The integrated perturbation theory for tensor fields

Refs.) TM, arXiv:2210.10435 (Paper I, basic formulations)
arXiv:2210.11085 (Paper II, loop corrections)
in prep. (Paper III, projection effects)

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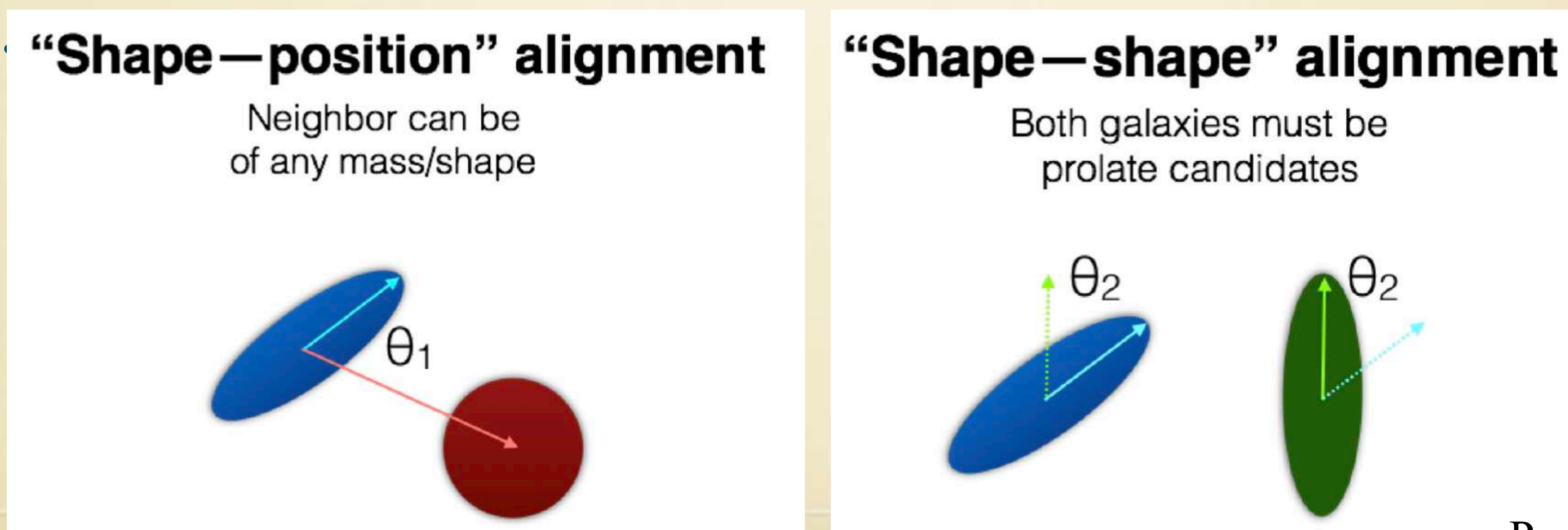
Intrinsic alignment of galaxies

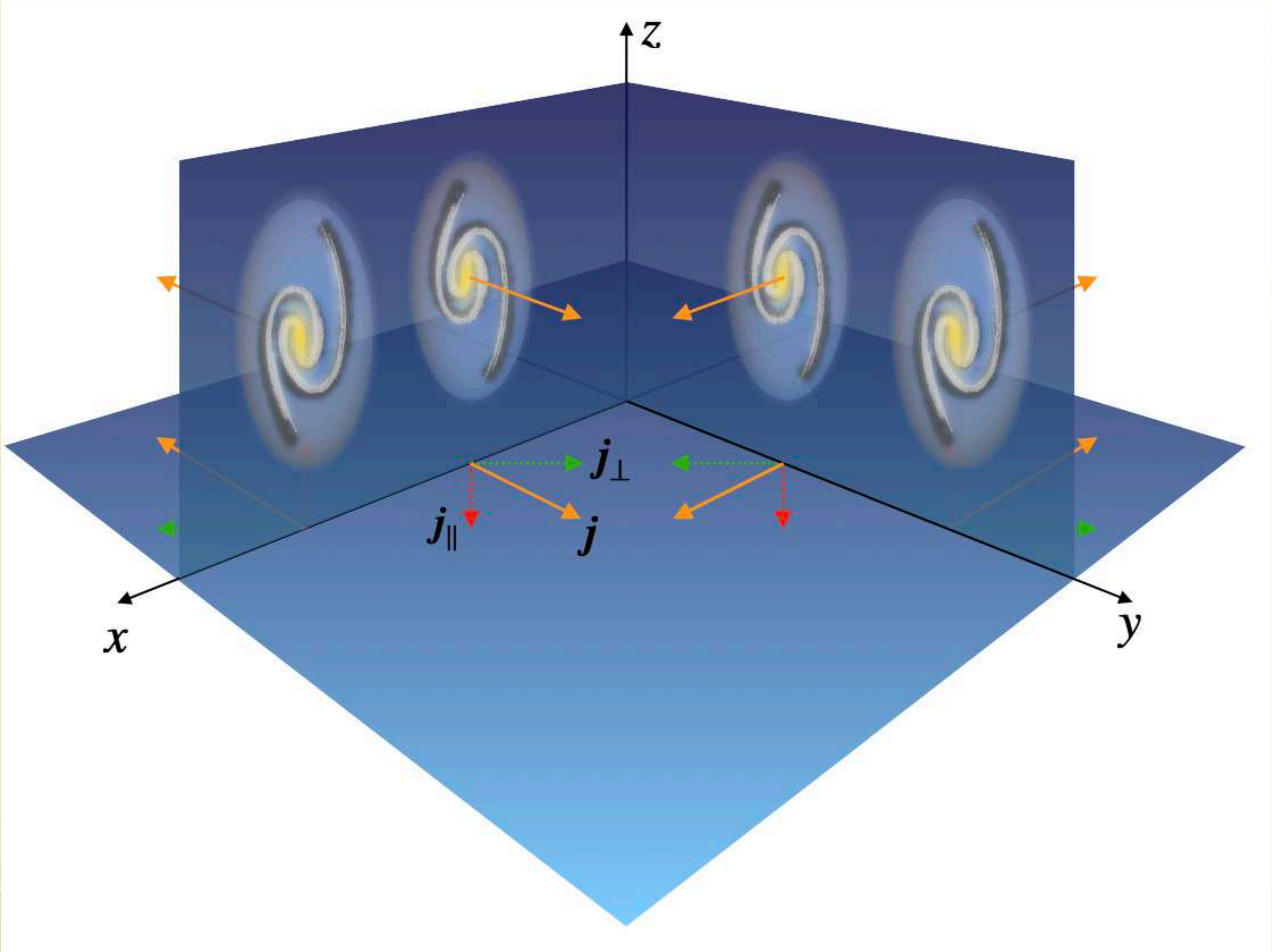
- Nonlinear phenomena:
  - The galaxy alignments are considered as results of complicated, nonlinear galaxy evolution precesses
- The spatial patterns:
  - The alignments are statistically correlated to the initial condition of the Universe, and thus to the large-scale structure of the universe



Pandya+ 2019

Galaxy spins as a probe of vector field

- Galaxy shapes: rank-2 or higher tensor
- Galaxy spins: rank-1 tensor (vector)



Motloch+ 2020

Galaxy shapes as a tensor-valued bias

- Galaxy bias
    - Galaxy number density is also a result of complicated, nonlinear phenomena
        - ⇒ Scalar-valued bias
- Galaxy spins
        - ⇒ Vector-valued bias
- Galaxy shapes
        - ⇒ Tensor-valued bias
- We establish a general formalism to treat the tensor-valued cosmological field in perturbation theory
    - Greatly inspired by pioneering work by Vlah, Chisari & Schmidt 2020; 2021 using EFT and the spherical basis of rank-2 tensor for the shapes
    - Note: the present formalism fully respects rotational covariance, any rank of tensors, the integrated perturbation theory

Moments of galaxy shapes

- Galaxy shapes are characterized by the second moment of stellar density (luminosity)

$$I_{ij}(\boldsymbol{x}) = \frac{\int d^3x' \, (x'_i - x_i)(x'_j - x_j)\rho(\boldsymbol{x}')}{\int d^3x' \, \rho(\boldsymbol{x}')}$$

- The shape field of galaxies are defined by

$$I_{ij}(\boldsymbol{x}) = \sum_{a \in \mathrm{galaxies}} I_{ij}(\boldsymbol{x}_a)\delta^3_{\mathrm{D}}(\boldsymbol{x} - \boldsymbol{x}_a)$$

- Shape fluctuation field is defined by (traceless part)

$$S_{ij}(\boldsymbol{x}) = \frac{I_{ij}(\boldsymbol{x}) - \langle I_{ij} \rangle}{\mathrm{tr}\,\langle I_{..} \rangle} = \frac{I_{ij}(\boldsymbol{x})}{\bar{I}} - \frac{\delta_{ij}}{3}$$

Moments of galaxy shapes

- Similarly, one can define higher-order shape fields from higher-order moments (c.f. Kogai+ 2021)

$$I_{i_1\cdots i_l}(\boldsymbol{x}) = \frac{\int d^3x'\,(x'_{i_1} - x_{i_1})\cdots(x'_{i_l} - x_{i_l})\rho(\boldsymbol{x}')}{\int d^3x'\,\rho(\boldsymbol{x}')}$$

- The higher-order shape field

$$I_{i_i\cdots i_l}(\boldsymbol{x}) = \sum_{a\in\mathrm{galaxies}} I_{i_1\cdots i_l}(\boldsymbol{x}_a)\delta^3_{\mathrm{D}}(\boldsymbol{x} - \boldsymbol{x}_a)$$

  - (Normalization is arbitrary)

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Decomposition of tensors into the spherical basis

- Building blocks: the spherical basis (e.g., Varshalovich+ 1988)

$$\mathbf{e}_0 = \hat{\mathbf{e}}_3, \quad \mathbf{e}_\pm = \mp \frac{\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2}{\sqrt{2}}.$$

$$\mathbf{e}^0 \equiv \mathbf{e}_0^* = \hat{\mathbf{e}}_3, \quad \mathbf{e}^\pm \equiv \mathbf{e}_\pm^* = \mp \frac{\hat{\mathbf{e}}_1 \mp i\hat{\mathbf{e}}_2}{\sqrt{2}}.$$

- Metric tensor of spherical basis

$$g_{(l)}^{mm'} = g^{(l)}_{mm'} = (-1)^m \delta_{m,-m'}, \quad g_{(l)}^{mm''} g^{(l)}_{m''m'} = \delta^m_{m'}.$$

$$\mathbf{e}^m = g_{(1)}^{mm'} \mathbf{e}_{m'}, \quad \mathbf{e}_m = g^{(1)}_{mm'} \mathbf{e}^{m'}.$$

c.f.) Wigner's notation

$$(-1)^l g^{(l)}_{mm'} = \begin{pmatrix} & l & \\ m & & m' \end{pmatrix} = (-1)^l g_{(l)}^{mm'}.$$

- Definition of spherical basis of traceless tensors

$$\mathsf{Y}^{(0)} = \frac{1}{\sqrt{4\pi}}, \quad \mathsf{Y}_i^{(m)} = \sqrt{\frac{3}{4\pi}}\, \mathsf{e}^m{}_i$$

$$\mathsf{Y}_{ij}^{(0)} = \frac{1}{4}\sqrt{\frac{5}{\pi}} \left(3\mathsf{e}^0{}_i\, \mathsf{e}^0{}_j - \delta_{ij}\right),$$

$$\mathsf{Y}_{ij}^{(\pm 1)} = \frac{1}{2}\sqrt{\frac{15}{\pi}}\, \mathsf{e}^0{}_{(i}\, \mathsf{e}^\pm{}_{j)},$$

$$\mathsf{Y}_{ij}^{(\pm 2)} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\, \mathsf{e}^\pm{}_i\, \mathsf{e}^\pm{}_j$$

  - which satisfy, in general,

$$Y_{lm}(\theta, \phi) = \mathsf{Y}^{(m)*}_{i_1 i_2 \cdots i_l} n_{i_1} n_{i_2} \cdots n_{i_l}.$$

Decomposition of tensors into the spherical basis

- A symmetric tensor is uniquely decomposed into traceless tensors, e.g.,

$$T_{ij} = T^{(2)}_{ij} + \frac{1}{3}\delta_{ij}T^{(0)},$$

$$T_{ijk} = T^{(3)}_{ijk} + \frac{3}{5}\delta_{(ij}T^{(1)}_{k)},$$

$$T_{ijkl} = T^{(4)}_{ijkl} + \frac{6}{7}\delta_{(ij}T^{(2)}_{kl)} + \frac{1}{5}\delta_{(ij}\delta_{kl)}T^{(0)}.$$

- A traceless tensor is decomposed into the spherical basis:

$$T^{(l)}_{i_1 i_2 \cdots i_l} = i^l \frac{4\pi\, l!}{(2l+1)!!} T_{lm} \mathsf{Y}^{(m)}_{i_1 i_2 \cdots i_l},$$

$$T_{lm} = (-i)^l T^{(l)}_{i_1 i_2 \cdots i_l} \mathsf{Y}^{(m)*}_{i_1 i_2 \cdots i_l},$$

- Irreducibility: SO(3) rotation (R: passive rotation, D: Wigner's rotation matrix)

$$T_{lm} \to T'_{lm} = T_{lm'} D^{m'}_{(l)m}(R).$$

Irreducible decomposition of the tensor field

- For a given class of astronomical objects X, we define a Cartesian tensor field (n_X : mean number density of X)

$$F_{X\,i_1 i_2\cdots}(\boldsymbol{x}) = \frac{1}{\bar{n}_X}\sum_{a\in X} F^a_{i_1 i_2\cdots}\delta^3_{\mathrm{D}}(\boldsymbol{x}-\boldsymbol{x}_a)\,,$$

- Irreducible decomposition of the Cartesian tensor:

$$F_{Xlm}(\boldsymbol{x}) = \frac{1}{\bar{n}_X}\sum_{a\in X} F^a_{lm}\delta^3_{\mathrm{D}}(\boldsymbol{x}-\boldsymbol{x}_a)\,,$$

  - This tensor is weighted by the number density, it is useful to consider the unweighted field as well:

- $$F_{Xlm}(\boldsymbol{x}) = [1+\delta_X(\boldsymbol{x})]\,G_{Xlm}(\boldsymbol{x}),$$ $$1+\delta_X(\boldsymbol{x}) = \frac{1}{\bar{n}_X}\sum_{a\in X}\delta^3_{\mathrm{D}}(\boldsymbol{x}-\boldsymbol{x}_a)\,.$$

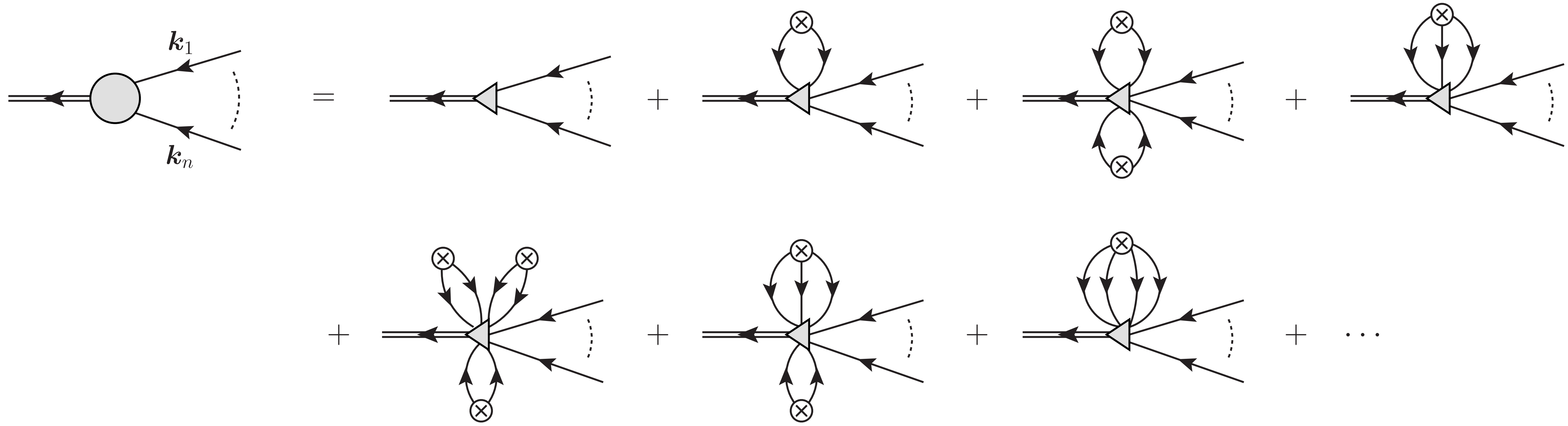
Multi-point propagator

TM (1995); Crocce & Scoccimarro (2006); Bernardeau et al. (2008); TM (2011)

- Define multi-point propagator of tensor field

$$\left\langle \frac{\delta^n \tilde{F}_{Xlm}(\boldsymbol{k})}{\delta\tilde{\delta}_{\rm L}(\boldsymbol{k}_1)\cdots\delta\tilde{\delta}_{\rm L}(\boldsymbol{k}_n)} \right\rangle = (2\pi)^{3-3n}\delta_{\rm D}^3(\boldsymbol{k}_1+\cdots+\boldsymbol{k}_n-\boldsymbol{k})\tilde{\Gamma}^{(n)}_{Xlm}(\boldsymbol{k}_1,\ldots\boldsymbol{k}_n),$$

$\Gamma^{(n)}_{\rm X}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n) =$



Renormalized bias functions

- The "renormalized bias function" is an essential piece in the iPT
  - Series of functions to characterize (nonlocal) biasing

$$c^{(n)}_{Xlm}(\boldsymbol{k}_1, \ldots \boldsymbol{k}_n) = (2\pi)^{3n} \int \frac{d^3k}{(2\pi)^3} \left\langle \frac{\delta^n F^{\mathrm{L}}_{Xlm}(\boldsymbol{k})}{\delta\delta_{\mathrm{L}}(\boldsymbol{k}_1) \cdots \delta\delta_{\mathrm{L}}(\boldsymbol{k}_n)} \right\rangle.$$

  - A counterpart of multi-point propagator for Lagrangian biasing
  - The models of the tensor field are generally represented by these functions

Decomposition of the renormalized bias functions into the spherical basis

- The decomposition is concisely represented by bipolar, tripolar spherical harmonics (Varshalovich+ 1988):

$$\left\{Y_{l_1}(\hat{\boldsymbol{k}}_1)\otimes Y_{l_2}(\hat{\boldsymbol{k}}_2)\right\}_{lm} \equiv (-1)^l\sqrt{2l+1}\,(l\,l_1 l_2)_m{}^{m_1 m_2}\,Y_{l_1 m_1}(\hat{\boldsymbol{k}}_1)Y_{l_2 m_2}(\hat{\boldsymbol{k}}_2).$$

$$\left\{Y_{l_1}(\hat{\boldsymbol{k}}_1)\otimes\left\{Y_{l_2}(\hat{\boldsymbol{k}}_2)\otimes Y_{l_3}(\hat{\boldsymbol{k}}_3)\right\}_L\right\}_{lm} = (-1)^{l+L}\sqrt{(2l+1)(2L+1)}\,(l\,l_1 L)_m{}^{m_1 M} \times (L l_2 l_3)_M{}^{m_2 m_3}\,Y_{l_1 m_1}(\hat{\boldsymbol{k}}_1)Y_{l_2 m_2}(\hat{\boldsymbol{k}}_2)Y_{l_3 m_3}(\hat{\boldsymbol{k}}_3).$$

  - A concise notation for 3j-symbols

$$(l_1\,l_2\,l_3)_{m_1 m_2 m_3} \equiv \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix},$$

$$(l_1\,l_2\,l_3)_{m_1}{}^{m_2 m_3} = g^{m_2 m_2'}_{(l_2)} g^{m_3 m_3'}_{(l_3)}\,(l_1\,l_2\,l_3)_{m_1 m_2' m_3'} = (-1)^{m_2+m_3}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & -m_2 & -m_3 \end{pmatrix},$$

- as

$$c^{(1)}_{Xlm}(\boldsymbol{k}_1) = c^{(1)}_{Xl}(k_1)Y_{lm}(\hat{\boldsymbol{k}}_1).$$

$$c^{(2)}_{Xlm}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \sum_{l_1,l_2} c^{(2)\,l}_{Xl_1 l_2}(k_1,k_2)\left\{Y_{l_1}(\hat{\boldsymbol{k}}_1)\otimes Y_{l_2}(\hat{\boldsymbol{k}}_2)\right\}_{lm}.$$

$$c^{(3)}_{Xlm}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \sum_{l_1,l_2,l_3,L} c^{(3)\,l;L}_{Xl_1 l_2 l_3}(k_1,k_2,k_3) \times \left\{Y_{l_1}(\hat{\boldsymbol{k}}_1)\otimes\left\{Y_{l_2}(\hat{\boldsymbol{k}}_2)\otimes Y_{l_3}(\hat{\boldsymbol{k}}_3)\right\}_L\right\}_{lm}.$$

Decomposition of the renormalized bias functions and propagators into the spherical basis

- Propagators (in real space) are similarly decomposed as

$$\hat{\Gamma}^{(1)}_{Xlm}(\boldsymbol{k}_1) = \hat{\Gamma}^{(1)}_{Xl}(k_1) Y_{lm}(\hat{\boldsymbol{k}}_1).$$

$$\hat{\Gamma}^{(2)}_{Xlm}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \sum_{l_1, l_2} \hat{\Gamma}^{(2)\,l}_{Xl_1l_2}(k_1, k_2) \left\{ Y_{l_1}(\hat{\boldsymbol{k}}_1) \otimes Y_{l_2}(\hat{\boldsymbol{k}}_2) \right\}_{lm}$$

$$\hat{\Gamma}^{(3)}_{Xlm}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \sum_{l_1, l_2, l_3, L} \hat{\Gamma}^{(3)\,l;L}_{Xl_1l_2l_3}(k_1, k_2, k_3) \times \left\{ Y_{l_1}(\hat{\boldsymbol{k}}_1) \otimes \left\{ Y_{l_2}(\hat{\boldsymbol{k}}_2) \otimes Y_{l_3}(\hat{\boldsymbol{k}}_3) \right\}_L \right\}_{lm}$$

- Redshift-space propagators: dependence on the line of sight is additionally present, and the decomposition is similar:

$$\hat{\Gamma}^{(1)}_{Xlm}(\boldsymbol{k}_1; \hat{\boldsymbol{z}}) = \sum_{l_z, l_1} \sqrt{\frac{4\pi}{2l_z + 1}} \hat{\Gamma}^{(1)\,l\,l_z}_{Xl_1}(k_1) \left\{ Y_{l_z}(\hat{\boldsymbol{z}}) \otimes Y_{l_1}(\hat{\boldsymbol{k}}_1) \right\}_{lm}.$$

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$$\hat{\Gamma}^{(2)}_{Xlm}(\boldsymbol{k}_1, \boldsymbol{k}_2; \hat{\boldsymbol{z}}) = \sum_{l_z, l_1, l_2, L} \sqrt{\frac{4\pi}{2l_z + 1}} \hat{\Gamma}^{(2)\,l\,l_z;L}_{Xl_1l_2}(k_1, k_2) \times \left\{ Y_{l_z}(\hat{\boldsymbol{z}}) \otimes \left\{ Y_{l_1}(\hat{\boldsymbol{k}}_1) \otimes Y_{l_2}(\hat{\boldsymbol{k}}_2) \right\}_L \right\}_{lm}$$

Sample calculations

- Results of lowest-order approximation (tree-level calculations)
  - Invariant 1st-order propagator
    - real space:

$$\hat{\Gamma}^{(1)}_{Xl}(k) = c^{(1)}_{Xl}(k) + 4\pi\, c^{(0)}_X \delta_{l0}.$$

    - redshift space:

$$\hat{\Gamma}^{(1)l\,l_z}_{Xl_1}(k) = c^{(1)}_{Xl}(k)\delta_{l_z0}\delta_{l_1l} + 4\pi\, c^{(0)}_X \delta_{l0}\left[\left(1+\frac{f}{3}\right)\delta_{l_z0}\delta_{l_10} + \frac{2f}{3}\delta_{l_z2}\delta_{l_12}\right].$$

  - Invariant 2nd-order propagator
    - real space:

$$\begin{aligned}\hat{\Gamma}^{(2)l}_{Xl_1l_2}(k_1,k_2) &= c^{(2)l}_{Xl_1l_2}(k_1,k_2)\\ &+ \sqrt{4\pi}c^{(1)}_{Xl}(k_1)\left[\delta_{l_1l}\delta_{l_20} + \frac{(-1)^l}{\sqrt{3}}\frac{k_1}{k_2}\delta_{l_21}\sqrt{2l_1+1}\begin{pmatrix} l & l_1 & 1\\ 0 & 0 & 0\end{pmatrix}\right]\\ &+ \sqrt{4\pi}c^{(1)}_{Xl}(k_2)\left[\delta_{l_10}\delta_{l_2l} + \frac{(-1)^l}{\sqrt{3}}\frac{k_2}{k_1}\delta_{l_11}\sqrt{2l_2+1}\begin{pmatrix} l & l_2 & 1\\ 0 & 0 & 0\end{pmatrix}\right]\\ &+ (4\pi)^{3/2}c^{(0)}_X\delta_{l0}\left[\frac{34}{21}\delta_{l_10}\delta_{l_20} - \frac{1}{\sqrt{3}}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\delta_{l_11}\delta_{l_21}\right.\\ &\left. + \frac{8}{21\sqrt{5}}\delta_{l_12}\delta_{l_22}\right]. \quad (191)\end{aligned}$$

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Sample calculations

- The power spectrum

$$\langle F_{X_1 l_1 m_1}(\boldsymbol{k}_1) F_{X_2 l_2 m_2}(\boldsymbol{k}_2)\rangle_{\rm c} = (2\pi)^3 \delta_{\rm D}^3(\boldsymbol{k}_1 + \boldsymbol{k}_2) P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k}_1),$$

  - Invariant spectrum:

$$P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k}) = (-1)^{l_1 + l_2} \sqrt{(2l_1+1)(2l_2+1)} \times \sum_l (-1)^l \sqrt{2l+1}\, \underbrace{(l_1\, l_2\, l)_{m_1 m_2}{}^{m}}_{\text{3j-symbol}}\, Y_{lm}(\hat{\boldsymbol{k}}) \underbrace{P^{l_1 l_2; l}_{X_1 X_2}(k)}_{\text{Invariant spectrum}},$$

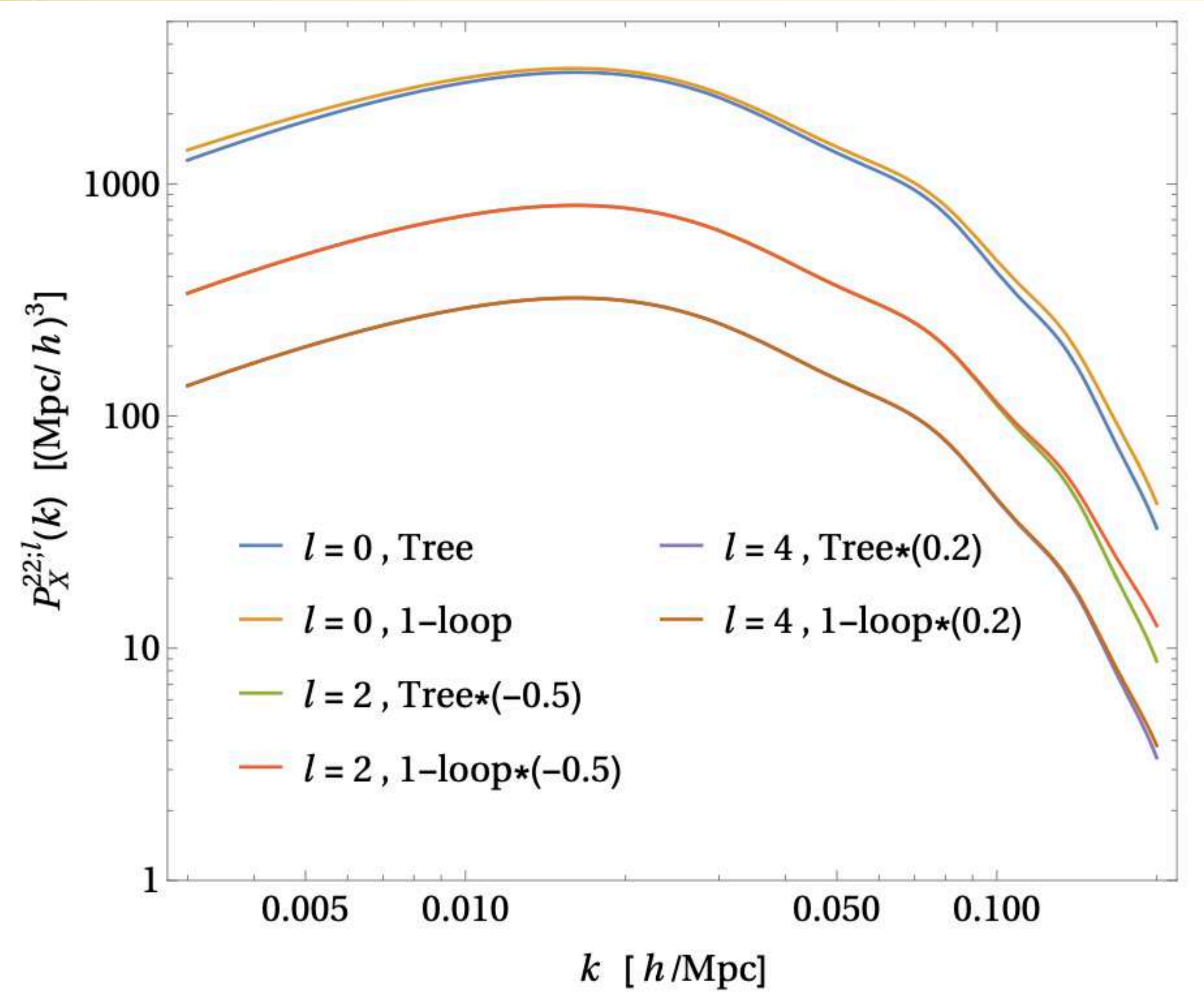
  - Result (tree-level)
    - real space & redshift space

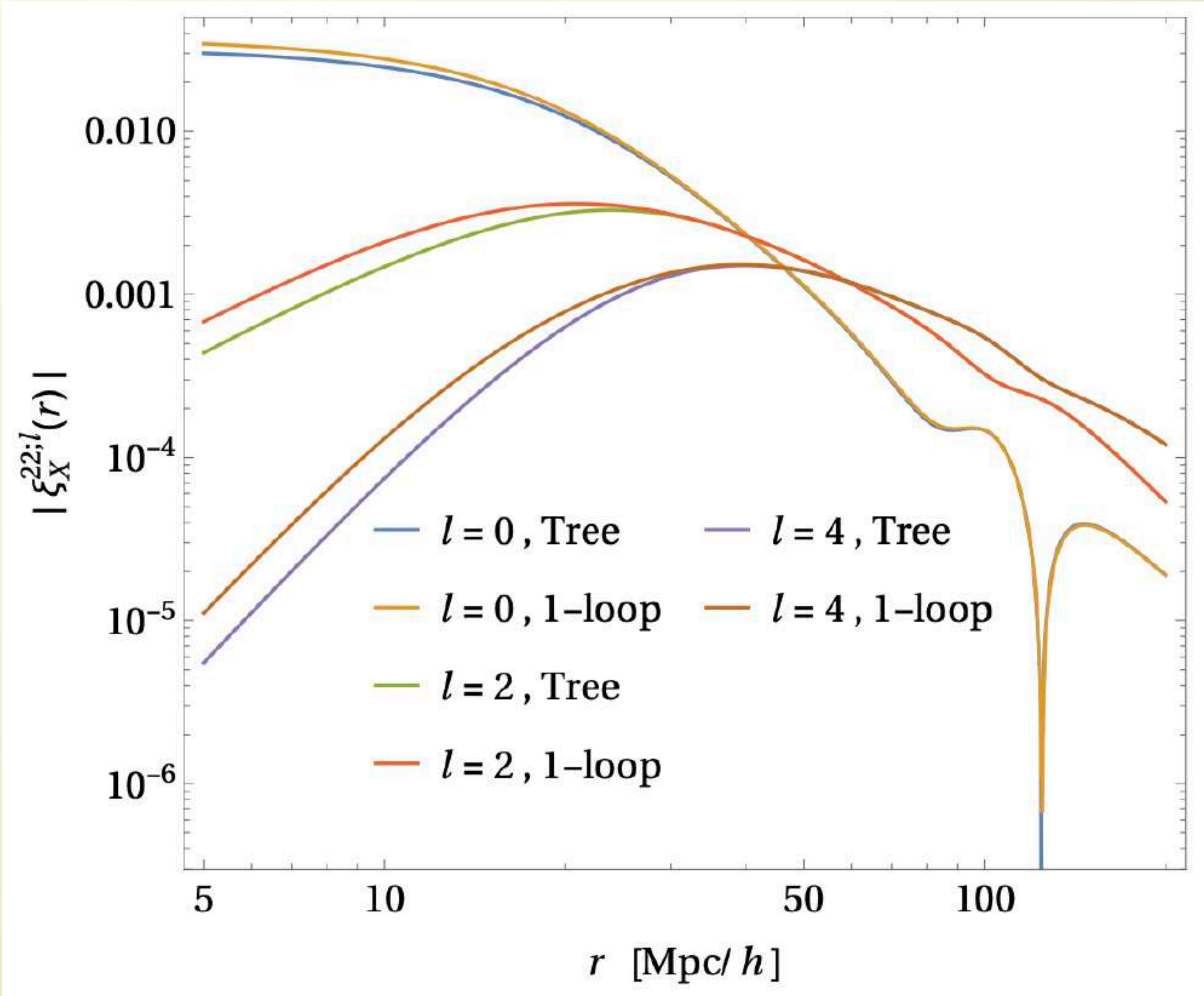
$$P^{l_1 l_2; l}_{X_1 X_2}(k) = \frac{(-1)^{l_2}}{\sqrt{4\pi}} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \Pi^2(k) \hat{\Gamma}^{(1)}_{X_1 l_1}(k) \hat{\Gamma}^{(1)}_{X_2 l_2}(k) P_{\rm L}(k).$$

$$P^{l_1 l_2; l\, l_z; L}_{X_1 X_2}(k, \mu) = \frac{1}{\sqrt{4\pi}} \Pi^2(k,\mu) P_{\rm L}(k) \sqrt{\{l_z\}\{L\}} \times \sum_{l_{z1}, l_{z2}, l'_1, l'_2} (-1)^{l'_2} \sqrt{\{l'_1\}\{l'_2\}} \begin{pmatrix} l_{z1} & l_{z2} & l_z \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l'_1 & l'_2 & l \\ 0 & 0 & 0 \end{pmatrix} \times \begin{Bmatrix} l_{z1} & l_{z2} & l_z \\ l'_1 & l'_2 & l \\ l_1 & l_2 & L \end{Bmatrix} \hat{\Gamma}^{(1) l_1 l_{z1}}_{X_1 l'_1}(k) \hat{\Gamma}^{(1) l_2 l_{z2}}_{X_2 l'_2}(k).$$

Sample calculations

- Tree & 1-loop power spectra & correlation functions
  - Assuming semi-local model for tensor bias





Sample calculations

- Scale-dependent bias (primordial non-Gaussianity, or cosmological collider physics)

$$P^{\mathrm{NG}\,l_1 l_2;l}_{X_1X_2}(k) = \frac{(-1)^l\left[1+(-1)^{l_2}\right]}{4\pi} f^{(l_2)}_{\mathrm{NL}} \frac{P_{\mathrm{L}}(k)}{\mathcal{M}(k)} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \times \left[c^{(1)}_{X_1 l_1}(k) + \delta_{l_1 0} c^{(0)}_{X_1}\right] \sum_{l',l''} (-1)^{l''} \sqrt{\{l'\}\{l''\}} \times \begin{pmatrix} l' & l'' & l_2 \\ 0 & 0 & 0 \end{pmatrix} \int \frac{p^2 dp}{2\pi^2} c^{(2)l_2}_{X_2 l' l''}(p,p) P_{\mathrm{L}}(p) + [(X_1 l_1) \leftrightarrow (X_2 l_2)].$$

$$\Delta b_{Xl}(k) = \sqrt{4\pi}(-1)^l \sqrt{\{l\}} \frac{P^{\mathrm{NG}\,l0;l}_{X\delta}(k)}{P_{\mathrm{L}}(k)}.$$

- For a local-type non-Gaussianity,

$$B_{\mathrm{L}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{2\mathcal{M}(k_3)}{\mathcal{M}(k_1)\mathcal{M}(k_2)} \times \sum_l f^{(l)}_{\mathrm{NL}} P_l(\hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2) P_{\mathrm{L}}(k_1) P_{\mathrm{L}}(k_2) + \mathrm{cyc.}$$

$$\Delta b_{Xl}(k) = \frac{1+(-1)^l}{\sqrt{4\pi}} \frac{f^{(l)}_{\mathrm{NL}}}{\mathcal{M}(k)} \sum_{l',l''} (-1)^{l''} \sqrt{\{l'\}\{l''\}} \times \begin{pmatrix} l' & l'' & l \\ 0 & 0 & 0 \end{pmatrix} \int \frac{p^2 dp}{2\pi^2} c^{(2)l}_{Xl'l''}(p,p) P_{\mathrm{L}}(p).$$

Summary & Prospects

- The basic framework of iPT for cosmological tensor fields is formulated
- Many people have done many numerical analyses
    - The comparison with numerical simulations
        - in progress with Akitsu, Taruya with halo-like model
    - Determinations of tensor bias functions
- Scale-dependent bias of tensor fields as new probes for primordial non-Gaussianity (Cosmological collider physics)
    - Generalization of Kogai et al. 2018; 2021 with Urakawa
- Projection effects (Paper III)

Projection effect (Paper III, in prep.)

- Projected field onto the sky

$$f_{Xi_1\cdots i_l} \equiv \mathcal{P}_{i_1 j_1} \cdots \mathcal{P}_{i_l j_l} F_{X j_1 \cdots j_l}$$

$$\mathcal{P}_{ij}(\hat{\boldsymbol{r}}) \equiv \delta_{ij} - \hat{r}_i \hat{r}_j,$$

- Traceless part of the projected field

$$f^{(s)}_{Xi_1\cdots i_s} = f^{(+s)}_X \mathrm{m}^+_{i_1} \cdots \mathrm{m}^+_{i_s} + f^{(-s)}_X \mathrm{m}^-_{i_1} \cdots \mathrm{m}^-_{i_s},$$

$$f^{(\pm s)}_X = (-1)^s f^{(s)}_{Xi_1\cdots i_s} \mathrm{m}^{\mp}_{i_1} \cdots \mathrm{m}^{\mp}_{i_s},$$

$$\mathbf{m}^{\pm} \equiv \mathbf{e}^{\pm} = \mp \frac{\hat{\mathbf{e}}_1 \pm \hat{\mathbf{e}}_2}{\sqrt{2}}$$

- Relation to the 3D irreducible tensor

- $$f^{(\pm s)}_X = i^s \sqrt{\frac{(2l+1)!!}{4\pi l!}}\, F_{Xs,\pm s},$$