Interaction and drag in non-equilibrium environments: from the study of granular materials

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Rheology of disordered particles-suspensions, glassy and granular particles-
Talk on June 21st, 2018
Outline

• Introduction: What do we know about drag in granular media?
• Previous study for pure 2D drag in granular media
• Previous studies on the interaction between intruders
• Simulations of two intruders in 2D granular media
  – Two intruders in a steady motion
  – Two intruders under an oscillation
• Phenomenological theory for the drag and the interactions for intruders
• Discussion & conclusions
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The disk is fixed by a wire.
See S. Takada’s talk yesterday.
Interactions of intruders in granular assemblies

**Brazil Nuts Effect (ED)**


- \( \rho_L < \rho_S \)
- \( \rho_L = \rho_S \)
- \( \rho_L > \rho_S \)

\[ \frac{\rho_L}{\rho_S} \approx 1.0 \Rightarrow \text{Attraction} \]

\( \rho_L, \rho_S \): Large/Small granular density

**Mechanism**

Lower density region

More collisions at region B than region A cause attraction (going to lower density)

**Rolling Cylinder (Exp.)**


- (a) \( f > f_{th} \) \Rightarrow \text{Separation}
- (b), (c) \( f < f_{th} \) \Rightarrow \text{Attraction}

\[ f < f_{th} \Rightarrow \text{Attraction} \]

\[ f > f_{th} \Rightarrow \text{Separation} \]

**Mechanism**

\[ f < f_{th} \Rightarrow \delta U^2 < 0 \]

(large fluctuation outside)

\[ \Rightarrow \delta P = \rho \frac{\delta U^2}{2} < 0 \]

Pressure difference makes attraction.

\[ \delta U^2 = -[U_{rms(out)}^2 + V_{rms(out)}^2 - (U_{rms(in)}^2 + V_{rms(in)}^2)] \]
Purpose of this talk

• This talk is dedicated to the theory for the drag force and the interaction of intruders mainly in granular media
• For this purpose, we review previous works.
• Then, we introduce recent numerical simulations by Tanabe.
• Finally, I will explain some phenomenological theories to understand the results of simulations and experiments.
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The streamline obtained from MD is well fitted by that of the *perfect fluid*.

Cavity behind the tracer (Dead water model can be used.)
Comparison with simulation result

Simulation results are well fitted \textbf{without any fitting parameters}! This is consistent with Chicago group for granular jet.

\[ F_{\text{drag}} = \left( \frac{3 + 2e}{2} - \frac{2}{3} \sin^2 \theta_0 \right) \sin \theta_0 D \rho V^2 \]
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Depletion force & adiabatic piston

- The problem is related to the depletion force (in a constriction region).
- Note that the conventional depletion force is the result of the violation of the pressure balance => acceleration.
- However, we are interested in the adiabatic motion under the pressure balance. => Casimir effect or adiabatic piston
The adiabatic piston

* The adiabatic piston problem is equivalent to fluctuating 1d depletion force problem.
* If the density is lower in the depletion region, the attractive interaction between pistons exist under the pressure balance as a result of non-Gaussian correction.

\[ V_{ad} \equiv \frac{\varepsilon \sqrt{\pi}}{4} v_T^2 \left( \frac{1}{v_{T_L}} - \frac{1}{v_{T_R}} \right) \]

\[ v_T \equiv \sqrt{2k_BT/M} \]

\[ \varepsilon \equiv \sqrt{m/M} \]
We begin with the generalized Fokker-Planck equation:

$$\partial_t P = \sum_{k=1}^{N} \left\{ \gamma^{-1} \nabla_k (P \nabla_k U) + \mathcal{L}_k P \right\}$$

\[
\mathcal{L}_k = \lambda \int \sum_{n=1}^{\infty} \frac{(-\mathbf{x} \cdot \nabla_k)^n}{n!} \mathcal{W} (\mathbf{x}) d\mathbf{x} = \lambda \int (e^{-\mathbf{x} \cdot \nabla_k} - 1) \mathcal{W} (\mathbf{x}) d\mathbf{x}.
\]

\[
\mathcal{L}_k = \lambda [\widetilde{\mathcal{W}} (i \nabla_k) - 1]. \quad \widetilde{\mathcal{W}} (\mathbf{q}) := \int d\mathbf{x} e^{-i\mathbf{q} \cdot \mathbf{x}} \mathcal{W} (\mathbf{x})
\]

If the hopping is nonlocal, we need higher order derivative in Fokker-Planck equation.
Effective interaction in steady state

\[
0 = \gamma^{-1} P_{ss} \nabla_k U + Db_{d\nu} \nabla_k \left(1 - a^2 c_{d\nu} \nabla_k^2 \right) P_{ss} \\

P_{ss} = \frac{e^{-\beta U}}{Z} \left[1 - a^2 \beta W + O(a^4) \right] \\

\beta = \frac{1}{(\gamma Db_{d\nu})} \\

W = c_{d\nu} \left\{ \beta \sum_{k=1}^{N} \nabla_k^2 U - \frac{3}{2} \sum_{\{k,\ell\}=1}^{N} \nabla_k U \cdot \nabla_\ell U + \beta^3 \int (\nabla_k U)^3 \, dr \right\}
\]
Both models have effective attractive interactions. Nevertheless, the AOUP exhibits strong phase separation but the non-Gaussian model does not.

AOUP = Active Ornstein Uhrenbeck process
Fodor et al., PRL 117, 038103 (2016).
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Simulation for two intruders

\[ m_i \ddot{r}_i = \sum_j F_{c,i,j}^{i,j} + F_{ex} \]

1. Repulsive force (only normal force)

\[ F_{c,i,j}^{i,j} = k_n \delta_{i,j} n_{i,j} - \gamma_n \nu_{n,i,j} \]

2. Driving force (follow tray oscillation)

\[ F_{ex} = -\mu(\nu_i - \nu_{ex}) \]

- Fixed flat wall \( L_y = 60D_S \)
- Periodic boundary \( L_x = 80D_S \)
- Two intruders are pinned in system
Explanation for simulations

\[ v_{ex}^x = V^* (D \sqrt{k_n/M_s}) \]

- Diameter of disks (density: const.)
  \[ D_S = D (1 \pm 0.1r) : \text{Small disks} \]
  (poly-dispersion: Uniform distr.)

- Intruders are fixed on system with distance \( L_0 \) for y-direction

- Control parameter: non-dimensional speed \( V^* \)

  - Fixed flat wall \( L_y = 60D_S \)
  - Periodic boundary \( L_x = 80D_S \)
  - Two intruders are pinned or mobile in system


\[ m_i \ddot{r}_i = \sum_j F_{c}^{i,j} + F_{ex} \]

- Disk index \( i \)
- Position \( r_i \)
- Mass \( m_i \)
- Drag and interaction of intruders
Separation angle: Low V differs from high V.

Flow time > relaxation time

\[ \frac{D}{v_{ex}} > \tau \Rightarrow \frac{v_{ex} \tau}{d} < 10: \text{Flow is different.} \]

\[ \frac{v_{ex} \tau}{d} = 4.71 \]

\[ \frac{v_{ex} \tau}{d} = 23.57 \]

No coalescence of wakes

2018/06/21 Drag and interaction of intruders
Granular flow: Velocity & Separation angle

Velocity field

Separation angle

$<V_{\theta}> / V_{ex}$

$\theta$

$R/d = 11.5$
$R/d = 12.5$
$R/d = 14.5$
$R/d = 30.5$
$R/d = 50.5$

$\theta_c$
$2\pi - \theta_o$

$V_{ex}$

$2\pi - \theta_o$

Drag and interaction of intruders

2018/06/21
Granular temperature (steady flow)

Fitting with $\cos^2 \theta$

One intruder

$$T_G(\phi) = \begin{cases} 
0, & |\phi + \Delta| < \phi_S \\
\gamma (\cos((\phi + \Delta) - \pi))^2, & \text{otherwise}
\end{cases}$$

$(\gamma = 0.47, \Delta = 0.203, \phi_S = 1.5691)$

($\phi_S$: separation angle)
Repulsive force between two intruders

• The effective repulsive interaction exists between two.

• $\Rightarrow$ This is contrast to perfect fluidity (Bernoulli).

![Graphs showing drag and interaction of intruders]
Both $F_{\text{drag}}(R)$ & $F_{\text{int}}(R)$ can be fit as $F(R) = a + bR^{-\alpha}$ ($\alpha \approx 1$).

$F_{\text{drag}}$ is suppressed when $R$ is small because the flow speed decreases.

$F_{\text{int}}$ approaches 0 as increasing $R$. 
Drag and interaction of intruders
Origin of attractive force

• There are wakes in front and back of intruders.
• Grains in the channel are squeezed out.
• Density in the channel is much lower than that outside.
• Pressure difference exists.
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Outline of the theory

• We assume that Enskog theory for binary systems can be used.
• Then, the environment is assumed to be Maxwellian.
• Then, we adopt Kramers-Moyal expansion.
• We can determine the drag force if we know the temperature.
• The temperature is determined by a phenomenological simple model.
• The interaction is also guessed from a simple crude theory.
Enskog theory for mixtures

The distribution function for $i$–species $f_i(V, t)$ for the velocity

$$\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) f_i(V, t) = \sum_{j=1}^{2} J_{ij}^E(V | f_i, f_j),$$

$$J_{ij}^E(V_1 | f_i, f_j) = d_{ij}g_{ij} \int dV_2 \int d\sigma \Theta(\sigma \cdot V_{12})(\sigma \cdot V_{12})$$

$$\times \left[ \frac{1}{e_{ij}^2} f_i(r, V_i''; t)f_j(r + \sigma, V_2''; t) - f_i(r, V_i; t)f_j(r - \sigma, V_2; t) \right]$$

with the radial distribution at contact $g_{ij}$ between $i$ and $j$, the precollisional velocities $V_i''$, $V_{12} = V_1 - V_2$, restitution coefficient $e_{ij}$ between $i$ and $j$.
intruder (mass $M$ and the diameter $D$) in another species (consisting of $N$ identical disks of the mass $m$

a small parameter $\epsilon \equiv \sqrt{m/M}$

the geometric factor

$$\chi = \frac{1}{2} n(d + D) g_{dD}(\varphi)$$

$$g_{dD}(\varphi) = \frac{1}{1 - \varphi} + \frac{9\varphi}{16(1 - \varphi)^2} z \quad z = 2dD\langle\sigma\rangle/((d + D)\langle\sigma^2\rangle)$$
• Boltzmann-Enskog equations for the intruder and grains are given by

\[
\frac{\partial P(\mathbf{V}, t)}{\partial t} = \int d\mathbf{V}' \left[ W_{\text{tr}}(\mathbf{V}|\mathbf{V}') P(\mathbf{V}', t) - W_{\text{tr}}(\mathbf{V}'|\mathbf{V}) P(\mathbf{V}, t) \right] \\
+ B_{\text{tr}} P(\mathbf{V}, t)
\]

\[
\frac{\partial p(\mathbf{v}, t)}{\partial t} = \int d\mathbf{v}' \left[ W_{\text{g}}(\mathbf{v}|\mathbf{v}') p(\mathbf{V}', t) - W_{\text{g}}(\mathbf{v}'|\mathbf{v}) p(\mathbf{v}, t) \right] \\
+ B_{\text{g}} p(\mathbf{v}, t) + \chi J_E[\mathbf{v}|p, p],
\]
Transition rate

- Transition rate is given by

\[ W_{tr}(V|V') = \chi \int dv' \int d\hat{\sigma} P(v', t) \Theta(-(V' - v') \cdot \hat{\sigma})(V' - v') \cdot \hat{\sigma} \delta \left( V - V' + \frac{\epsilon^2}{1 + \epsilon^2 (1 + \epsilon)} (V' - v') \cdot \hat{\sigma} \right) \]

and

\[ W_g(v|v') = \frac{\chi}{N} \int dv' \int d\hat{\sigma} P(V', t) \Theta(-(V' - v') \cdot \hat{\sigma})(V' - v') \cdot \hat{\sigma} \delta \left( v - v' + \frac{1}{1 + \epsilon^2 (1 + \epsilon)} (v' - V') \cdot \hat{\sigma} \right), \]

\[ V' = V - \frac{1 + \epsilon}{e} \frac{M}{M + m} \hat{\sigma} \hat{\sigma} \cdot (V - v) = V + (1 + \epsilon) \frac{1}{1 + \epsilon^2} \hat{\sigma} \hat{\sigma} \cdot (V' - v'). \]

\( \hat{\sigma} \) is the unit normal at contact and \( \Theta(x) \) is Heaviside’s step function.

If we assume \( p(v) = (m/2\pi T)^{d/2} \exp\left[-mv^2/(2T)\right] \), one can rewrite

\[ W_{tr}(v'_1|v_1) = \frac{\chi |\Delta v|^{2-d}}{k(\epsilon, e)^2} \sqrt{\frac{m}{2\pi T}} \exp\left[-mv^2_{2\sigma}/2T\right]. \]

If we are interested in the case of \( d = 2 \),

\[ W_{tr}(V'|V) = \frac{\chi}{k(\epsilon, e)^2} \sqrt{\frac{m}{2\pi T}} \exp \left[-\frac{m[V'_\sigma - V_\sigma + k(\epsilon, e)V_\sigma]^2}{2Tk(\epsilon, e)^2} \right], \]
\[
\frac{\partial P(V, t)}{\partial t} = \mathcal{L}_{\text{gas}} P(V, t) + \mathcal{B}_{\text{tr}} P(V, t),
\]

Introducing \( W(V; \nu) \equiv W_{\text{tr}}(V' | V) \)

\[
\mathcal{L}_{\text{gas}} P(V, t) = - \int d\nu W(V; \nu) P(V, t) + \int d\nu W(V - \nu; \nu) P(V - \nu, t)
\]

\[
= - \int d\nu W(V; \nu)d\nu P(V, t) + \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left( \frac{\partial}{\partial V} \right)^n \cdot \int d\nu \nu^n W(V; \nu) P(V, t)
\]

\[
= \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \left( \frac{\partial}{\partial V} \right)^n \cdot \int d\nu \nu^n W(V; \nu) P(V, t),
\]

where we have used the formal relation

\[
f(V - \nu) = \sum_{n=0}^{\infty} \frac{(-\nu)^2}{n!} \cdot \left( \frac{\partial}{\partial V} \right)^n f(V) = \exp[-\nu \cdot \frac{\partial}{\partial V}]f(V)
\]
Summary of Kramers-Moyal expansion

\[
\mathcal{L}_{\text{gas}} P(V, t) = \sum_{n=1}^{\infty} \frac{(-1)^n \partial^n}{\partial V_{\alpha_1} \cdots \partial V_{\alpha_n}} D_{\alpha_1 \ldots \alpha_n}^{(n)}(V) P(V, t)
\]

\[
D_{\alpha_1 \ldots \alpha_n}^{(n)}(V) = \frac{1}{n!} \int dV' (V'_{\alpha_1} - V_{\alpha_1}) \cdots (V'_{\alpha_n} - V_{\alpha_n}) W_{\text{tr}}(V'|V).
\]
Thermal system

• Drag law

\[ F_{ex} = -M D_x^{(1)}(V). \]

• If we assume that the thermal speed is larger than pulling speed, we obtain

\[ D_x^{(1)} \approx -2(1 + e)e^2/(3(1 + e^2)) \chi \sqrt{2\pi T/mV_x} \text{ for } V_x \ll \sqrt{T/m}, \]

• Then, we reach

\[ F_{ex} \approx \frac{\chi(1+e)}{3(1+e^2)} \sqrt{2\pi mT V_x} \]
Granular or athermal system

• We need to determine the temperature for athermal systems.
• For this, we adopt the simple picture:
Mean flow and the temperature

- The mean flow is the tangential projection of particle velocity.
- The fluctuation is the normal velocity:

\[ \delta v_n = \pm v \cos \theta \]

\[ T(\theta) \approx m \langle (\Delta v)^2 \rangle = mv^2 \cos^2 \theta. \]
Using the average temperature

\[ T \approx \overline{T(\theta)} = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\theta T(\theta) = m \frac{V^2}{\pi n_0} \int_{-\pi/2}^{\pi/2} d\theta \cos^2 \theta = m\frac{V^2}{2} \]

We obtain

\[ F_{\text{drag}} = \frac{\sqrt{\pi(1+e)e^2}}{6(1+e^2)} n(d + D) g_d D(\varphi) e^{-1/2} (5I_0(1/2) + 3I_1(1/2)) V^2, \]
Interaction between two intruders

- Let us consider a steady flow:
A simplified model

- Two-body problem can be reduced to one-body problem:

\[
\begin{align*}
\Delta T & \approx m \left( \frac{V^2}{4} \right) \sin^2 (2\theta_0), \\
\Delta T(\theta) & \approx \left( \frac{1}{4} \right) m \dot{V}^2 \sin^2 (2\theta_0) \exp[-2(\theta - \theta_0)^2]
\end{align*}
\]

The perpendicular component to the wall is \( u \sin(\pi/2 - \theta_0) = u \cos \theta_0 = (V_0/2) \sin 2\theta_0 \).
Two-body interaction

• Induced pressure

\[ \Delta P(\theta) \approx n g d_d(\varphi) \Delta T(\theta). \]

• Total induced force is repulsive:

\[ F_{\text{tot}} \approx (1/4) n g d_D(\varphi) m V^2 \sin^2(2\theta_0) \int_{-\infty}^{\infty} d\psi e^{-2\psi^2} = (1/4) \sqrt{\pi/2} n g d_D(\varphi) m V^2 \sin^2(2\theta_0). \]

the separation angle \( \theta_0 \) \( \theta_0 \approx 2\pi/5 \) in the simulation.

\[ F_{\text{tot}} = 0 \text{ if } \theta_0 = \pi/2 \]

![Graph showing action of intruders](image-url)
Separation angle?

- Although the perfect fluid model cannot explain the repulsive interaction, the separation angle seems to have a similar tendency to that observed in the simulation, if we assume that the separation takes place at zero pressure.

\[ \omega_-(\text{Inviscid Flow}) \quad \psi_c (\text{granular flow}) \]
Application of perfect fluid model to two single bodies

\[ F_{\text{drag}} = \frac{D}{2} \rho V^2 \left[ \left( e + \frac{3}{2} \right) - \frac{2}{3} \sin^2 \psi_c \right] \sin \psi_c + \left( e + \frac{5}{6} \right) \]

\[ F_{\text{int}} = \frac{D}{2} \rho V^2 \left( - \left( e - \frac{1}{2} \right) - \frac{2}{3} \cos^2 \psi_c \right) \cos \psi_c \]

Drag and resistance anaginst separation angle \( \psi_c \) where \( \psi_o = \frac{3\pi}{2} \)

Note that pure two body calculation gives attractive interaction even if there exists the separation.
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The origin of the attractive interaction in oscillatory systems can be understood qualitatively:

- There are wakes (cavities) in front and back of the intruders.
- Grains in the channel can be squeezed out because the wake regions do not have pressure.
- The density in the channel is much lower than the outside density.
- The pressure also has the same tendency.
- So the intruders approach with each other.
Conclusions

• Perfect fluidity seems to work for the drag force in pure 2D.
• However, the interaction between two intruders cannot be expressed by the perfect fluid model.
• Instead, a simple model can be used for the drag force and the interaction in a steady flow.
• The qualitative picture for the interaction in an oscillatory flow exists, but there is no quantitative description.
THANK YOU