YITP Workshop “Rheology of Disordered Particles...”, June 27–29 2018

UNIFIED RHEOLOGY OF GRANULAR AND GAS-SOLID SUSPENSIONS

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June 27, 2018

NONLINEAR THEORY FOR GRANULAR AND GAS-SOLID SUSPENSIONS

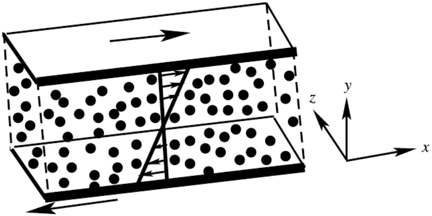
Dry Granular Fluid

- "Nonlinear" hydrodynamics and rheology of granular fluid
- Uniform Shear Flow (USF): from dilute to (moderately) dense
- Saha and Alam (J. Fluid Mech., vol. 757, 2014; vol. 795, 2016; Preprint (2018b))

Gas-Solid Suspension under USF

- Moderately-dense gas-solid suspension [Preprint (2018a)]
- Hysteresis in dilute gas-solid suspension
- Saha and Alam (J. Fluid Mech., vol. 833, 2017)

UNIFORM SHEAR FLOW AND INELASTIC COLLISION



USF : $\boldsymbol{u} = (2\dot{\gamma}y, 0, 0)$, $\quad 2\dot{\gamma} = \frac{du}{dy} \Longrightarrow$ Uniform Shear Rate.

- Binary collision

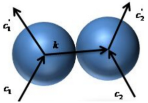


FIGURE: Collision of two spheres

Collision rule: $(\boldsymbol{g}' \cdot \boldsymbol{k}) = -e\,(\boldsymbol{g} \cdot \boldsymbol{k})$

Smooth spheres : $|\boldsymbol{g}' \times \boldsymbol{k}| = |\boldsymbol{g} \times \boldsymbol{k}|$

Change in kinetic energy :

$\Delta E = -\frac{m}{4}(1 - e^2)(\boldsymbol{g} \cdot \boldsymbol{k})^2$

$e \in [0, 1]$

$e = 1$ elastic collision

$e = 0$ sticky collision

MOTIVATION: NORMAL STRESS DIFFERENCES (NSD)

$$\mathcal{N}_1 = \frac{P_{xx} - P_{yy}}{p}$$
$$p = \frac{P_{xx} + P_{yy}}{2}$$

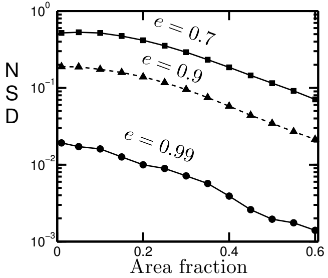


FIGURE: Variation of first normal stress difference

- Non-vanishing first NSD : $\mathcal{N}_1 \neq 0$
- M. Alam and S Luding, J. Fluid Mech., 476 (2003)
- O. R. Walton, J. Rheology (1986)

MOTIVATION: NORMAL STRESS DIFFERENCES (NSD)

$$\mathcal{N}_1 = \frac{P_{xx} - P_{yy}}{p}$$

$$\mathcal{N}_2 = \frac{P_{yy} - P_{zz}}{p}$$

$$p = \frac{P_{xx} + P_{yy} + P_{zz}}{3}$$

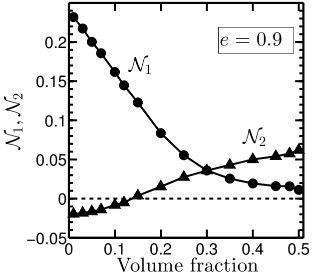


FIGURE: Variations of two normal stress differences

- Non-vanishing 1st and 2nd NSDs : $\mathcal{N}_1 \neq 0, \mathcal{N}_2 \neq 0$
- M. Alam and S Luding, Powders and Grains, 1141 (2005)

FROM KINETIC THEORY TO HYDRODYNAMICS OF GRANULAR FLUID

- Enskog-Boltzmann Equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{c}\cdot\nabla\right) f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t) = J(f^{(2)}) \tag{1}$$

Legacy: Savage and Jenkins (1983–) , Goldhirsch, Brey, Santos, Dufty, (1995–), ...

- Field Variables

1. Mass Density

$$\rho(\boldsymbol{x},\ t) \equiv mn(\boldsymbol{x},\ t) = m\int f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t)d\boldsymbol{c} \tag{2}$$

2. Hydrodynamic Velocity

$$\boldsymbol{u}(\boldsymbol{x},\ t) \equiv \langle \boldsymbol{c}\rangle = \frac{1}{n(\boldsymbol{x},\ t)}\int \boldsymbol{c} f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t)d\boldsymbol{c} \tag{3}$$

3. Second-Moment Tensor

$$\boldsymbol{M}(\boldsymbol{x},t) \equiv \langle \boldsymbol{C}\boldsymbol{C}\rangle = \frac{1}{n(\boldsymbol{x},t)}\int \boldsymbol{C}\boldsymbol{C} f^{(1)}(\boldsymbol{c},\boldsymbol{x},t)d\boldsymbol{c} \tag{4}$$

$\boldsymbol{C} \equiv \boldsymbol{c} - \boldsymbol{u}$ is peculiar/fluctuation velocity.

4. Granular Temperature

$$T(\boldsymbol{x},\ t) = \frac{1}{3}\langle \boldsymbol{C}\cdot\boldsymbol{C}\rangle = \frac{1}{3n(\boldsymbol{x},t)}\int \boldsymbol{C}^2 f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t)d\boldsymbol{c} \tag{5}$$

GRAD-LEVEL MOMENT EQUATIONS

- 10-moment System

$$\left.\begin{aligned} \frac{D\rho}{Dt} &= -\rho u_{\alpha,\alpha} \\ \rho\frac{Du_\alpha}{Dt} &= -P_{\alpha\beta,\alpha} \\ \rho\frac{DM_{\alpha\beta}}{Dt} &= -Q_{\gamma\alpha\beta,\gamma} - P_{\delta\beta}u_{\alpha,\delta} - P_{\delta\alpha}u_{\beta,\delta} + \aleph_{\alpha\beta} \end{aligned}\right\} \quad (6)$$

$$\frac{3}{2}\rho\frac{DT}{Dt} = -q_{\alpha,\alpha} - P_{\alpha\beta}u_{\beta,\alpha} - \mathcal{D}$$

$$\aleph_{\alpha\beta} = \aleph[mC_\alpha C_\beta] \quad (7)$$

$$\mathcal{D} = -\frac{1}{2}\aleph_{\alpha\alpha} \sim (1-e^2) \quad (8)$$

$$\boldsymbol{P} = \boldsymbol{P}^k + \boldsymbol{P}^c = \rho\boldsymbol{M} + \Theta(m\boldsymbol{C}) \quad (9)$$

- Harold Grad, Commun. Pure Appl. Math. **2**, 331 (1949)
- J. T. Jenkins and M. W. Richman, Arch. Rat. Mech. Anal. **87**, 647 (1985)

14-MOMENT EQUATIONS

- Additional Hydrodynamic Fields

$$q^k_\alpha(\boldsymbol{x}, t) = \frac{m}{2}\int C^2 C_\alpha f^{(1)}(\boldsymbol{c}, \boldsymbol{x}, t) d\boldsymbol{c} = \frac{\rho}{2}\langle C^2 C_\alpha \rangle \equiv \frac{\rho}{2} M_{\alpha\beta\beta}$$

$$M_{\alpha\alpha\beta\beta}(\boldsymbol{x}, t) = \int C^4 f^{(1)}(\boldsymbol{c}, \boldsymbol{x}, t) d\boldsymbol{c} = \langle C^4 \rangle$$

- 14-moment System

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\rho = -\rho u_{\alpha,\alpha}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)u_\alpha = -P_{\alpha\beta,\beta}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)M_{\alpha\beta} = -Q_{\gamma\alpha\beta,\gamma} - P_{\gamma\beta}u_{\alpha,\gamma} - P_{\gamma\alpha}u_{\beta,\gamma} + \aleph_{\alpha\beta}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)M_{\alpha\beta\beta} = -Q_{\gamma\alpha\beta\beta,\gamma} + 3M_{(\alpha\beta}P_{\beta)n,n} - 3Q_{n(\alpha\beta}u_{\beta),n} + \aleph_{\alpha\beta\beta}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)M_{\alpha\alpha\beta\beta} = -Q_{\gamma\alpha\alpha\beta\beta,\gamma} + 4M_{(\alpha\alpha\beta}P_{\beta)n,n} - 4Q_{n(\alpha\beta}u_{\beta),n} + \aleph_{\alpha\alpha\beta\beta}$$

GRAD MOMENT EXPANSION (GME)

- Single-particle distribution function is expanded around the Maxwellian:

$$f^{(1)} = \frac{n}{(2\pi T)^{\frac{3}{2}}} e^{-C^2/2T} \left\{ 1 + \frac{1}{2\rho T^2} P^k_{\langle ij \rangle} C_i C_j + \frac{q^k_i}{5\rho T^3}\left(C^2 C_i - 5TC_i\right) + \left(\frac{15}{8} - \frac{5}{4T}C^2 + \frac{C^4}{8T^2}\right)\mathfrak{a}_2 \right\}$$

- Excess kurtosis

$$\mathfrak{a}_2 = \frac{\langle C^4 \rangle}{\langle C^4 \rangle_M} - 1 = \frac{M_{\alpha\alpha\beta\beta} - M_{\alpha\alpha\beta\beta|M}}{M_{\alpha\alpha\beta\beta|M}},$$

- Molecular chaos ansatz:

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{x} - \sigma \boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{x}) = g_0(\nu) f^{(1)}(\boldsymbol{c}_1, \boldsymbol{x} - \sigma \boldsymbol{k}) f^{(1)}(\boldsymbol{c}_2, \boldsymbol{x})$$

$$g_0(\nu) = \frac{(1 - 7\nu/16)}{(1-\nu)^2}, \qquad \nu = n\pi\sigma^2/4$$

$$= \frac{(1 - \nu/2)}{(1-\nu)^3}, \qquad \nu = n\pi\sigma^3/6$$

- N. F. Carnahan and K. E. Starling, J. Chem. Phys., **51**, 635 (1969)

2ND-ORDER "NON-LINEAR" GME THEORY: DISSIPATION RATE

- Energy Balance

$$\frac{3}{2}\rho\frac{DT}{Dt} = -q_{\alpha,\alpha} - P_{\alpha\beta}u_{\beta,\alpha} - \mathcal{D}$$

- Dissipation Rate (with "2nd-order" nonlinearity)

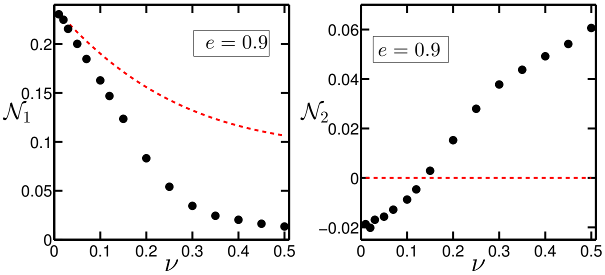
$$\begin{aligned}
\mathcal{D} &= \mathcal{D}_0 + \mathcal{D}_u(\nabla\cdot\boldsymbol{u}) \\
&+ \mathcal{D}_{uu}\Big(\nabla\boldsymbol{u}:\nabla\boldsymbol{u} + \nabla\boldsymbol{u}:\nabla\boldsymbol{u}' + (\nabla\cdot\boldsymbol{u})^2\Big) \\
&+ \mathcal{D}_q\nabla\cdot\boldsymbol{q}^k + \mathcal{D}_{u\Pi}(\nabla\boldsymbol{u}:\boldsymbol{\Pi}) + \mathcal{D}_{q\Pi}\Big((\boldsymbol{q}^k\nabla:\boldsymbol{\Pi}) + (\nabla\boldsymbol{q}^k:\boldsymbol{\Pi})\Big) \\
&+ \mathcal{D}_{q\rho}(\boldsymbol{q}^k\cdot\nabla\rho) + \mathcal{D}_{q\mathfrak{a}_2}(\boldsymbol{q}^k\cdot\nabla\mathfrak{a}_2) \\
&+ \mathcal{D}_\rho\nabla^2\rho + \mathcal{D}_T\nabla^2 T + \mathcal{D}_\Pi\Big(\nabla\cdot(\nabla\cdot\boldsymbol{\Pi})\Big) + \mathcal{D}_{\mathfrak{a}_2}\nabla^2\mathfrak{a}_2 \\
&+ \mathcal{D}_{\rho T}\nabla\rho\cdot\nabla T + \mathcal{D}_{\rho\Pi}\nabla\rho\cdot(\nabla\cdot\boldsymbol{\Pi}) + \mathcal{D}_{\rho\mathfrak{a}_2}\nabla\rho\cdot\nabla\mathfrak{a}_2 \\
&+ \mathcal{D}_{TT}(\nabla T)^2 + \mathcal{D}_{T\Pi}\nabla T\cdot(\nabla\cdot\boldsymbol{\Pi}) + \mathcal{D}_{T\mathfrak{a}_2}\nabla T\cdot\nabla\mathfrak{a}_2 \\
&+ \mathcal{D}_{uq}\Big(\nabla\boldsymbol{u}:\nabla\boldsymbol{q}^k + \nabla\boldsymbol{u}:\nabla\boldsymbol{q}^{k'} + (\nabla\cdot\boldsymbol{q}^k)^2\Big).
\end{aligned} \tag{10}$$

2ND-ORDER EXPRESSION FOR $\mathcal{D}$: WHOLE RANGE OF DENSITY

$$
\begin{aligned}
\mathscr{D} &= -\frac{1}{2}\aleph_{\alpha\alpha} \\
&= \underbrace{\frac{12\rho\nu g_0(1-e^2)T^{\frac{3}{2}}}{\pi^{\frac{1}{2}}\sigma}\left(1+\frac{3}{16}\mathfrak{a}_2+\frac{9}{1024}\mathfrak{a}_2^2\right)}\underbrace{-3\rho\nu g_0(1-e^2)T(\nabla\cdot\boldsymbol{u})} \\
&- \frac{3}{10}(1-e^2)\nu g_0(2+21\mathfrak{a}_2)\nabla\cdot\boldsymbol{q}^k+\frac{3\nu g_0(1-e^2)}{5\pi^{\frac{1}{2}}\sigma\rho T^{\frac{1}{2}}}\boldsymbol{\Pi}:\boldsymbol{\Pi}+\frac{3\nu g_0(1-e^2)}{50\pi^{\frac{1}{2}}\rho\sigma T^{\frac{3}{2}}}(\boldsymbol{q}^k\cdot\boldsymbol{q}^k) \\
&- \frac{6}{5}\nu g_0(1-e^2)(\nabla\boldsymbol{u}:\boldsymbol{\Pi})-\frac{399}{175\rho T}\nu g_0(1-e^2)\Big((\boldsymbol{q}^k\nabla:\boldsymbol{\Pi})+(\nabla\boldsymbol{q}^k:\boldsymbol{\Pi})\Big) \\
&+ \frac{3}{5\rho}\nu g_0(1-e^2)(\boldsymbol{q}^k\cdot\nabla\rho)-\frac{63}{10}\nu g_0(1-e^2)(\boldsymbol{q}^k\cdot\nabla\mathfrak{a}_2) \\
&+ \frac{\rho\nu(1-e^2)\sigma}{16\sqrt{\pi}T^{\frac{3}{2}}} \\
&\quad\times\Bigg[g_0(\nu)\Bigg\{32\left(\frac{T^3}{\rho}\right)\nabla^2\rho+24T^2\nabla^2T+\frac{48}{5}\left(\frac{T^2}{\rho}\right)\Big(\nabla\cdot(\nabla\cdot\boldsymbol{\Pi})\Big)+3T^3\nabla^2\mathfrak{a}_2 \\
&\qquad+\cdots \\
&\qquad+\frac{\partial g_0}{\partial\rho}\nabla\rho\cdot\Bigg\{32\left(\frac{T^3}{\rho}\right)\nabla\rho+24T^2\nabla T+3T^3\nabla\mathfrak{a}_2+\frac{48}{5}\left(\frac{T^2}{\rho}\right)(\nabla\cdot\boldsymbol{\Pi})\Bigg\}\Bigg],
\end{aligned}
\tag{11}
$$

"LINEAR-ORDER" GME THEORY: PREDICTIONS FOR USF

- V. Garzo, Phys. Fluids **25** (2013)
- J. T. Jenkins and M. W. Richman, Arch. Rat. Mech. Anal. **87**, 647 (1985)



$\mathcal{N}_2 \equiv 0 \Leftarrow$ previous work

"NON-LINEAR" GME THEORY: PRESENT WORK

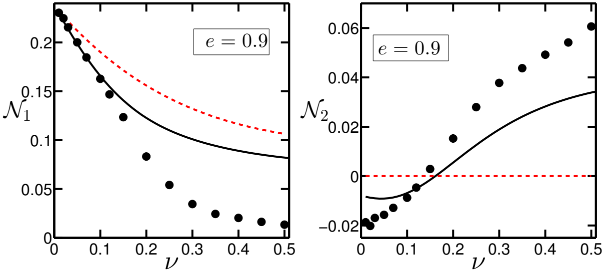


FIGURE: "—": present nonlinear theory [Saha & Alam, preprint (2018b)]

- Quantitative prediction for $\mathcal{N}_1$ and $\mathcal{N}_2$ are not good (even at $e = 0.9$)

Maximum Entropy Principle and Extended Hydrodynamics

Hydrodynamic fields:

$$\left.\begin{array}{lcl} \rho(\boldsymbol{x},t) & \equiv & mn(\boldsymbol{x},t) = m\int f(\boldsymbol{c},\boldsymbol{x},t)\mathrm{d}\boldsymbol{c} \\ \boldsymbol{u}(\boldsymbol{x},t) & \equiv & \langle \boldsymbol{c}\rangle = \frac{1}{n(\boldsymbol{x},t)}\int \boldsymbol{c}f(\boldsymbol{c},\boldsymbol{x},t)\mathrm{d}\boldsymbol{c} \\ \boldsymbol{M}(\boldsymbol{x},t) & \equiv & \langle \boldsymbol{CC}\rangle = \frac{1}{n(\boldsymbol{x},t)}\int \boldsymbol{CC}f(\boldsymbol{c},\boldsymbol{x},t)d\boldsymbol{c}. \end{array}\right\} \quad (13)$$

Optimum distribution function is such that it maximizes the uncertainty about the velocity, subject to the compatibility conditions of hydrodynamic fields in (13). Entropy is defined as (Saha & Alam 2017, JFM)

$$S = -\int f(\boldsymbol{c},\boldsymbol{x},t)\ln f(\boldsymbol{c},\boldsymbol{x},t)\mathrm{d}\boldsymbol{c} \quad (14)$$

Variation of entropy can be written as

$$\delta S = -\int \delta f \underbrace{\left(\ln f + 1 - \alpha - \alpha_i c_i - \alpha_{ij}C_iC_j\right)} \mathrm{d}\boldsymbol{c}, \quad (15)$$

For maximum entropy, the variation δS must be equal to zero, yielding

$$f = \exp(\alpha - 1 + \alpha_i c_i + \alpha_{ij}C_iC_j). \quad (16)$$

Solution for Lagrange multipliers $\{\alpha, \alpha_i, \alpha_{ij}\}$ follows from Eq. (13):

$$\alpha = 1 + \ln n - \frac{1}{2}\ln\left(8\pi^3|\boldsymbol{M}|\right), \quad \alpha_i = 0, \quad \text{and} \quad \alpha_{ij} = -\frac{1}{2}\left(\boldsymbol{M}^{-1}\right)_{ij}. \quad (17)$$

ANISOTROPIC MAXWELLIAN AND USF

- Single-particle distribution function is an anisotropic Maxwellian:

$$f^{(1)}(\boldsymbol{c},\boldsymbol{x},t) = \frac{n}{(8\pi^3|\boldsymbol{M}|)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\boldsymbol{C}\cdot\boldsymbol{M}^{-1}\cdot\boldsymbol{C}\right) \equiv f_{AM} \qquad (18)$$

- "Isotropic" $\boldsymbol{M} = T\delta_{\alpha\beta} \quad \Rightarrow$ Maxwellian distribution function

$$f^{(1)}(\boldsymbol{c},\boldsymbol{x},t) = \frac{n}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{C^2}{2T}\right) \equiv f_M \qquad (19)$$

- Eqn. (18) follows from "Maximum Entropy Principle" (Jaynes 1957)

$$\delta S = -\int \delta f\left(\ln f + 1 - \alpha - \alpha_i c_i - \alpha_{ij}C_iC_j\right)\mathrm{d}\boldsymbol{c}, \qquad (20)$$

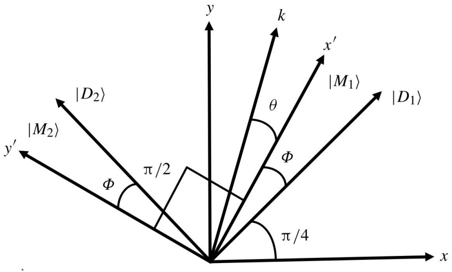
- f_{AM} holds exactly for USF
- P. Goldreich and S. Tremaine, Icarus (1978); Araki & Tremaine (1986)
- J. T. Jenkins & M. W. Richman, JFM (1988); Richman, J. Rheol. (1989)

UNIFORM SHEAR FLOW

- Velocity gradient tensor can be decomposed as

$$\nabla \boldsymbol{u} = \boldsymbol{D} + \boldsymbol{W} \equiv \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (21)$$

- Eigenvalues of $\boldsymbol{D}$ are $(\dot{\gamma}, -\dot{\gamma}, 0)$, with corresponding eigenvectors: $|D_1\rangle = (\cos\frac{\pi}{4}, \sin\frac{\pi}{4}, 0)$, $|D_2\rangle = (-\sin\frac{\pi}{4}, \cos\frac{\pi}{4}, 0)$ and $|D_3\rangle = (0, 0, 1)$,



CONSTRUCTION OF SECOND-MOMENT TENSOR IN USF

- Second Moment Tensor:

$$\boldsymbol{M} = \langle \boldsymbol{C}\boldsymbol{C} \rangle = T\boldsymbol{I} + \widehat{\boldsymbol{M}} \tag{22}$$

- Eigenvalues of $\boldsymbol{M}$ are $T(1+\xi)$, $T(1+\varsigma)$ and $T(1+\zeta)$, such that

$$\xi + \varsigma + \zeta = 0. \tag{23}$$

  and the eigen-directions are $|M_1\rangle$, $|M_2\rangle$ and $|M_3\rangle$, respectively.

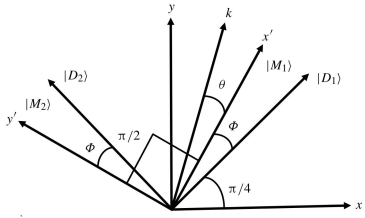
- Second-moment tensor can be represented in terms of its eigen-basis

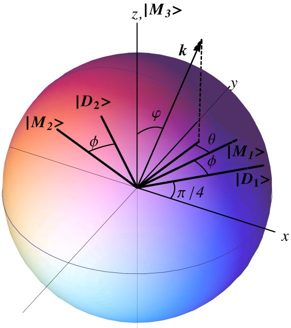
$$\boldsymbol{M} = T(1+\xi)|M_1\rangle\langle M_1| + T(1+\varsigma)|M_2\rangle\langle M_2| + T(1+\zeta)|M_3\rangle\langle M_3|. \tag{24}$$

- $|M_1\rangle$, $|M_2\rangle$ and $|M_3\rangle$ are chosen as, with unknown $\phi \equiv |D_1\rangle \measuredangle |M_1\rangle$

$$|M_1\rangle = \begin{bmatrix} \cos\left(\phi + \frac{\pi}{4}\right) \\ \sin\left(\phi + \frac{\pi}{4}\right) \\ 0 \end{bmatrix}, \quad |M_2\rangle = \begin{bmatrix} -\sin\left(\phi + \frac{\pi}{4}\right) \\ \cos\left(\phi + \frac{\pi}{4}\right) \\ 0 \end{bmatrix} \quad \text{and} \quad |M_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

CONSTRUCTION OF SECOND-MOMENT TENSOR IN USF

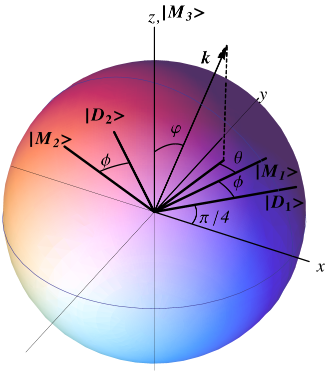




Second-moment tensor :

$$\boldsymbol{M} \equiv \langle \mathbf{CC} \rangle = T \begin{bmatrix} 1+\lambda^2+\eta\sin 2\phi & -\eta\cos 2\phi & 0 \\ -\eta\cos 2\phi & 1+\lambda^2-\eta\sin 2\phi & 0 \\ 0 & 0 & 1-2\lambda^2 \end{bmatrix}, \quad (25)$$

LEGACY OF JENKINS & RICHMAN (1988-) ARAKI, GOLDREICH, TREMAINE (1978–)...



- $\eta \propto T_x - T_y \sim M_2 - M_1$,
- $\phi \equiv |D_1\rangle \measuredangle |M_1\rangle$,
- $\lambda^2 \propto T - T_z$,
- $R = \frac{\dot{\gamma}\sigma}{4\sqrt{T}} = \frac{v_{sh}}{v_{th}}$

- η, ϕ, λ and R completely describe $\boldsymbol{M}$
- $\Rightarrow$ All transport coefficients are functions of $(\eta, \phi, \lambda, R; \nu, e)$

STRESS TENSOR: ANALYTICAL RESULTS FOR ALL DENSITY

- For USF, solve

$$P_{\delta\beta}u_{\alpha,\delta} + P_{\delta\alpha}u_{\beta,\delta} = \aleph_{\alpha\beta}$$

  - Exact-solution for unknowns $(\eta, \phi, \lambda^2, R)$ has been found at 2nd-order for all density (Saha & Alam 2016)

- Up-to super-super-Burnett order $O(\dot{\gamma}^4)$:

$$\mu^* = \frac{\nu\sqrt{T^*}}{8}\left[\frac{\eta\cos 2\phi}{R} + \frac{4(1+e)\nu g_0}{105\sqrt{\pi}}\left(21\left\{8+\sqrt{\pi}\frac{\eta\cos 2\phi}{R}\right\} \underbrace{+48\lambda^2 + 128R^2 - 4\eta^2\left\{2+(1+2\cos^2 2\phi)\right\}}\right)\right]$$

$$p^* = \nu T^*\left[1 + \frac{2(1+e)\nu g_0}{315}\left\{315 \underbrace{+672R^2 + \frac{8}{\sqrt{\pi}}\eta R\cos 2\phi(42+3\eta^2-32R^2-12\lambda^2)}\right\}\right]$$

$$\mathscr{D}_{inelastic} = \frac{\rho\nu g_0(1-e^2)T^{\frac{3}{2}}}{70\sigma\sqrt{\pi}}\left[840 + \underbrace{\left(4+\sqrt{\pi}\frac{\eta}{R}\cos 2\phi\right)R^2 + 84\eta^2}\right]$$

- Transport coefficients at Navier-Stokes order, $O(\dot{\gamma})$, are recovered by removing higher-order terms.

NORMAL-STRESS DIFFERENCES

$$\begin{aligned} P^*_{xx} - P^*_{yy} &= 2\eta \sin(2\phi)\nu T^* \\ &+ \frac{8\nu^2 g_0(1+e)T^*}{105}\left(21\eta\sin 2\phi - \frac{8}{\sqrt{\pi}}R\eta^2 \sin 2\phi \cos 2\phi\right) \\ P^*_{yy} - P^*_{zz} &= \left(3\lambda^2 - \eta \sin 2\phi\right)\nu T^* \\ &+ \frac{4(1+e)\nu^2 g_0 T^*}{1155}\left[33(32R^2 - 7\eta\sin 2\phi + 21\lambda^2) \right. \\ &\left. + \frac{8}{\sqrt{\pi}}\eta R \cos 2\phi\left\{66 + 6\eta^2 - 64R^2 - 33\lambda^2 + 11\eta\sin 2\phi\right\}\right] \end{aligned}$$

$$(\nu \to 0) \Rightarrow \boxed{\mathcal{N}_1 = 2\eta\sin 2\phi;\; \mathcal{N}_2 = 3\lambda^2 - \eta\sin 2\phi} \sim 2(1-e) \sim O(\dot{\gamma}^2)$$

1. NSD's are nonlinear/Burnett-order effects (Sela & Goldhirsch 1998)
2. $\mathcal{N}_1 \sim \eta, \phi \quad \Rightarrow$ (shear-plane anisotropy)
3. $\mathcal{N}_2 \sim \lambda^2 \sim (T - T_z) = T^{ex} \quad \Rightarrow$ ("excess" temperature)

RESULTS FOR DISKS

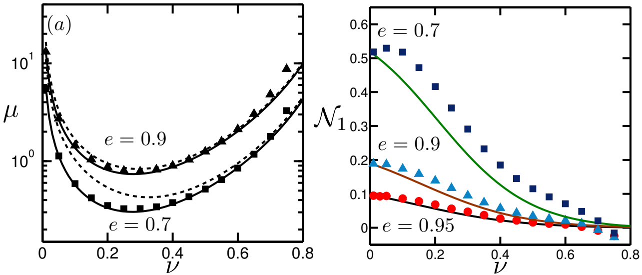


FIGURE: •: Simulation results; "– –": Navier-Stokes order; "–": present theory

- S. Saha and M. Alam, JFM **757** (2014)
- Lutsko, Garzo and Dufty, PRE (1999, 2005)

RESULTS FOR SPHERES: GME

- Predictions from several Grad-level theories

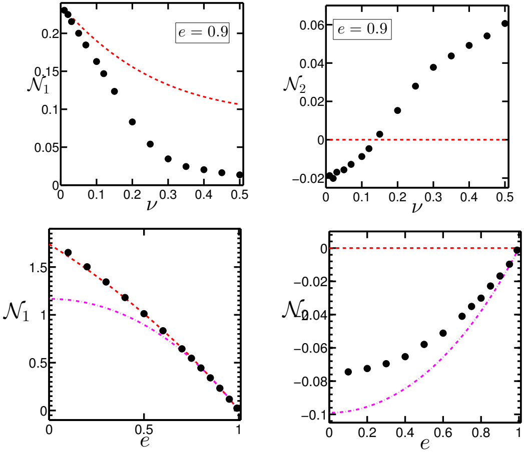


FIGURE: ●: Simulation; "——": Grad-level theories (JenkinsRichman1985, Garzo 2013); ". — ." Super-Burnett dilute solution (Sela & Goldhirsch 1998)

RESULTS FOR SPHERES: AME

- Improvement over other Grad-level theories

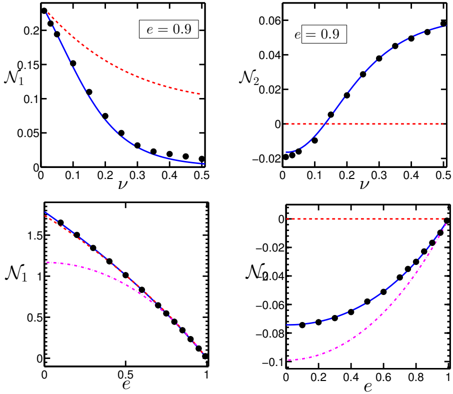
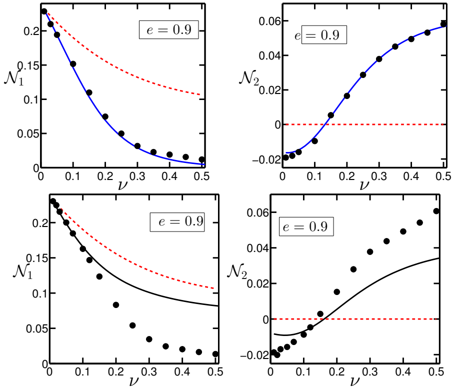


FIGURE: ●: Simulation results; "– –": existing Grad level theories (Grazo 2013; JenkinsR1985); "—": present theory

- S. Saha and M. Alam, J. Fluid Mech. **795** (2016)

RECAP ON "GME" NON-LINEAR THEORY VERSUS AME



- Predictions of AME are better than GME at any density.

CONCLUSIONS FOR DRY GRANULAR FLUID

1. Grad-level 10-moment equations are analysed using anisotropic Maxwellian
2. Analytical expressions for all transport coefficients, up-to super-Burnett order, have been derived for whole range of density.
3. Excellent agreement with simulation is found over whole range of density
4. Origin of NSDs is tied to anisotropies of second-moment tensor
5. Ref. Saha & Alam (2014, 2016), JFM
6. Developed a 14-moment Theory for dense granular fluid [Saha & Alam 2018a, Preprint]
7. Breakdown of Onsager's reciprocity relations? [Alam & Saha (2018c) Preprint]

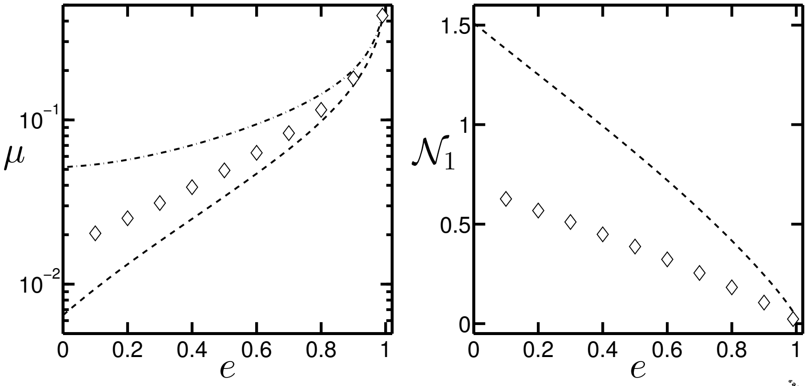
Part 2

Gas-Solid Suspension

- 'Dry' granular flow $\Rightarrow$ $St \to \infty$
- Effect of the interstitial fluid has been neglected
- How to include effects of interstitial fluid?
- Would the predictions of "anisotropic Maxwellian" hold at small values of St?

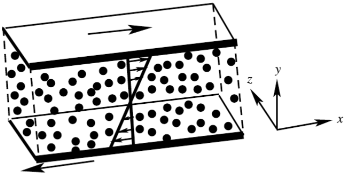
PREDICTIONS OF EXISTING THEORIES: MOTIVATION?

- $\nu = 0.2, \quad St = 10$
- dashed lines: GME (Sangani et al. 1996)
- dot-dash line: Navier-Stokes-order Theory



EQUATIONS FOR GAS PHASE

Gas-Solid Suspension at Steady State



- $Re = \frac{\rho\dot{\gamma}\sigma^2}{2\mu_f} \ll 1,$
- $St = 2\dot{\gamma}\tau_{vis}$
- $\tau_{vis} = \frac{m}{3\pi\mu_f\sigma}, \quad \tau_{coll} \sim O(\sigma/\langle C^2\rangle^{1/2})$
- "Ignited" state $\tau_{coll} \ll \tau_{vis}$ (analog of 'rapid' granular fluid)
- Absence of gravity

Gas Phase:

- Stokes equations of motion

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\mu_g \nabla^2 v_i = \frac{\partial p_g}{\partial x_i}$$

EQUATIONS FOR PARTICLE PHASE

Particle Phase

Collision rule: $(\boldsymbol{g}' \cdot \boldsymbol{k}) = -e\,(\boldsymbol{g} \cdot \boldsymbol{k})$

Smooth spheres : $|\boldsymbol{g}' \times \boldsymbol{k}| = |\boldsymbol{g} \times \boldsymbol{k}|$

Change in kinetic energy:

$\Delta E = -\frac{m}{4}(1-e^2)(\boldsymbol{g} \cdot \boldsymbol{k})^2$

$e = 1$ elastic collision;

$e = 0$ sticking collision

- Binary collision

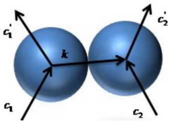


FIGURE: Collision of two spheres

- Enskog-Boltzmann Equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{c} \cdot \nabla\right) f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t) + \nabla_{\boldsymbol{c}} \cdot (\dot{\boldsymbol{c}} f^{(1)}) = \left(\frac{\partial f^{(1)}}{\partial t}\right)_{coll} \qquad (26)$$

- $\nabla_{\boldsymbol{c}} \cdot (\dot{\boldsymbol{c}} f^{(1)})$: rate of change of $f^{(1)}$ due to particle acceleration
- Effective Stokes Drag: $\frac{d\boldsymbol{c}}{dt} \propto -(\boldsymbol{c} - \boldsymbol{v})$
- Previous Work: Tsao & Koch (1995); Sangani et al. (1996), ...

HYDRODYNAMIC VARIABLES

- Hydrodynamic Variables
  1. Mass Density

$$\rho(\boldsymbol{x},\ t) \equiv mn(\boldsymbol{x},\ t) = m\int f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t)d\boldsymbol{c} \tag{27}$$

  2. Coarse-grained Velocity

$$\boldsymbol{u}(\boldsymbol{x},\ t) \equiv \langle \boldsymbol{c}\rangle = \frac{1}{n(\boldsymbol{x},\ t)}\int \boldsymbol{c}f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t)d\boldsymbol{c} \tag{28}$$

  3. Full Second Moment Tensor

$$\boldsymbol{M}(\boldsymbol{x},t) \equiv \langle \boldsymbol{C}\boldsymbol{C}\rangle = \frac{1}{n(\boldsymbol{x},t)}\int \boldsymbol{C}\boldsymbol{C}f^{(1)}(\boldsymbol{c},\boldsymbol{x},t)d\boldsymbol{c} \tag{29}$$

  $\boldsymbol{C} \equiv \boldsymbol{c} - \boldsymbol{u}$ is the peculiar velocity.

  4. Granular Temperature

$$T(\boldsymbol{x},\ t) = \frac{1}{3}\langle \boldsymbol{C}\cdot\boldsymbol{C}\rangle = \frac{1}{3n(\boldsymbol{x},t)}\int \boldsymbol{C}^2 f^{(1)}(\boldsymbol{c},\ \boldsymbol{x},\ t)d\boldsymbol{c} \tag{30}$$

MOMENT EQUATIONS

- Balance Equations for Particle-phase

$$\left.\begin{aligned} \frac{D\rho}{Dt} &= -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} \\ \rho\frac{D\boldsymbol{u}}{Dt} &= -\boldsymbol{\nabla}\cdot\boldsymbol{P} + \mathbf{F}_{drag} \\ \rho\frac{D\boldsymbol{M}}{Dt} &= -\boldsymbol{\nabla}\cdot\boldsymbol{Q} - \boldsymbol{P}\cdot\boldsymbol{\nabla u} - (\boldsymbol{P}\cdot\boldsymbol{\nabla u})^T - \frac{4\dot\gamma}{St_d}\boldsymbol{P}^k + \boldsymbol{\aleph} \end{aligned}\right\} \quad (31)$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{M} = -\nabla\cdot\boldsymbol{Q} - \boldsymbol{P}\cdot\nabla\boldsymbol{u} - (\boldsymbol{P}\cdot\nabla\boldsymbol{u})^T - \frac{4\dot\gamma}{St_d}\rho\langle\boldsymbol{CC}\rangle \underbrace{- \frac{2\dot\gamma}{St_d}\rho\langle(\boldsymbol{u}-\boldsymbol{v})\boldsymbol{C}\rangle - \frac{2\dot\gamma}{St_d}\rho\langle\boldsymbol{C}(\boldsymbol{u}-\boldsymbol{v})\rangle} + \boldsymbol{\aleph} \quad (32)$$

- $\mathbf{F}_{drag} \equiv \rho\langle\frac{d\mathbf{c}}{dt}\rangle = \rho\langle-\frac{\mathbf{c}-\mathbf{v}}{\tau}\rangle \propto (\boldsymbol{u}-\boldsymbol{v})$
- $St_d = f(\nu)St$, with $f(\nu\to 0)\to 1$ (Sangani et al 1996, JFM)
- Balance of Energy

$$\frac{3}{2}\rho\frac{DT}{Dt} + q_{\alpha,\alpha} = -P_{\alpha\beta}u_{\beta,\alpha} - \mathscr{D}_{\text{viscous}} - \mathscr{D}_{\text{inelastic}} \quad (33)$$

(34)

viscous heating ⇒ inelastic collision + dissipation due to fluid drag

“IGNITED” STATE

- Collision time $\tau_{coll} \ll$ viscous relaxation time τ_{vis}
- Particles have large fluctuation velocity: $T/(\dot{\gamma}\sigma)^2 \gg 1$
- Ignited state is analogous to “rapid” granular state
- Distribution function

$$f^{(1)}(\boldsymbol{c}, \boldsymbol{r}, t) = \frac{n}{(8\pi^3|\boldsymbol{M}|)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\boldsymbol{C}\cdot\boldsymbol{M}^{-1}\cdot\boldsymbol{C}\right).$$

- USF: drag due to gas-phase is zero

$$\boxed{\boldsymbol{u} = \boldsymbol{v} = 2\dot{\gamma}y\hat{\mathbf{x}}}$$

$$\boldsymbol{P}\cdot\nabla\boldsymbol{u} + (\boldsymbol{P}\cdot\nabla\boldsymbol{u})^T + \frac{4\dot{\gamma}}{St_d}\boldsymbol{P}^k = \aleph \qquad (35)$$

SOLUTION FOR ANISOTROPY

- Solution for (η, ϕ, λ) is

$$\left.\begin{aligned} \eta^2 &= \frac{30(1-e^2)St_d\nu g_0 + 60\sqrt{\pi}R - 32(1+e)(1+3e)St_d\nu g_0 R^2}{40\sqrt{\pi}R + 3(1+e)(11-3e)St_d\nu g_0} \\ \phi &= \frac{1}{2}\sin^{-1}\left[\frac{5}{\{5-2(1+e)(1-3e)\nu g_0\}}\eta\right] \\ \frac{\eta}{R}\cos(2\phi) &= \frac{\sqrt{\pi}}{\left\{3(1+e)(3-e)\nu g_0 + 10\sqrt{\pi}\left(\frac{R}{St_d}\right)\right\}}\cos^2(2\phi)\{5+2(1+e)(3e-1)\nu g_0\} \end{aligned}\right\},$$

- $R(\nu, e; St_d)$ is the real positive root of the quadratic equation

$$\begin{aligned} &\Big[200(23-11e)\pi + 250(1-e)\pi St_d^2 - 96(3-e)^2(1+e)^2(1+3e)St_d^2\nu^2 g_0^2 \\ &\qquad -(11-3e)\pi St_d^2\{5-2(1+e)(1-3e)\nu g_0\}^2\Big]R^2 \\ &+60(1+e)(3-e)(19-13e)\sqrt{\pi}(St_d)\nu g_0 R \\ &+90(1+e)(1-e^2)(3-e)^2 St_d^2\nu^2 g_0^2 = 0. \end{aligned} \qquad (37)$$

UNIFIED RHEOLOGY: FROM GAS-SOLID TO GRANULAR SUSPENSION

- Solution of second-moment equation yields

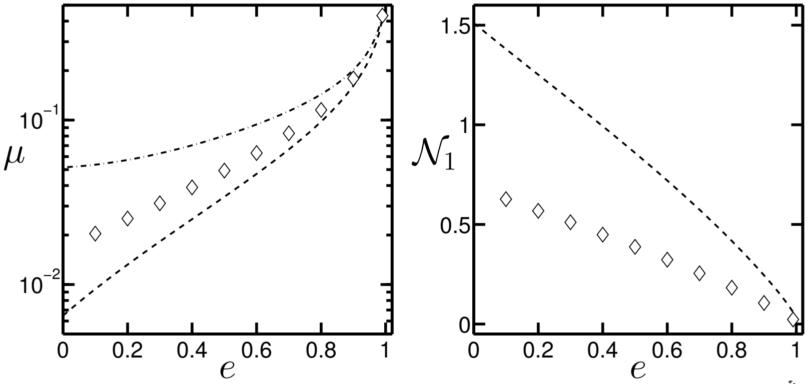
$$
\begin{aligned}
\eta &= \eta(\nu, e; St_d) \\
\phi &= \phi(\nu, e; St_d) \\
\lambda &= \lambda(\nu, e; St_d) \\
R &= R(\nu, e; St_d)
\end{aligned}
$$

- Dry Granular limit: $St_d \to \infty$
- Same expressions for all transport coefficients
- e.g. Viscosity

$$
\begin{aligned}
\mu^* &= \frac{\nu\sqrt{T^*}}{8}\left[\frac{\eta\cos 2\phi}{R} + \frac{4(1+e)\nu g_0}{105\sqrt{\pi}}\left(21\left\{8+\sqrt{\pi}\frac{\eta\cos 2\phi}{R}\right\}\right.\right. \\
&\qquad \left.\left. \underbrace{+\,48\lambda^2 + 128R^2 - 4\eta^2\left\{2+(1+2\cos^2 2\phi)\right\}}\right)\right]
\end{aligned}
\qquad (38)
$$

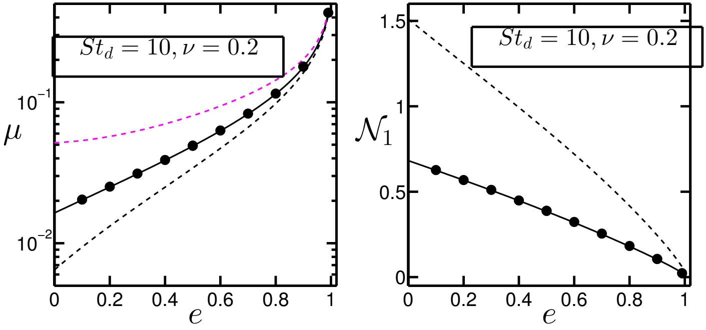
RECAP: PREDICTIONS OF EXISTING THEORIES

- $\nu = 0.2, \quad St = 10$
- dashed lines: GME (Sangani et al. 1996)
- dot-dash line: Navier-Stokes-order Theory



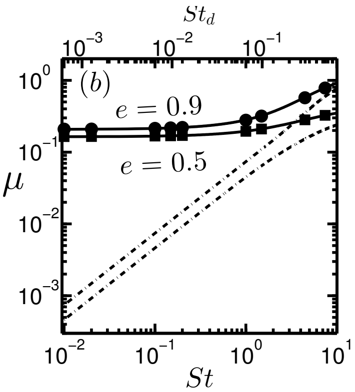
PRESENT THEORY

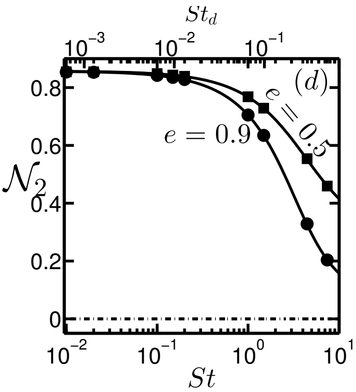
- $\nu = 0.2, \quad St_d = 10$
- Solid line: AME (present theory)
- Symbols: DSMC simulation



DOES THEORY HOLD AT SMALL STOKES NUMBERS?

- How low can you go in terms of *St*? (Jim Jenkins@KITP 2018 March)
- $\frac{\rho_p}{\rho_g} = \frac{9}{2}\frac{St}{Re}$
- $\nu = 0.5$ [Comparison with Sangani et al. (1996), "dot-dash" lines]





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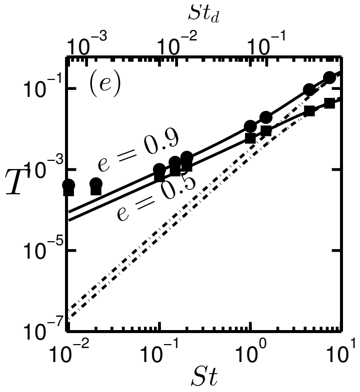


- Good agreement up-to $St \sim 0.1$!
- With $Re \sim 1$, theoretical predictions likely to hold for $\rho_p \sim \rho_g$.

Hysteresis (DST?) in Dilute Suspension

- Dealt with “Ignited” state $\Rightarrow$ $\tau_{vis} \gg \tau_{coll}$
- What happens $\tau_{vis} < \tau_{coll}$ or $\tau_{vis} \sim \tau_{coll}$? $\Rightarrow$ Quenched state
- Ref. Saha & Alam (2017), JFM, vol. 833

QUENCHED STATE

Particle inertia is small and they follow fluid-motion (Tsao and Koch 1995)

- Viscous relaxation time $\tau_{vis} \ll$ collision time τ_{coll}
- Particle agitation is small: $T/\dot{\gamma}\sigma \ll 1$
- Particle velocity is equal to the local fluid velocity $\Rightarrow \boldsymbol{c} \approx \boldsymbol{u} \Rightarrow \boldsymbol{C} \approx \boldsymbol{0}$

Velocity distribution function (leading order):

$$f = n\delta(\boldsymbol{C}). \tag{39}$$

$$\aleph^{qs}_{\alpha\beta} = \rho_p \dot{\gamma}^3 \sigma^2 \frac{(1+e)^2 \nu}{16} \begin{bmatrix} \frac{512}{315\pi} & -\frac{16}{35} & 0 \\ -\frac{16}{35} & \frac{512}{315\pi} & 0 \\ 0 & 0 & \frac{128}{315\pi} \end{bmatrix}. \tag{40}$$

COMBINING QUENCHED AND IGNITED STATES

- Ansatz: Both variance-driven and shear-induced collisions are important
- Second moment balance:

$$P_{\delta\beta}u_{\alpha,\delta} + P_{\delta\alpha}u_{\beta,\delta} + \frac{2\dot{\gamma}}{St}P_{\alpha\beta} = \aleph^{qs}_{\alpha\beta} + \aleph^{is}_{\alpha\beta} \tag{41}$$

- Transition between ignited and quenched states?
- ‘Quantitative’ prediction of hydrodynamics and rheology at small *St*?

GRANULAR TEMPERATURE: "ASYMPTOTIC" SOLUTION

$$\mathcal{G} \equiv a_{10}\xi^{10} + a_9\xi^9 + a_8\xi^8 + a_7\xi^7 + a_6\xi^6 + a_5\xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 = 0$$

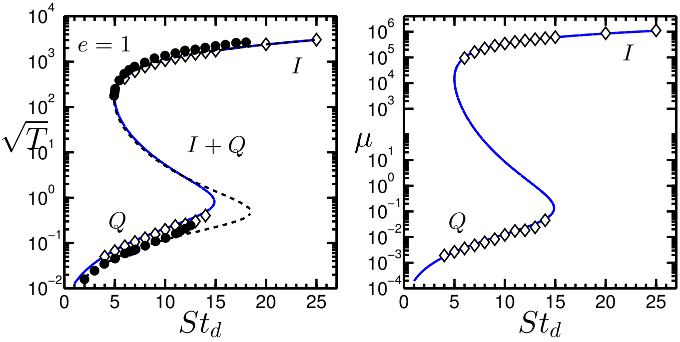
$\xi = \sqrt{T}$, $\nu \ll 1, St \gg 1$ and $St^3\nu \ll 1$

- $$\sqrt{T_{qs}} = \sqrt{\frac{32(1+e)^2}{945\pi}} St^{3/2}\nu^{1/2} \overset{e=1}{\equiv} \frac{8\sqrt{2}}{3\sqrt{105\pi}} St^{3/2}\nu^{1/2}; \quad St < St_{c_2}$$
- $$\sqrt{T_{is}} = \frac{5(1+e)^{-1}(1691 + 539e - 1223e^2 + 337e^3)\sqrt{\pi}}{48(3-e)(12607 - 19952e + 10099e^2 - 1746e^3)} \left(\frac{St}{\nu}\right); \quad St > St_{c_1}$$
- $$\sqrt{T_{us}} = \frac{840\sqrt{\pi}}{(1+e)(107+193e)} \left(\frac{1}{St^3\nu}\right) \overset{e=1}{\equiv} \frac{7\sqrt{\pi}}{5} \left(\frac{1}{St^3\nu}\right); \quad St_{c_1} < St < St_{c_2}$$

Granular Temperature: Comparison with simulation

Temperature equation is solved numerically; $T \equiv T(St, \nu, e)$

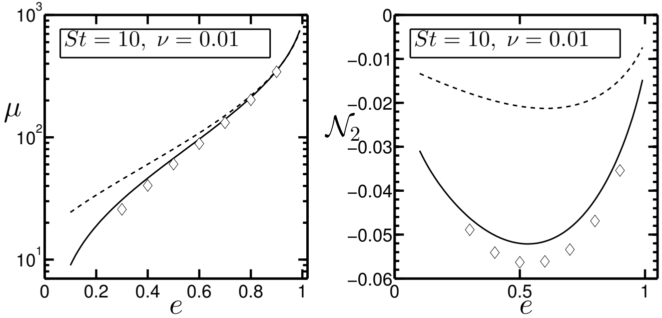
$\nu = 5 \times 10^{-4}, \quad e = 1$ Viscosity



- dashed line (GME): Sangani et al. (1996)
- solid line (AME): present theory

NSDs AND VISCOSITY IN THE DILUTE LIMIT

Dilute Limit Rheology ("ignited" state):

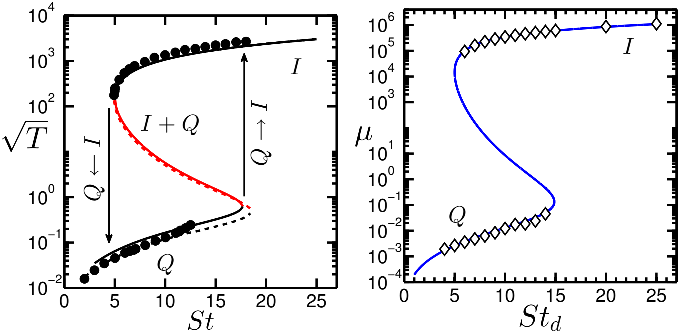


- Solid line (AME): Saha and Alam, J. Fluid Mech **833** (2017)
- Dashed line (GME): Sangani et al. (1996) [$\mathcal{N}_1$]; Tsao & Koch (1995) and Chamorro et al. (2015) [$\mathcal{N}_2$]

Granular Temperature: Hysteresis

Temperature equation is solved numerically; $T \equiv T(St, \nu, e)$

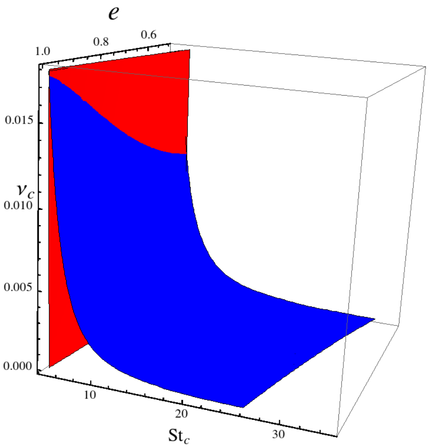
$\nu = 5 \times 10^{-4}, \quad e = 1$ Viscosity



- dashed line: Sangani et al. (1996)

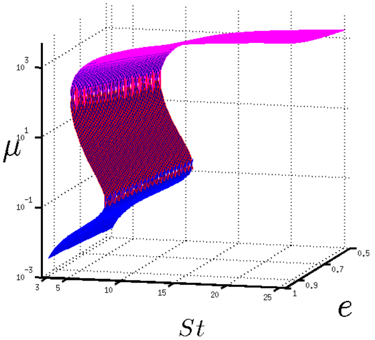
CRITICAL STOKES NUMBERS AND MASTER PHASE DIAGRAM

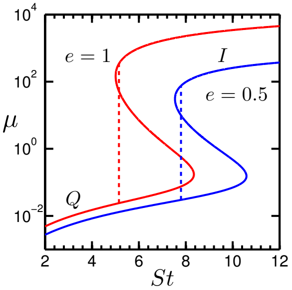
$$St_{c_1} \approx 9.9 - 4.91e, \qquad St_{c_2}^3 \nu_c = \left(\frac{3087000\pi^2}{(1+e)^4(107+193e)^2} \right)^{\frac{1}{3}}. \qquad (42)$$



SHEAR VISCOSITY: SHEAR-THICKENING

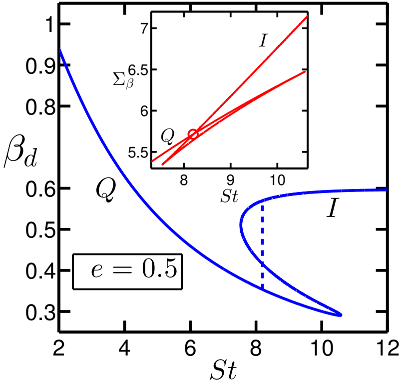
$\nu = 0.005$





$$\textit{Scaling}: \quad \mu_{is} \sim \left(\frac{St}{\nu^2}\right), \ \mu_{qs} \sim \left(\nu^2 St^2\right), \ \mu_{us} \sim \left(\nu^{-2} St^{-7}\right) \tag{43}$$

SELECTED BRANCH?



- Dynamic Friction: $\beta_d = \frac{P_{xy}}{p}$
- $\Sigma_\beta(\dot\gamma) = \int_0^{\dot\gamma} \beta_d(\dot\gamma) d\dot\gamma$
- Selected branch: $\sup \Sigma_\beta$ ('Massieu-like' function?)
- Maxwell's equal-area rule?

CONCLUSIONS: GAS-SOLID SUSPENSION

- Hysteresis in transport coefficients (in dilute regime) is tied to competition between “variance-driven” and “shear-induced” collisions.
- Same expressions for all transport coefficients for both granular and gas-solid suspensions.
- Dependence on St is implicit via second-moment anisotropy (η, ϕ, λ^2)
- Limit of $St \to \infty$, results for dry granular flows are recovered
- Ref. Saha & Alam (2017), JFM, vol. 833
- Excellent predictions of AME with simulation even at $St = 0.1$ over whole range of density $\nu \in (0, 0.5)$ (Saha et al 2018a)!

- Acknowledgement
  - Department of Science and Technology, Govt. of India
  - Prof. Hisao Hayakawa, YITP, Kyoto University

Thank You