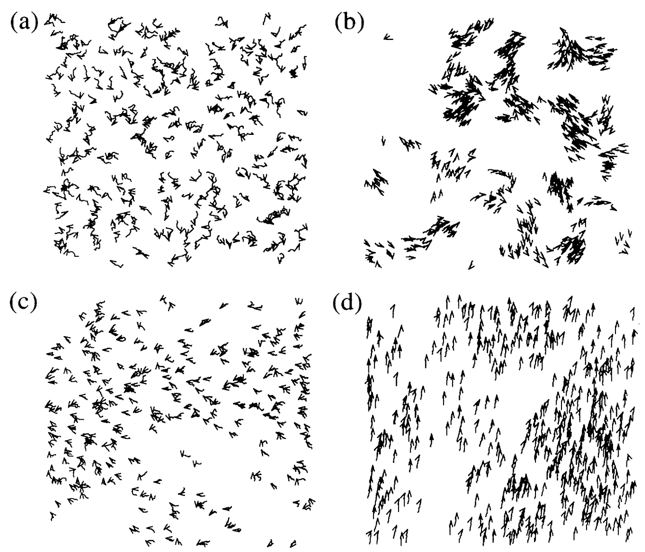
**Several non-equilibrium systems exhibit the continuous symmetry breaking even in $d \leq 2$**

### 2D Vicsek model

Vicsek (1994)
**Ikeda, arXiv:2403.02086(2024)**

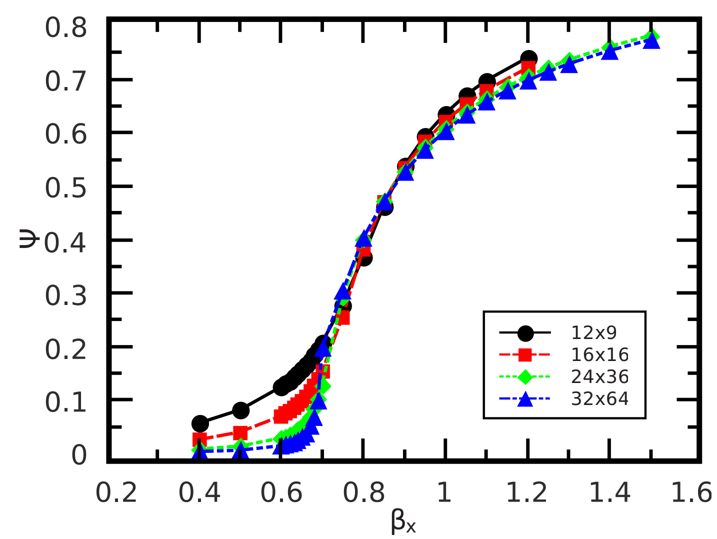
$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t\,.$$

$$\theta(t+1) = \langle\theta(t)\rangle_r + \Delta\theta\,,$$




### 2d XY model with anistropic noise

M. D. Reichl *et al.* (2010)




### 2d O(n) model with shear

Nakano *et al.* (2021),
Ikeda arXiv:2401.01603 (2024)

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right]\varphi_a = -\Gamma\frac{\delta\Phi[\boldsymbol{\varphi}]}{\delta\varphi_a} + \eta_a$$

### Anticorreltaed noise

Galliano *et al.* (2023)
Ikeda, PRE **108**(6), 064119 (2023)
Ikeda, arXiv: 2309.03155 (2023)
Kuroda *et al.* (2024)

# Introduction

## Continuous symmetry breaking far from equilibirum

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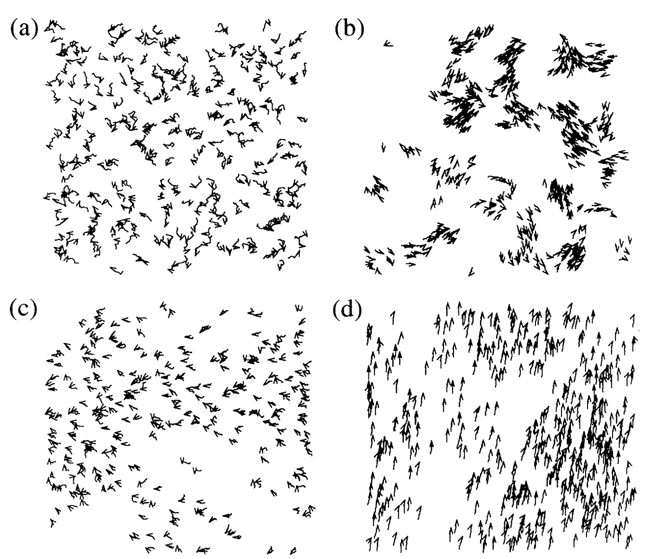


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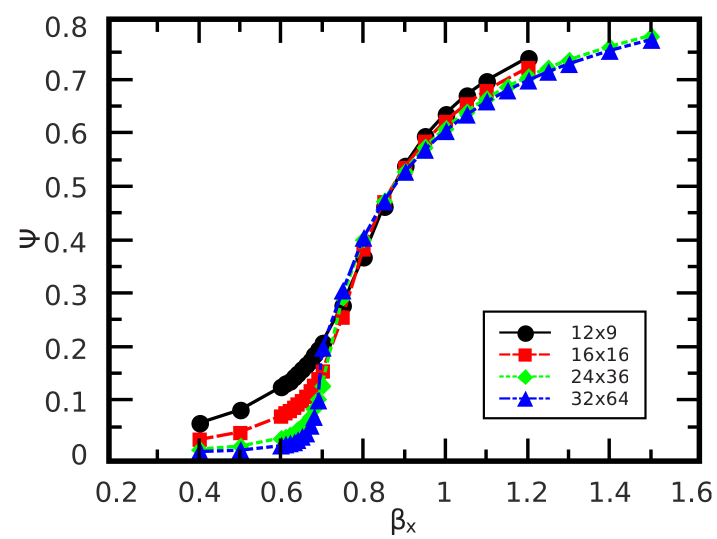


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# Introduction

## Vicsek model

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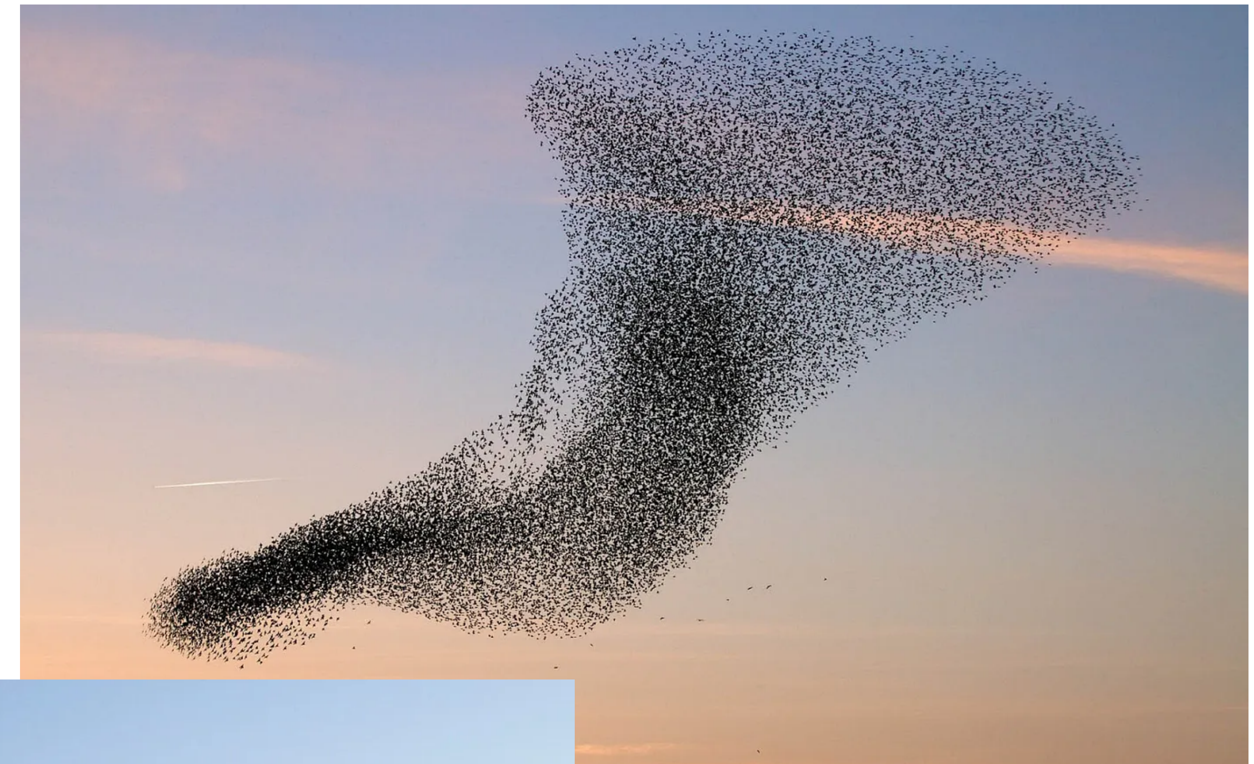
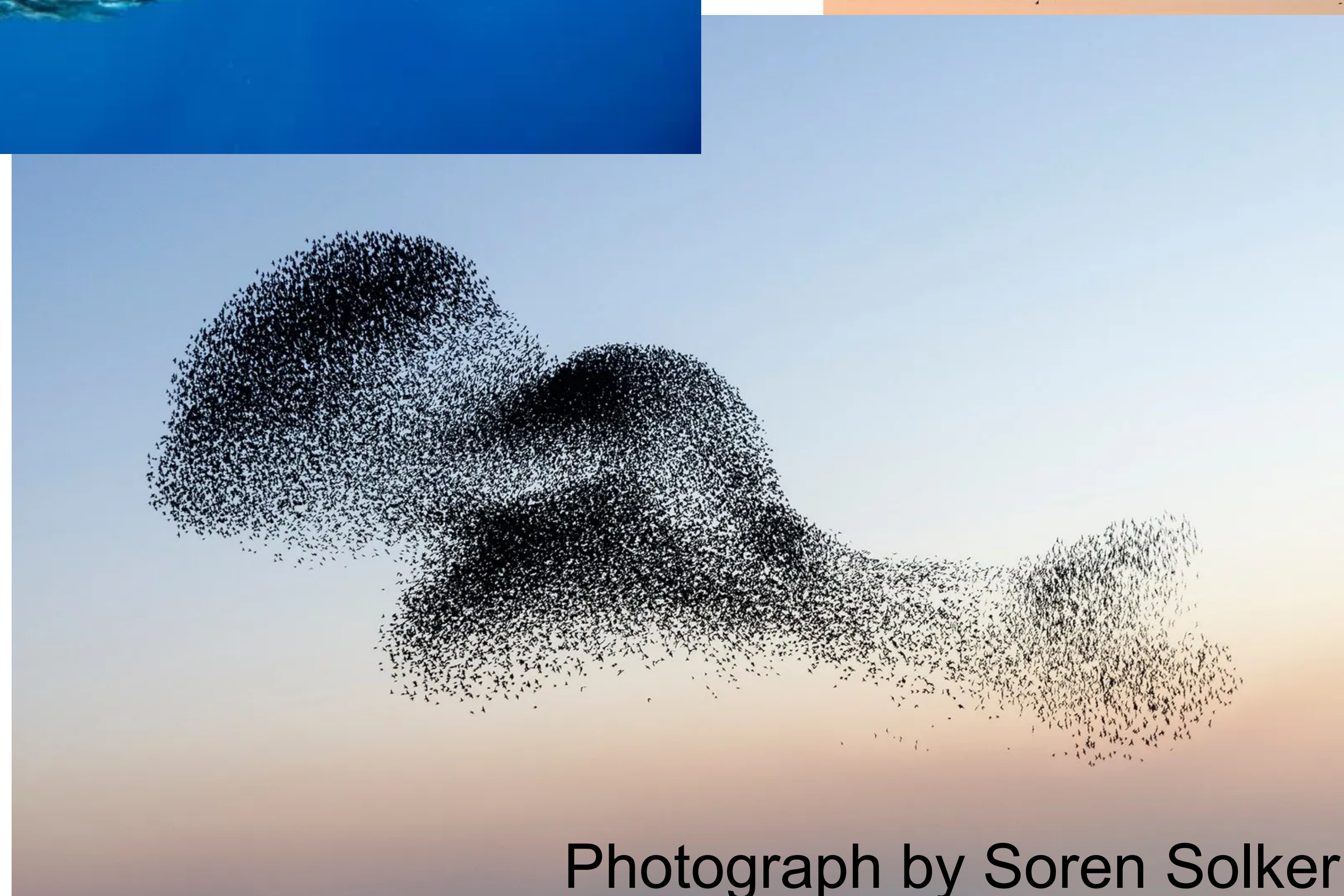


MENAHEM KAHANA
via Getty Images



Photograph by Soren Solker

# Introduction

## Vicsek model

**Model** Vicsek *et al.* (1995)

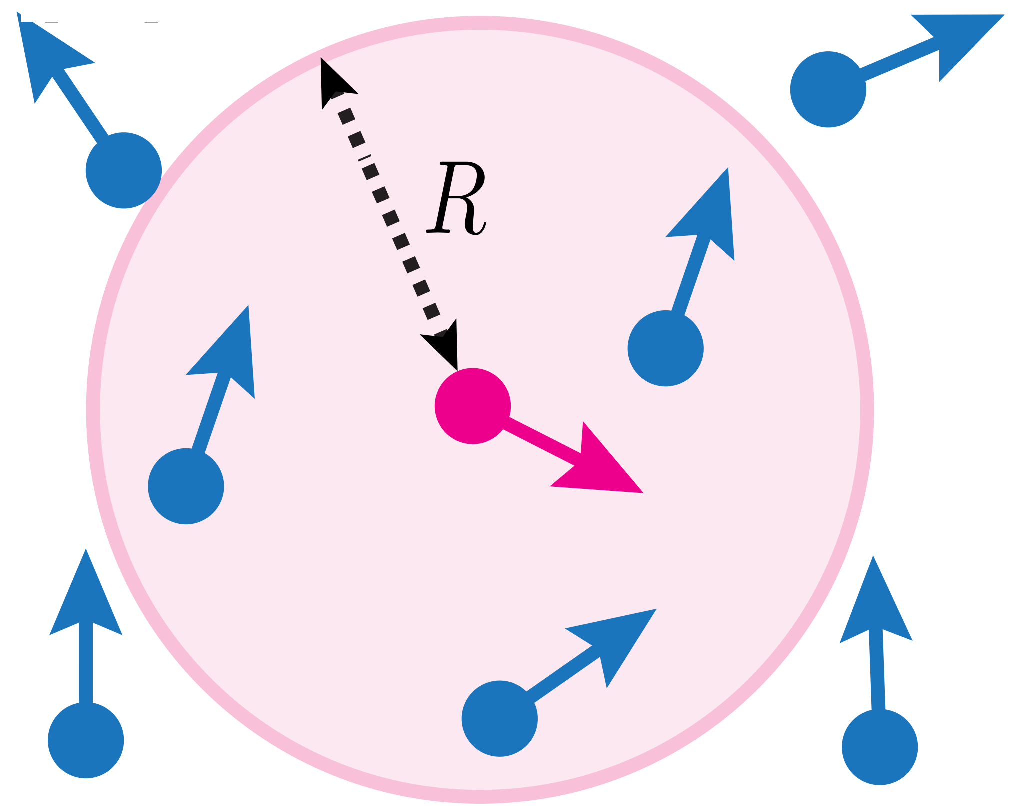
$$\theta_j^{t+1} = \arg \sum_{k\sim j} e^{i\theta_k^t} + \eta_j^t,$$

Allingnment interaction to average direction

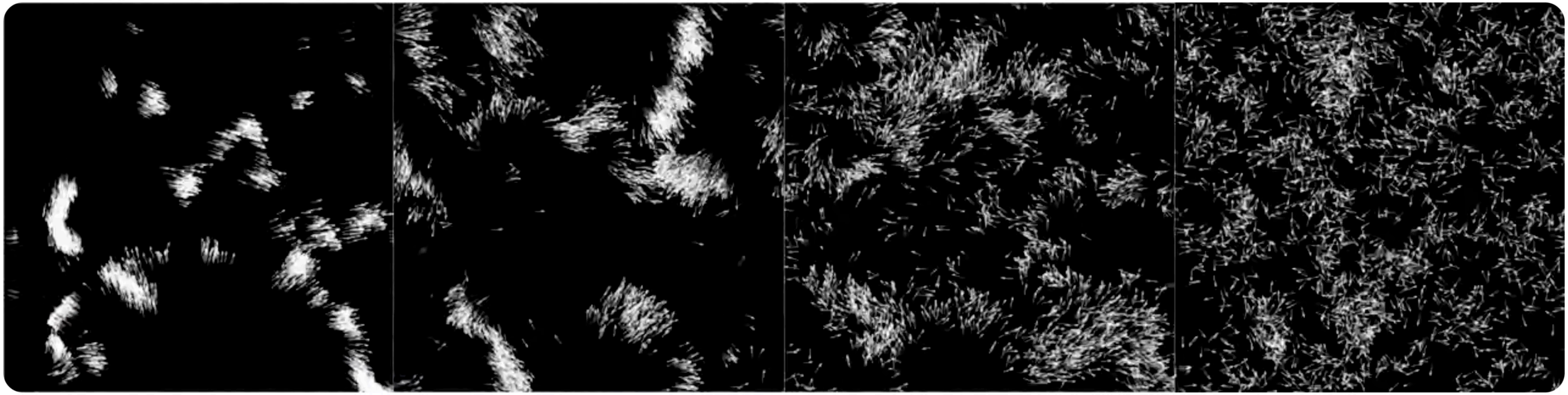
Noise

$$\boldsymbol{r}_j^{t+1} = \boldsymbol{r}_j^t + v_0 \boldsymbol{e}_{\theta_j^{t+1}},$$

Nishiguchi (2023)




**Small noise** Damian Sowinski's YouTube chanel **Large noise**



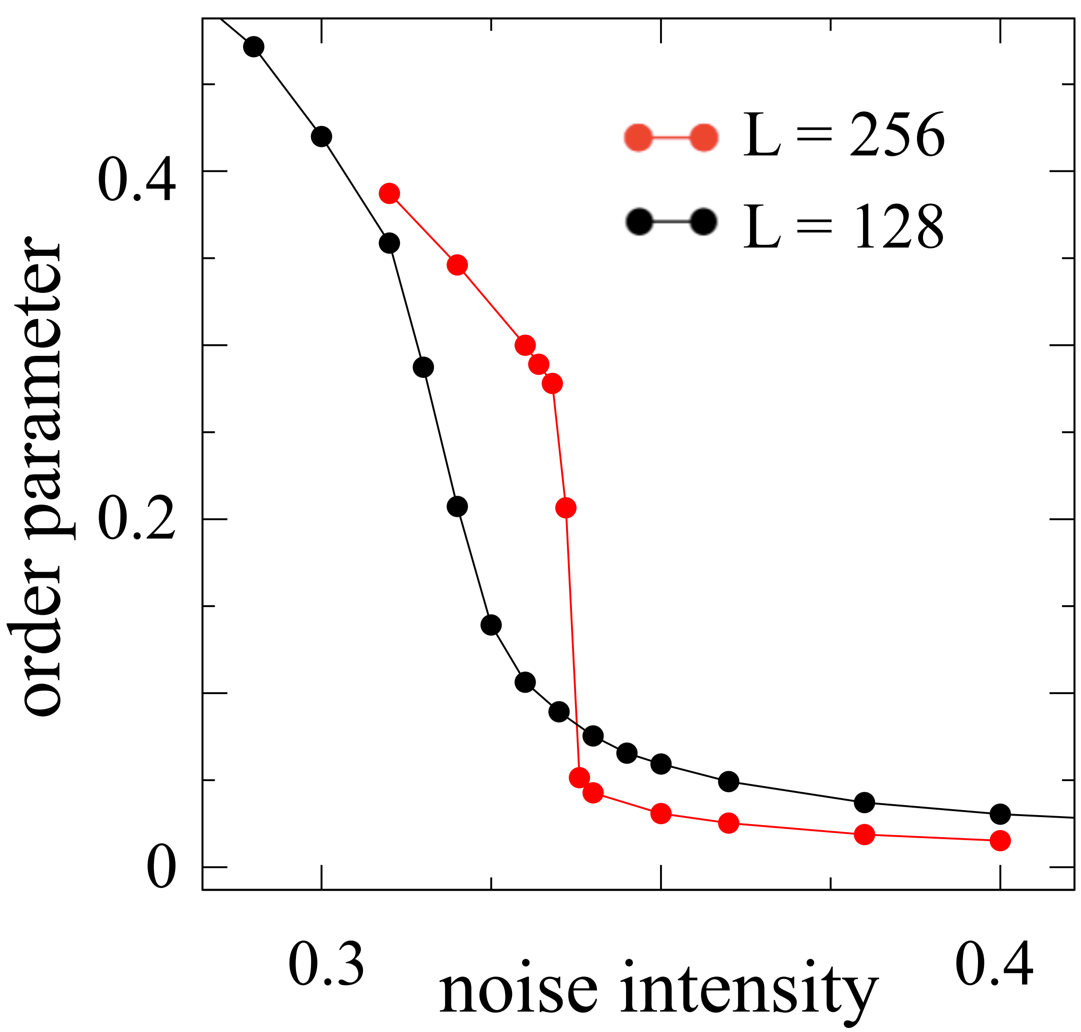
# Introduction

## Vicsek model

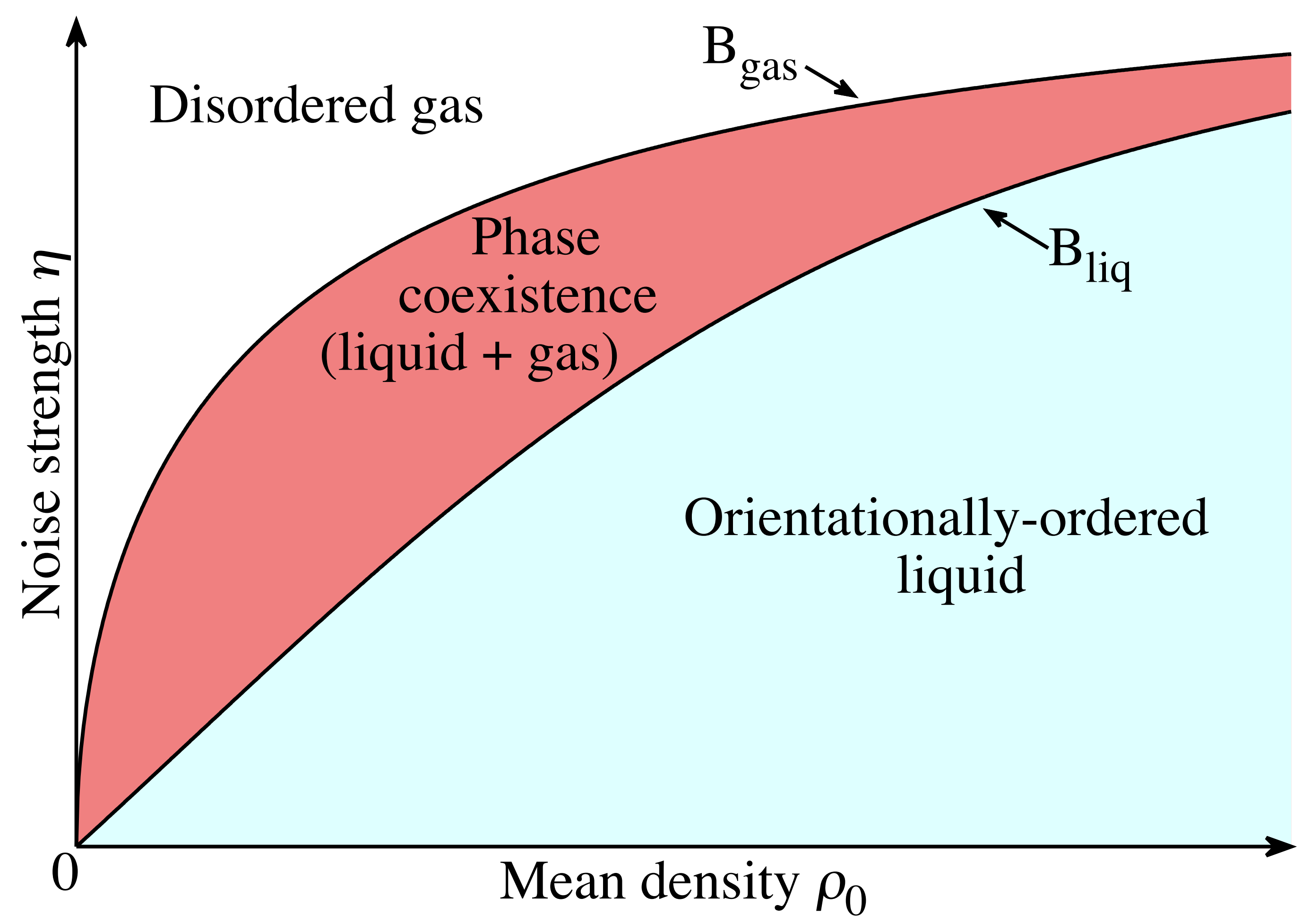
XY model like
Order paramter

**Order parameter**

$$C = \left| \left\langle \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i} \right\rangle \right| \quad \boldsymbol{v}_i(\theta_i) = \begin{pmatrix} v_i \cos\theta_i \\ v_i \sin\theta_i \end{pmatrix}$$



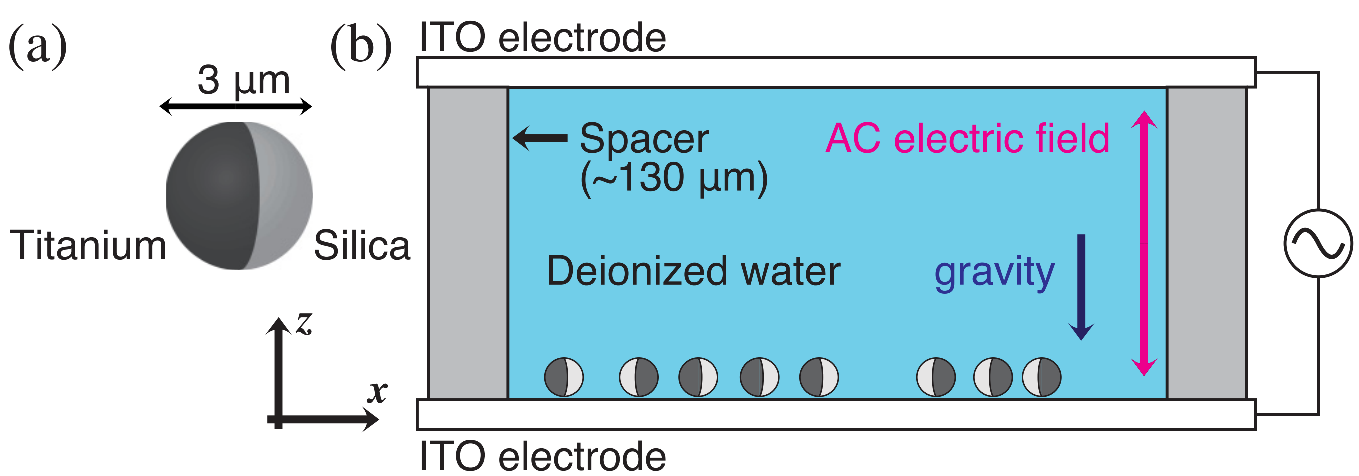

Chaté and Mahault (2019)



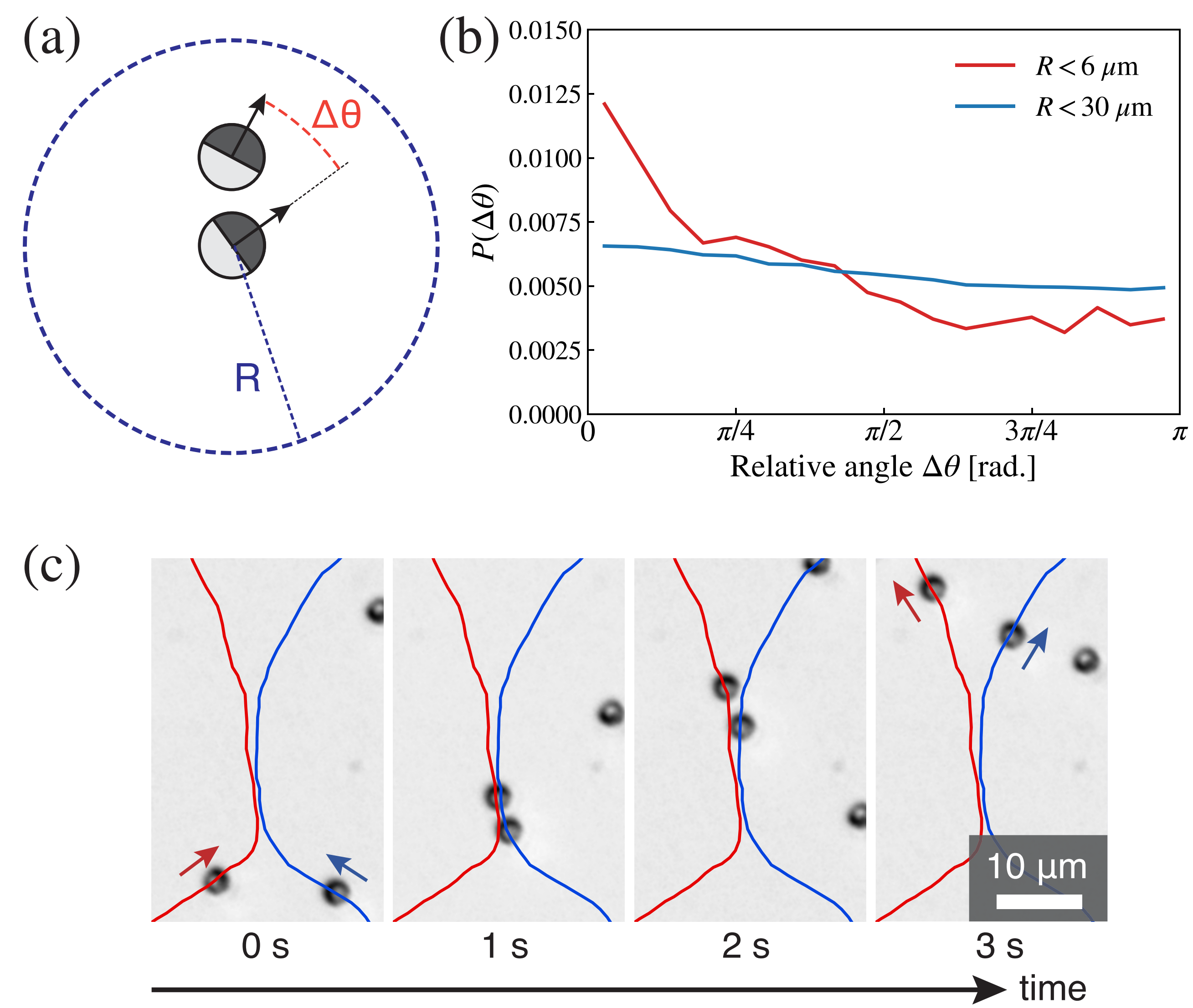
# Introduction

## Vicsek like model in experiment

### Janus particles in AC electric field




Iwasawa, Nishiguchi, and Sano (2021)



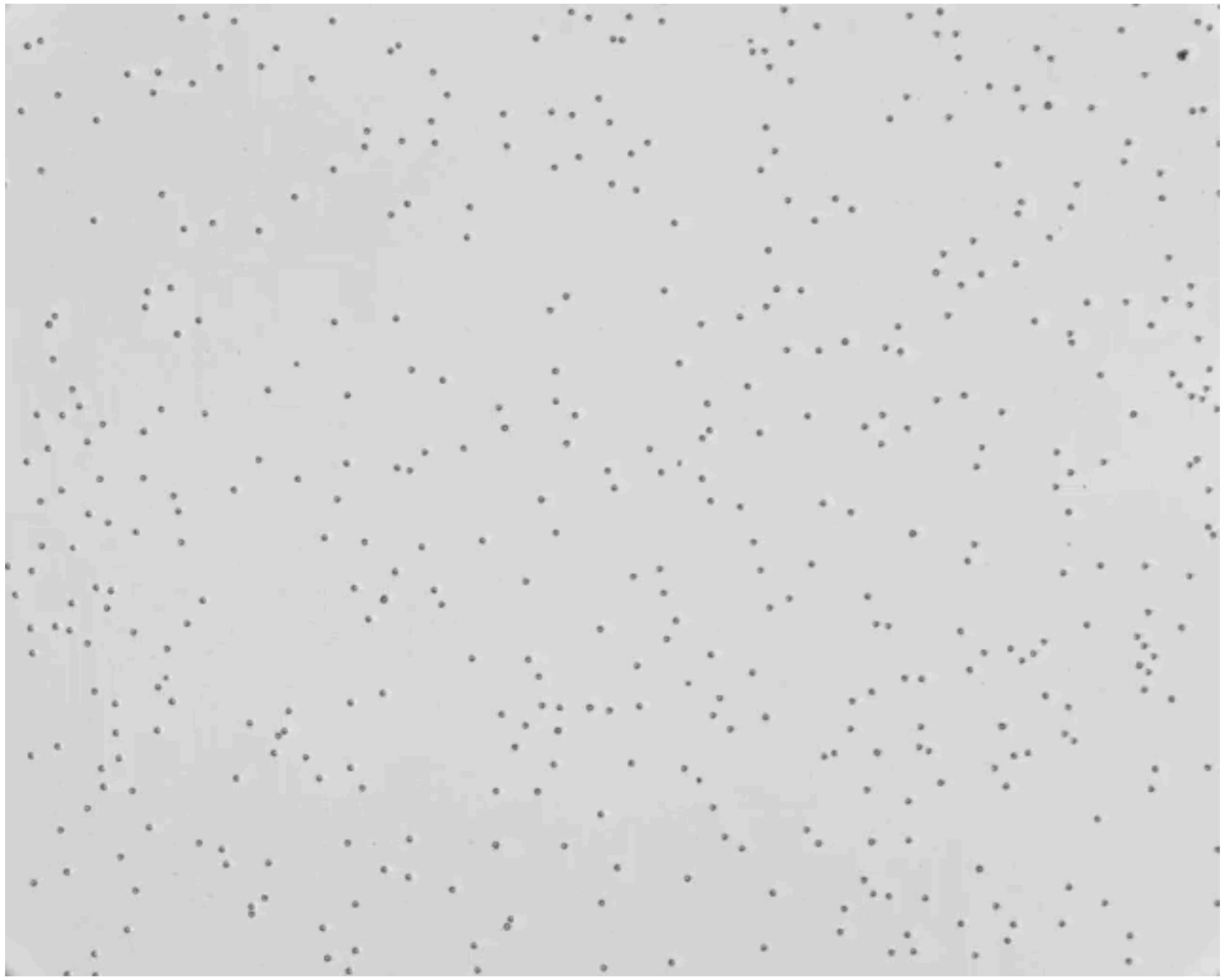

Effective Vicsek like alingment interaction

# Introduction

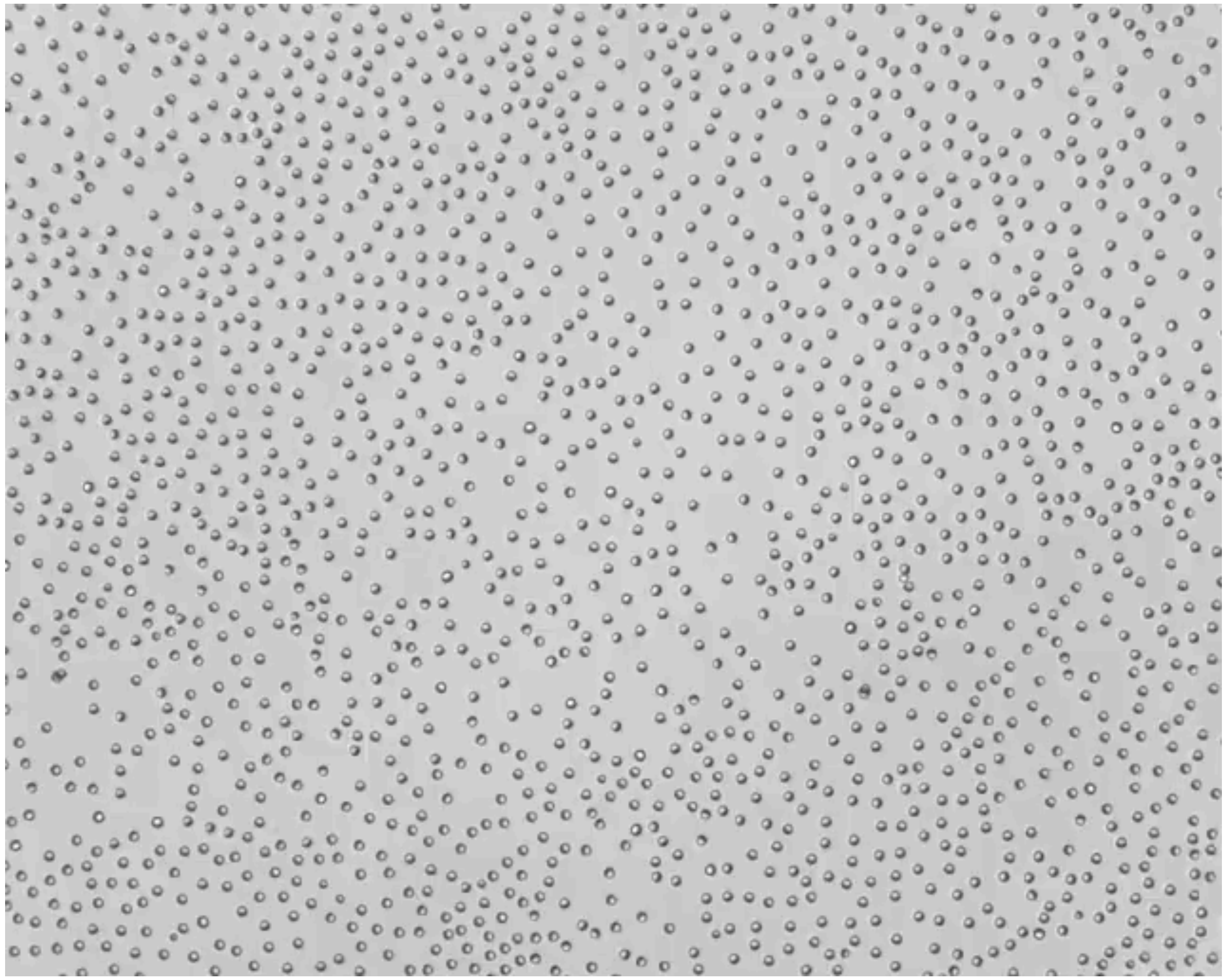
## Vicsek like model in experiment

**Janus particles in AC electric field**

Low density, disordered phase



High density, ordered phase

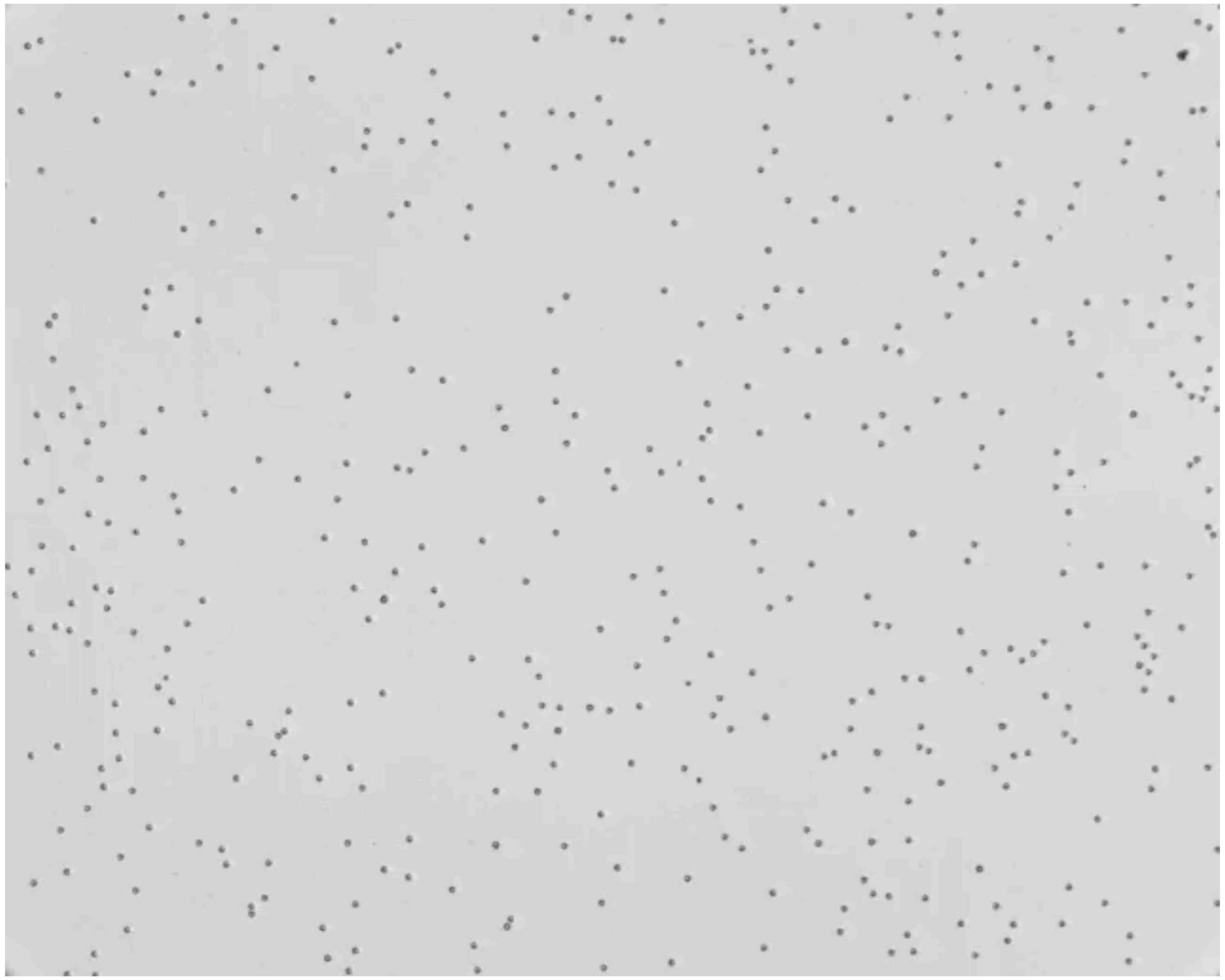


Iwasawa, Nishiguchi, and Sano (2021)

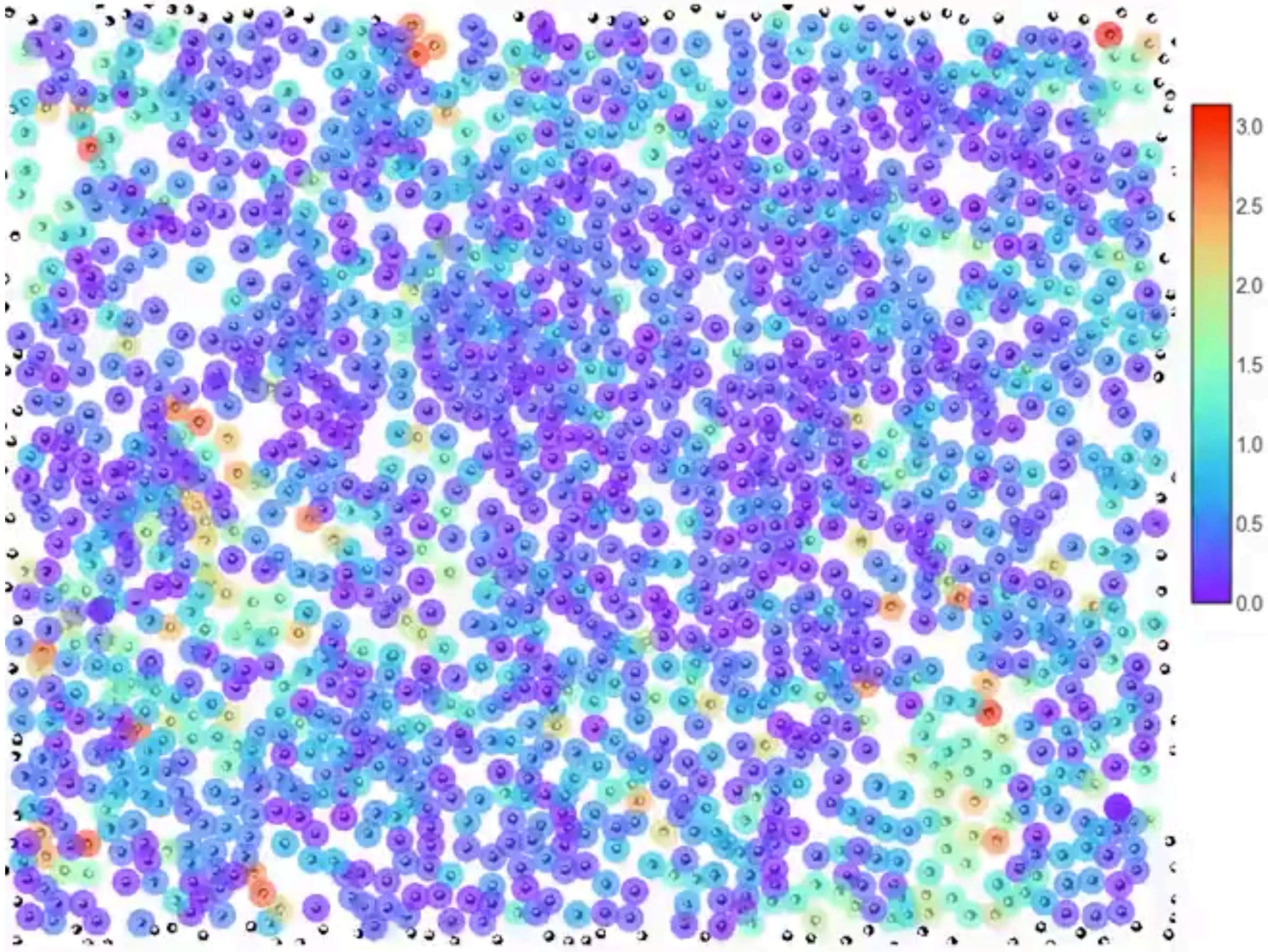
## Vicsek like model in experiment

**Janus particles in AC electric field**

Low density, disordered phase



High density, ordered phase




Iwasawa, Nishiguchi, and Sano (2021)

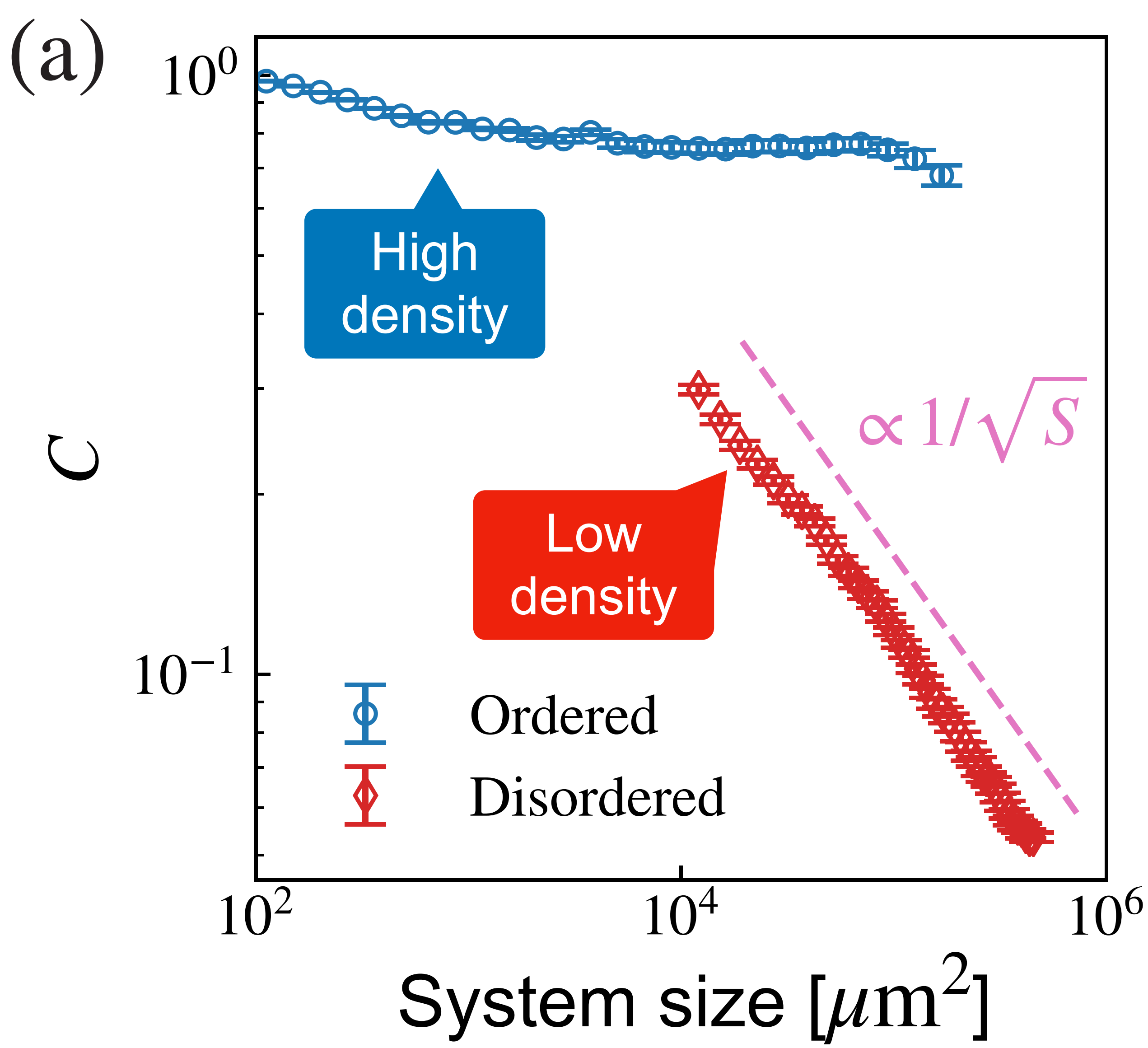
# Introduction

## Vicsek like model in experiment

Iwasawa, Nishiguchi, and Sano (2021)

XY model like
Order paramter

$$C = \langle \left| \langle e^{i\theta} \rangle_S \right| \rangle_t$$




True long-range order emerge even in d=2!!!

# Overview

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- Previous research: Toner-Tu theory
- Model
- Theory
- Comparison with numerical simulations
- Summary and discussions

# Overview

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# Previous reserach: Toner-Tu thoery

## Hydrodynamic description

### Question

Why does the Vicsek model exhibit the continuous symmetry breaking even in d=2?

**To answer, this question, Toner and Tu constructed and investigated a coarse grained hydrodynamic equation for the Vicsek model**

# Previous reserach: Toner-Tu thoery

## Hydrodynamic description

**Slow (hydrodynamic) variables of the Vicsek Model**

$\rho(\boldsymbol{x}, t)$: local density, $\boldsymbol{v}(\boldsymbol{x}, t)$: local velocity

### Navier-Stokes equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \kappa \nabla^2 \boldsymbol{v} - \frac{1}{\rho} \nabla P$$

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \boldsymbol{v})$$

Pressure
$P = f(\rho, \boldsymbol{v})$

**Standard model to describe hydrodynamic variables.**

### O(n) model

$$F(\boldsymbol{\phi}) = \int d\boldsymbol{x} \left[ a \frac{|\boldsymbol{\phi}|^2}{2} + b \frac{|\boldsymbol{\phi}|^4}{4} \right]$$

$$\boldsymbol{\phi} = \begin{pmatrix} \phi \cos\theta \\ \phi \sin\theta \end{pmatrix}$$

**Standard model to describe Continuous symmetry breaking**

# Previous reserach: Toner-Tu thoery

## Toner-Tu hydrodynamic description

**Navier-Stokes equation**

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \kappa \nabla^2 \boldsymbol{v} - \frac{1}{\rho}\nabla P$$

**O(n) model**

$$F(\boldsymbol{\phi}) = \int d\boldsymbol{x} \left[ a\frac{|\boldsymbol{\phi}|^2}{2} + b\frac{|\boldsymbol{\phi}|^4}{4} \right]$$

**Toner-Tu equation**

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \kappa \nabla^2 \boldsymbol{v} - \frac{1}{\rho}\nabla P - \gamma \frac{\delta F[\boldsymbol{v}]}{\delta \boldsymbol{v}} + \boldsymbol{\xi}$$

Noise

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{v})$$

Toner and Tu (1995)

# Previous reserach: Toner-Tu thoery

## Scaling behaviors

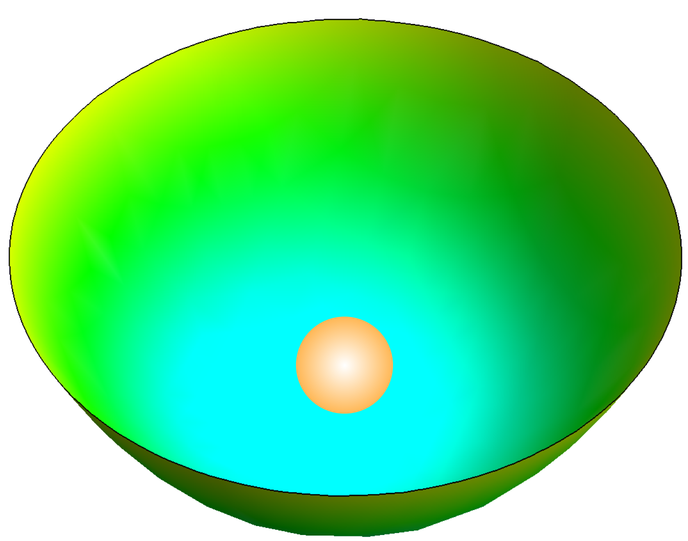
**Toner-Tu equation** Toner and Tu (1995)

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \kappa \nabla^2 \boldsymbol{v} - \frac{1}{\rho}\nabla P - \gamma \frac{\delta F[\boldsymbol{v}]}{\delta \boldsymbol{v}} + \boldsymbol{\xi}$$

$$F[\boldsymbol{v}] = \int d\boldsymbol{x} \left[ \frac{a}{2} |\boldsymbol{v}|^2 + \frac{b}{4} |\boldsymbol{v}|^4 \right]$$

**Disordered phae**

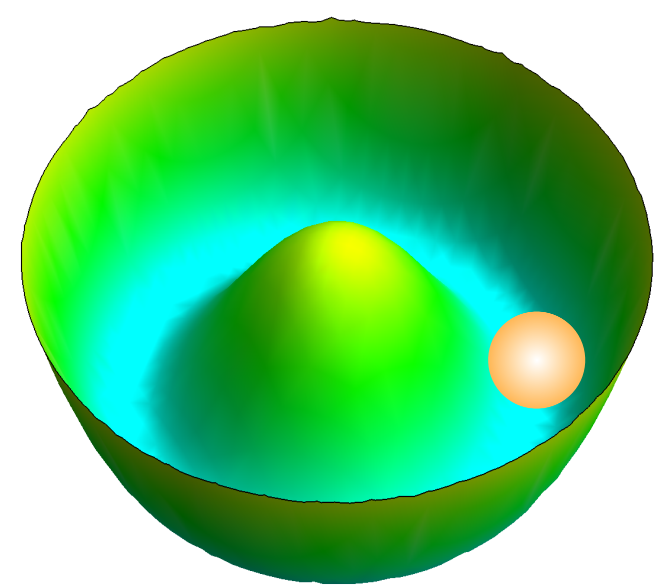
$a > 0$



$\langle \boldsymbol{v} \rangle = 0$

**Ordered phae**

$a < 0$



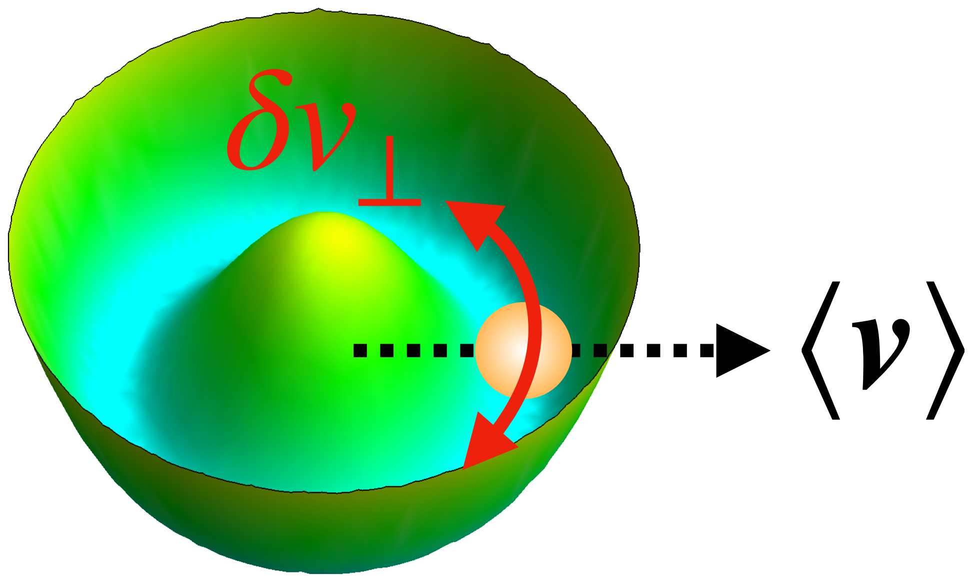
$\langle \boldsymbol{v} \rangle \neq 0$

# Previous reserach: Toner-Tu thoery

## Scaling behaviors

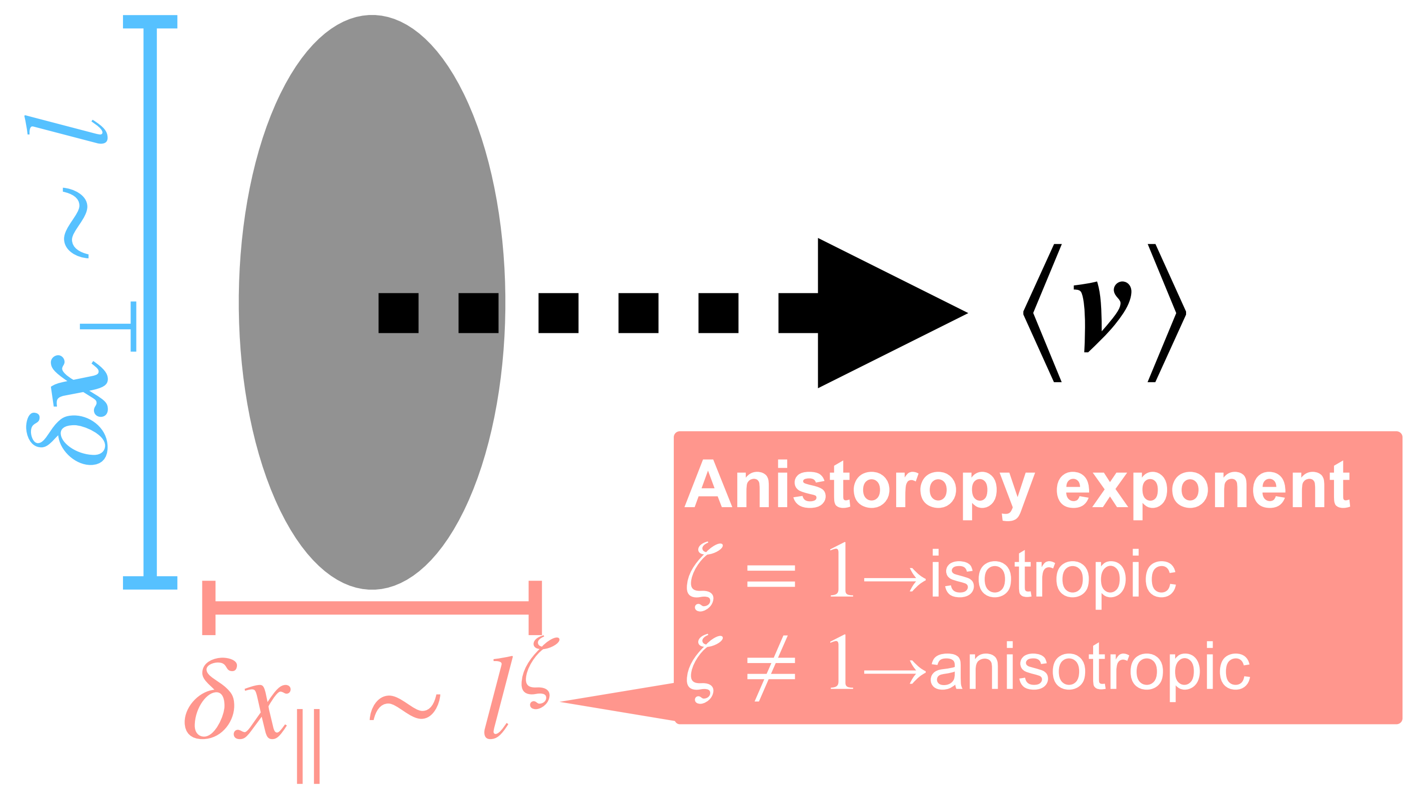
### Ordered phae

$a < 0$




$\delta v_\perp$: The velocity component orthogonal to the mean-veloctiy $\langle \boldsymbol{v} \rangle$ is the NG mode.

### Fluctuation of NG mode in the ordered phae




### Anisotropic scaling transformations

$$\boldsymbol{x}_\perp \to l\boldsymbol{x}_\perp,\ x_\parallel \to l^\zeta x_\parallel,\ t \to l^z t,\ \delta\boldsymbol{v}_\perp \to l^\chi \delta\boldsymbol{v}_\perp$$

# Previous reserach: Toner-Tu thoery

## Scaling behaviors

**Toner-Tu equation** Toner and Tu (1995)

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \kappa \nabla^2 \boldsymbol{v} - \frac{1}{\rho} \nabla P - \gamma \frac{\delta F[\boldsymbol{v}]}{\delta \boldsymbol{v}} + \boldsymbol{\xi}$$

**Anisotropic scaling transformations**

$$\boldsymbol{x}_\perp \to l\boldsymbol{x}_\perp,\ x_\parallel \to l^\zeta x_\parallel,\ t \to l^z t,\ \delta\boldsymbol{v}_\perp \to l^\chi \delta\boldsymbol{v}_\perp$$

**Renormalization group Calculations**

**Scaling exponents (TT95)**

$$z = \frac{2(1+d)}{5},\ \zeta = \frac{d+1}{5},\ \chi = \frac{3-2d}{5}$$

**Fluctuation**

$$\chi < 0 \text{ for } d > 3/2$$

$$\langle v_\perp^2 \rangle \sim l^{2\chi} \xrightarrow{l\to\infty} \begin{cases} \infty & d \le 3/2 \\ 0 & d > 3/2 \end{cases}$$

Long range order can exist in d=2

# Previous reserach: Toner-Tu thoery

## Comarison with numerical simulation

### Scaling exponents (TT95)

$$z = \frac{2(1+d)}{5},\ \zeta = \frac{d+1}{5},\ \chi = \frac{3-2d}{5}$$

### Comparison with simulation

B. Mahault *et al.* (2019)

| | | $d=2$ Vicsek | $d=2$ TT95 | $d=3$ Vicsek | $d=3$ TT95 |
|---|---|---|---|---|---|
| $\boldsymbol{v}_\perp$ | $\chi$ | -0.31(2) | -0.2 | -0.62 | -0.6 |
| $x_\parallel$ | $\zeta$ | 0.95(2) | 0.6 | 1 | 0.8 |
| $t$ | $z$ | 1.33(2) | 1.2 | 1.77 | 1.6 |

Numerical results of the Vicsek model are inconsistent with TT95.
In particular, the numerical results suggest the (almost) isotropic scaling $\zeta \fallingdotseq 1$

# Previous reserach: Toner-Tu thoery

## What was the problem?

In 2012, J. Toner, one of the authoer of the Toner-Tu thery, performed reanalysis of the hydrodynamic equation. J. Toner (2012)

**The most general EOM for slow variables allowed by symmetry**

$$\partial_t \vec{v}_\perp + \gamma \partial_\parallel \vec{v}_\perp + \lambda_1^0 (\vec{v}_\perp \cdot \vec{\nabla}_\perp)\vec{v}_\perp$$
$$= -g_1 \delta\rho \partial_\parallel \vec{v}_\perp - g_2 \vec{v}_\perp \partial_\parallel \delta\rho - \frac{c_0^2}{\rho_0} \vec{\nabla}_\perp \delta\rho - g_3 \vec{\nabla}_\perp (\delta\rho^2)$$
$$+ D_B \vec{\nabla}_\perp (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + D_T \nabla_\perp^2 \vec{v}_\perp + D_\parallel \partial_\parallel^2 \vec{v}_\perp$$
$$+ \nu_t \partial_t \vec{\nabla}_\perp \delta\rho + \nu_\parallel \partial_\parallel \vec{\nabla}_\perp \delta\rho + \vec{f}_\perp \qquad (2.18)$$

$$\partial_t \delta\rho + \rho_o \vec{\nabla}_\perp \cdot \vec{v}_\perp + w_1 \vec{\nabla}_\perp \cdot (\vec{v}_\perp \delta\rho) + v_2 \partial_\parallel \delta\rho$$
$$= D_{\rho_\parallel} \partial_\parallel^2 \delta\rho + D_{\rho_\perp} \nabla_\perp^2 \delta\rho + D_{\rho v} \partial_\parallel (\vec{\nabla}_\perp \cdot \vec{v}_\perp)$$
$$+ \phi \partial_t \partial_\parallel \delta\rho + w_2 \partial_\parallel (\delta\rho^2) + w_3 \partial_\parallel (|\vec{v}_\perp|^2),$$

Preivously overlooked relevant terms

**In parinciple, this equation probides the correct scaling exponents.**

However the EOM is so complicated…

# Previous reserach: Toner-Tu thoery

## Historical summary

- 1995: Vicsek model (Vicsek *et al.*)
- 1995: Toner-Tu theory (Toner and Tu)
- 2012: Re-analysis revealed some missing terms in TT95 (Toner)
- 2019: Extensive numerical simulation ($N \sim 10^9$) of the Vicsek model reported different values of scaling exponents (Mahault *et al.* )
- **2024: Correct scaling exponents (this work)**

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# Model

## Origin of the discrepancy between TT95 and simulation

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**The most general EOM for slow variables allowed by symmetry**

$$\partial_t \vec{v}_\perp + \gamma \partial_\parallel \vec{v}_\perp + \lambda_1^0 (\vec{v}_\perp \cdot \vec{\nabla}_\perp) \vec{v}_\perp$$
$$= -g_1 \delta\rho \partial_\parallel \vec{v}_\perp - g_2 \vec{v}_\perp \partial_\parallel \delta\rho - \frac{c_0^2}{\rho_0} \vec{\nabla}_\perp \delta\rho - g_3 \vec{\nabla}_\perp (\delta\rho^2)$$
$$+ D_B \vec{\nabla}_\perp (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + D_T \nabla_\perp^2 \vec{v}_\perp + D_\parallel \partial_\parallel^2 \vec{v}_\perp$$
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$$+ \phi \partial_t \partial_\parallel \delta\rho + w_2 \partial_\parallel (\delta\rho^2) + w_3 \partial_\parallel (|\vec{v}_\perp|^2),$$

Preivously overlooked relevant terms

In parinciple, this equation probides the exact scaling exponents. However the EOM is so complicated that we can not solve exactly.

**Here instead, we porpose a simple phenomenological equation of motions (EOM) for slow modes.**

# Model

## Equations for slow variables in d=2

### Slow variables

Velocity fluctuations perpendicular to $\langle \boldsymbol{v} \rangle$

$$\delta v_\perp(\boldsymbol{x}, t), \quad \delta\rho(\boldsymbol{x}, t)$$

Density fluctuations

**First order gradient expansion for equation of motion**

$$\begin{pmatrix} \delta\dot{\rho} \\ \delta\dot{v}_\perp \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \begin{pmatrix} \nabla\delta\rho \\ \nabla\delta v_\perp \end{pmatrix} + \cdots$$

Diagonalization
in the Fourier space

**Eigenmodes** Tonear and Tu (1995, 1998)

$$\dot{\tilde{u}}_\pm(\boldsymbol{q}, t) = iqc_\pm(\hat{q})\tilde{u}_\pm(\boldsymbol{q}, t) + O(q^2, u^2)$$

Non-degenerated sound velocities $c_+(\hat{q}) \neq c_-(\hat{q})$

## Eigenmodes

Tonear and Tu (1995, 1998)

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$$\dot{\tilde{u}}_\pm(\boldsymbol{q},t) = iqc_\pm(\hat{q})\tilde{u}_\pm(\boldsymbol{q},t) + O(q^2, u^2)$$

Non-degenerated sound velocities $c_+(\hat{q}) \neq c_-(\hat{q})$

**Interaction representation** $\tilde{u}_\pm(\boldsymbol{q},t) = u_\pm(\boldsymbol{q},t)e^{iqc_\pm t}$

## Spatial correlation

$$C_{\alpha\beta}(\boldsymbol{x}) = \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\langle \tilde{u}_\alpha(\boldsymbol{q},t)\tilde{u}^*_\beta(\boldsymbol{q},t)\rangle = \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\langle u_\alpha(\boldsymbol{q},t)u^*_\beta(\boldsymbol{q},t)\rangle e^{iq(c_\alpha - c_\beta)t}$$

Fast oscillation

The correlation decouples for difference modes $\alpha \neq \beta$, due to the fast oscillation of $e^{iq(c_\alpha - c_\beta)t}$, when $c_\alpha \neq c_\beta$

$u_+$ and $u_-$ can be treated separately!

Chaté and Solon (2024)

**We shall construct a EOM for $u_\pm$, separately.**

# Model

## Phenomelorogical EOM for $u_{\pm}$

**We shall construct a EOM for $u_{\pm}$, separately.**

# Model

## Phenomelorogical EOM for $u_\pm$

**We shall construct a EOM for $u_\pm$, separately.**

- Since $u_\pm$ have the same symmetry, it is sufficient to consider one of them. Hereafter, we omit the subscritp $u_\pm \to u$
- We must take into account ALL the relevant terms allowed by symmetry.

### The most general EOM

$$\dot{u} = \nabla^2 u + u a \cdot \nabla u + \xi$$

Noise

Advection $c \cdot \nabla u$ do not appear in the interaction representation.

The lowest-order non-linearlity

(Higher order terms $\ddot{u}, \nabla^3 u, (\nabla u)^2, \cdots$ are irrelevant )

# Overview

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# Theory

## Linear analysis

We first investigate the linear model
to revisit the derivation of the scaling exponents.

**The most general EOM**

$$\dot{u} = \nabla^2 u + u\boldsymbol{a} \cdot \nabla u + \xi$$

Neglect non-linear term

$$u\boldsymbol{a} \cdot \nabla u$$

**Linear model**

$$\dot{u} = \nabla^2 u + \xi$$

# Linear model

$$\dot{u} = \nabla^2 u + \xi$$

# Linear model

$$\dot{u} = \nabla^2 u + \xi$$

**Anistoropy exponent**
$\zeta = 1$→isotropic
$\zeta \neq 1$→anisotropic

## Scaling transformation

$$\boldsymbol{x}_\perp \to l\boldsymbol{x}_\perp,\ x_\parallel \to l^\zeta x_\parallel,\ t \to l^z t,\ u \to l^\chi u$$

$$l^{\chi-z}\dot{u} = l^{\chi-2}\nabla_\perp^2 u + l^{\chi-2\zeta}\partial_\parallel^2 u + l^{-(d-1+\zeta+z)/2}\xi$$

**Scaling relations** $\chi - z = \chi - 2 = \chi - 2\zeta = -(d-1+\zeta+z)/2$

## Scaling exponents

$$\zeta = 1,\ z = 2,\ \chi = \frac{2-d}{2}$$

Fluctuations destroy long-range order in $d \leq 2$

$$\langle u^2 \rangle \sim l^{2\chi} \xrightarrow{l\to\infty} \begin{cases} 0 & d > 2 \\ \infty & d \leq 2 \end{cases}$$

The lower-critical dimension of the linear model is $d_{\mathrm{low}} = 2$

**What will happen for non-linear model?**

# Theory

## Vicsek Model

### Non-linear EOM

$$\dot{u} = \nabla^2 u + u\boldsymbol{a} \cdot \nabla u + \xi$$

### Pseudo-Galilean invariance

$$u \to u + U, \; \boldsymbol{x} \to \boldsymbol{x} + U\boldsymbol{a}t$$

$\boldsymbol{x} = \{x_\parallel, \boldsymbol{x}_\perp\}$ and $ut$

have the same scaling dimension

$$[x_\perp] = [ut] \to \; 1 = \chi + z$$

$$[x_\parallel] = [ut] \to \; \zeta = \chi + z$$

### Non-linear term

$$u\boldsymbol{a} \cdot \nabla u = \nabla \cdot (\boldsymbol{a}u^2/2)$$

**Perterbative RG** $\to \nabla$ (**Some functions**)

does not affect the scaling exponents of the terms without ∇

$$\dot{u} \sim l^{\chi - z}, \; \xi \sim l^{-(d-1+\zeta+z)/2}$$

$$\chi - z = -(d-1+\zeta+z)/2$$

# Theory

## Vicsek Model

### Scaling relations

$$1 = \chi + z \quad \zeta = \chi + z \quad \chi - z = -(d - 1 + \zeta + z)/2$$

## Vicsek Model

### Scaling relations

$$1 = \chi + z \quad \zeta = \chi + z \quad \chi - z = -(d - 1 + \zeta + z)/2$$

**New Scaling exponents**

$$\zeta = 1,\ z = \frac{2+d}{3},\ \chi = \frac{1-d}{3}$$

**Different from TT95**

Fluctuations destroy long-range order in $d \leq 1$

$$\langle u^2 \rangle \sim l^{2\chi} \xrightarrow{l \to \infty} \begin{cases} \infty & d \leq 1 \\ 0 & d > 1 \end{cases}$$

The non-linear term stabilizes the long-range order even in d=2.

**We get new scaling exponents distrinct from the Toner-Tu theory.**

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# Comparison with numerical simulation

## Scaling exponents

B. Mahault, F. Ginelli, and H. Chaté (2019)

| | | $d=2$ Vicsek | TT95 | This work | $d=3$ Vicsek | TT95 | This work |
|---|---|---|---|---|---|---|---|
| $u$ | $\chi$ | -0.31(2) | -0.2 | -0.33 | -0.62 | -0.6 | -0.67 |
| $x_{\parallel}$ | $\zeta$ | 0.95(2) | 0.6 | 1 | 1 | 0.8 | 1 |
| $t$ | $z$ | 1.33(2) | 1.2 | 1.33 | 1.77 | 1.6 | 1.67 |

- **In d=2, we get almost perfect agreement.**
- **In d=3, the both theories work well, and it is difficult to judge which theory works better.**

# Comparison with numerical simulation

## Correlation function

### Scalings

$$\boldsymbol{x}_\perp \sim l,\ x_\parallel \sim l^\zeta,\ t \sim l^z,\ u \sim l^\chi$$

### Correlation function in real space

$$C(\boldsymbol{x}_\perp, x_\parallel) = = \langle u(\boldsymbol{x}_\perp, x_\parallel) u(\boldsymbol{0},0)\rangle = l^{2\chi} C(l^{-1}\boldsymbol{x}_\perp, l^{-\zeta}x_\parallel) = \begin{cases} |\boldsymbol{x}_\perp|^{2\chi} & |\boldsymbol{x}_\perp| \gg x_\parallel \\ |x_\parallel|^{2\chi/\zeta} & |\boldsymbol{x}_\perp| \ll x_\parallel \end{cases}$$

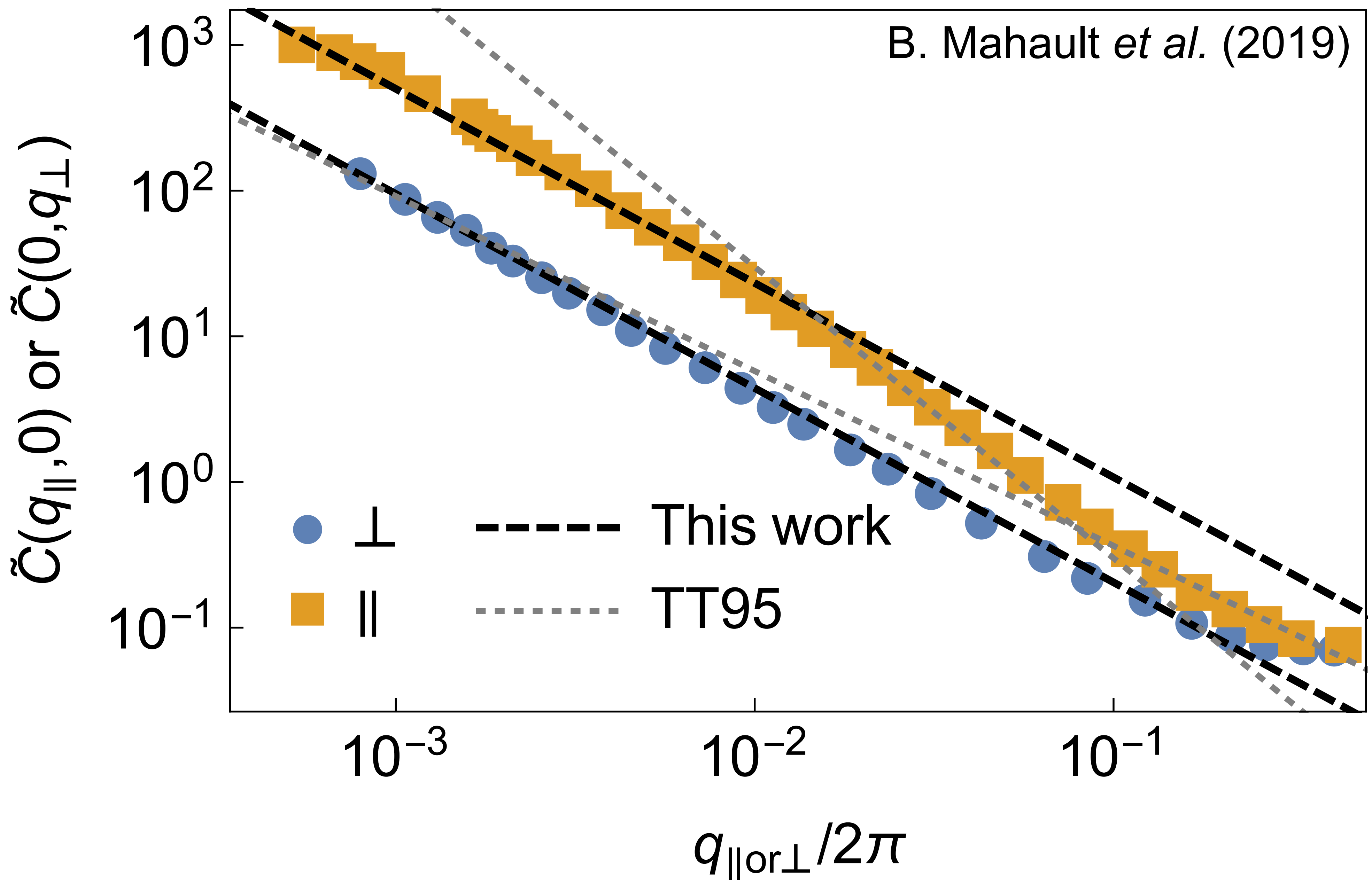
### Correlation function in Fourier space

$$\tilde{C}(\boldsymbol{q}_\perp, q_\parallel) = \begin{cases} |\boldsymbol{q}_\perp|^{-z} & |\boldsymbol{q}_\perp| \gg q_\parallel \\ |q_\parallel|^{-z/\zeta} & |\boldsymbol{q}_\perp| \ll q_\parallel \end{cases}$$

# Comparison with numerical simulation

## Correlation function

### Velocity correlation in d=2

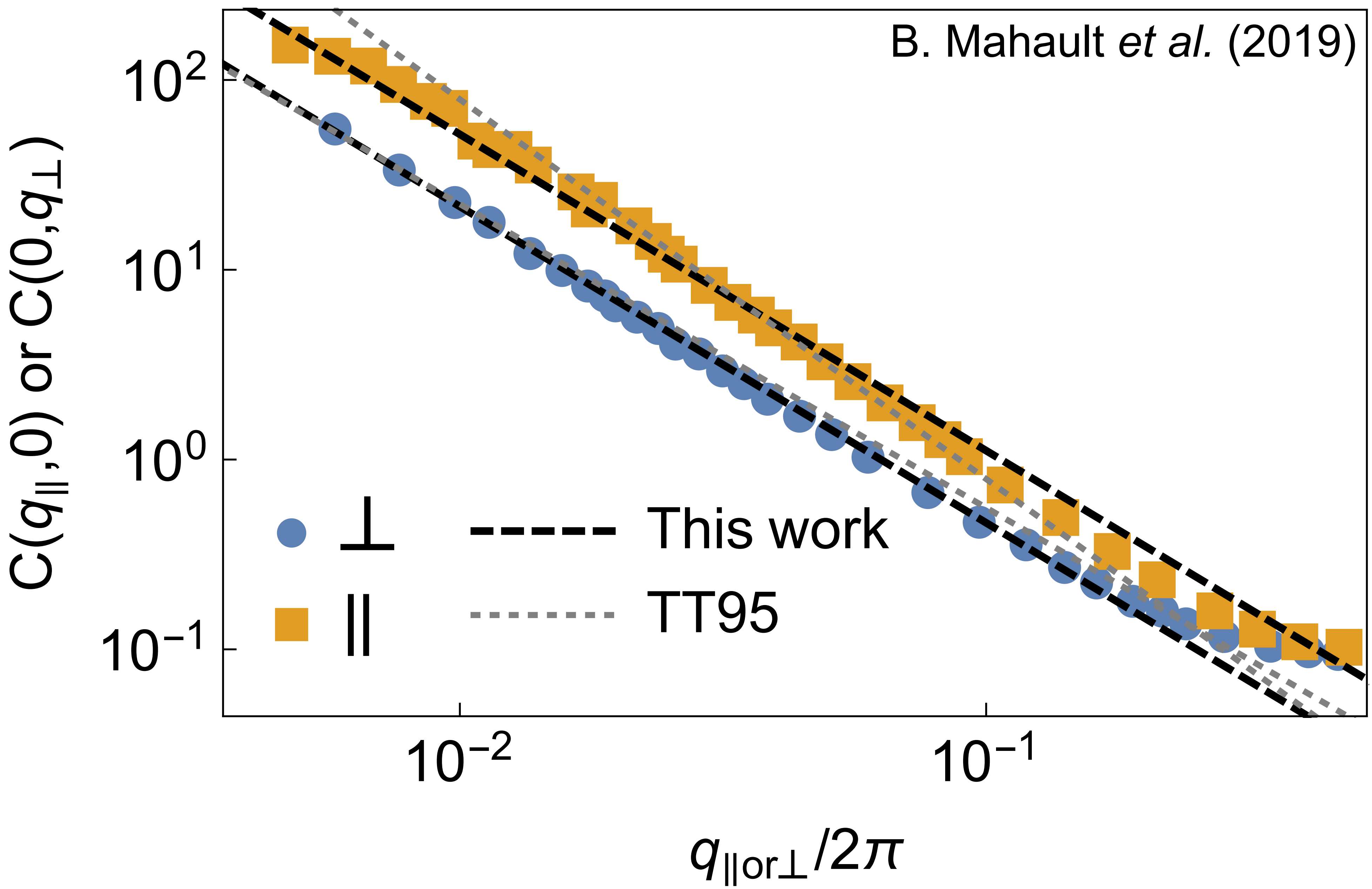



**Our theory works better than the Toner-Tu theory (1995).**

# Comparison with numerical simulation

## Correlation function

### Velocity correlation in d=3




**Both theories work well in d=3.**

Further studies are necessary to judge the correct theory.

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# Discussion

## Connection with the Toner-Tu theory

Here we brifely revisit the derivation of the scaling exponents of the Toner-Tu model.

### Non-linear EOM

$$\dot{u} = \nabla^2 u + u\boldsymbol{a}_\perp \cdot \nabla_\perp u + u a_\parallel \partial_\parallel u + \xi$$

### Slow mode for Toner-Tu theory

The Toner-Tu theory neglects (non-linear) coupling between velocity and density fluctuations. Toner (2012)

$$\Rightarrow \quad u \propto \boldsymbol{v}_\perp$$

Velocity perpendicular to the order parameter $\langle \boldsymbol{v} \rangle$

### Symmetry

$$\boldsymbol{v}_\perp(-\boldsymbol{x}_\perp) = -\boldsymbol{v}_\perp(\boldsymbol{x}_\perp) \quad \Rightarrow \quad u\boldsymbol{a}_\perp \cdot \nabla_\perp u + \cancel{u a_\parallel \partial_\parallel u}$$

Prohibited by symmetry

## Non-linear EOM

$$\dot{u} = \nabla^2 u + u\boldsymbol{a}_\perp \cdot \nabla_\perp u + u a_\parallel \partial_\parallel u + \xi$$

Prohibited by symmetry

## Non-linear EOM

$$\dot{u} = \nabla^2 u + u\boldsymbol{a}_\perp \cdot \nabla_\perp u + u a_\parallel \partial_\parallel u + \xi$$

Prohibited by symmetry

## Galilean invariance

$$u \to u + U,\ \boldsymbol{x}_\perp \to \boldsymbol{x}_\perp + U\boldsymbol{a}_\perp t$$

$\boldsymbol{x}_\perp$ and $ut$
have the same scaling dimension

$$1 = \chi + z$$

## Conservative non-linear term

$$u\boldsymbol{a}_\perp \cdot \nabla_\perp u = \nabla_\perp \cdot (\boldsymbol{a}_\perp u^2/2)$$

This term does not affect the scaling exponents of the terms without $\nabla_\perp$

$$\dot{u} \sim l^{\chi - z},\ \partial_\parallel^2 u \sim l^{\chi - 2\zeta},\ \xi \sim l^{-(d+z)/2}$$

$$\chi - z = \chi - 2\zeta = -(d - 1 + \zeta + z)/2$$

# Discussion

## Connection with the Toner-Tu theory

### Scaling relations

$$1 = \chi + z \qquad \chi - z = \chi - 2\zeta = -(d - 1 + \zeta + z)/2$$

# Discussion

## Connection with the Toner-Tu theory

### Scaling relations

$$1 = \chi + z \qquad \chi - z = \chi - 2\zeta = -(d - 1 + \zeta + z)/2$$

**Scaling exponents**

$$\zeta = \frac{1+d}{5},\ z = \frac{2(1+d)}{5},\ \chi = \frac{3-2d}{5}$$

**Consisten with TT95**

Fluctuations destroy long-range order in $d \leq 3/2$

$$\langle u^2 \rangle \sim l^{2\chi} \sim \begin{cases} \text{finite} & d > 3/2 \\ \infty & d \leq 3/2 \end{cases}$$

The non-linear term stabilizes the long-range order even in d=2.

**What will happen for Vicsek Model, where the density fluctutaions couple with velocity fluctutaions?**

# Discussion

## Connection with the Toner-Tu theory

### Non-linear EOM

$$\dot{u} = \nabla^2 u + u\boldsymbol{a} \perp \cdot \nabla_\perp u + u a_\parallel \partial_\parallel u + \xi$$

### Slow mode for the Vicsek Model

$u$ is a mixed mode between $\delta\rho$ and $\boldsymbol{v}_\perp$

### ~~Symmetry~~

$$\boldsymbol{v}(-\boldsymbol{x}_\perp) = -\boldsymbol{v}_\perp(\boldsymbol{x}_\perp)$$

There is no-reason to prohibit $u a_\parallel \partial_\parallel u$

**When constructing phenomenological EOM, it is important to consider ALL relevant terms allowed by symmetry.**

# Summary

- We constructed a phenomenological EOM for the slow modes of the Vicsek Model and calculated the scaling exponents.
- Our scaling exponents well agree with the numericla simulation of the Vicsek Model in d=2.
- Our theory and TT95 equally well fit the numerical data in d=3.