Fluidity, Anisotropy, and Velocity Correlations in Collisional Granular Shearing Flows

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Kamrin, et al. have introduced fluidity, $g \equiv \dot{\gamma} p / s$, as a measure of a granular material's resistance to flow.

$\dot{\gamma}$ shear rate ($u_1 = \dot{\gamma} x_2$), p pressure, s shear stress

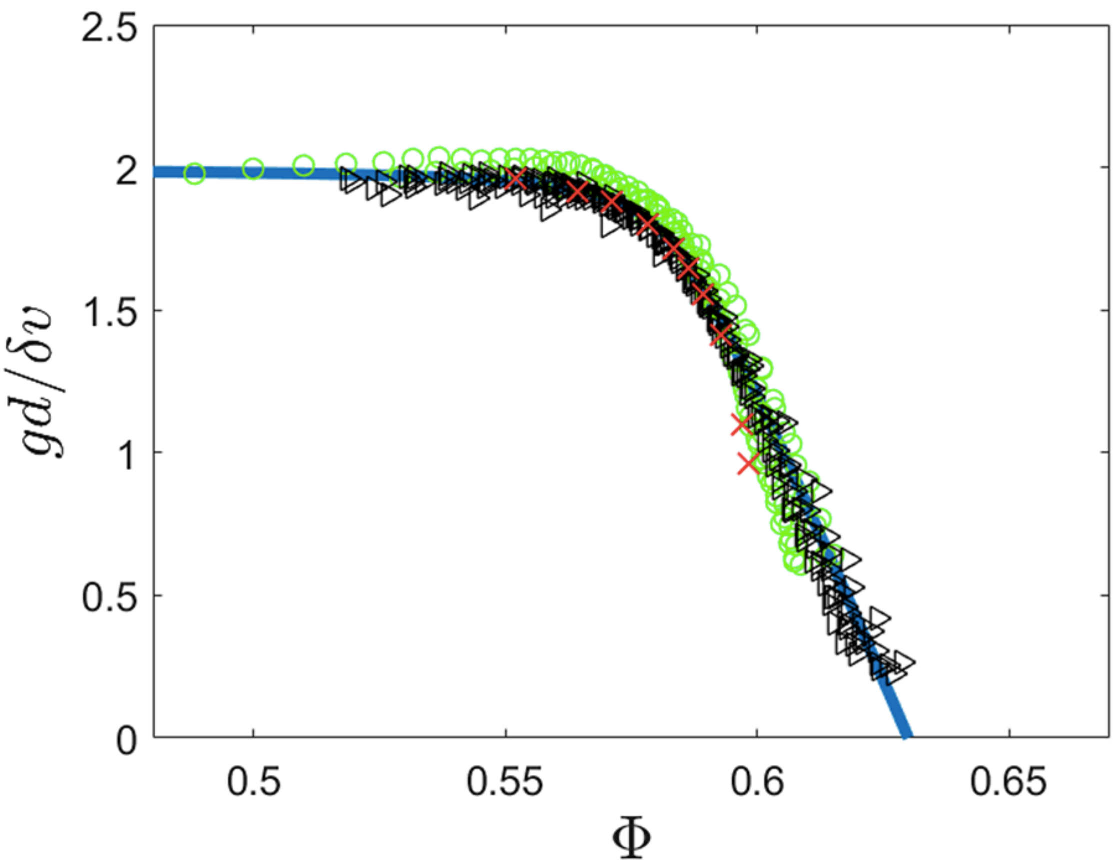
Show that fluidity of frictionless spheres in steady, homogeneous shearing flows, made appropriately dimensionless, is predicted by existing kinetic theories.

Provide an explanation for the deviations of the predictions from the results of discrete numerical simulations in the limit of random close packing.

Fluidity: $g = \dot{\gamma}\frac{p}{s}$

Zhang & Kamrin, PRL 118, 058001 (2017):

Make the fluidity dimensionless with the sphere diameter d and the strength of the velocity fluctuations, $\delta v = \sqrt{\langle C^2 \rangle / 3}$, and plot simulation results versus volume fraction, Φ.



Dimensionless fluidity, $gd / \delta v$, collapses the data from different discrete numerical simulations of steady, inhomogeneous shearing flows of frictional spheres.

Kinetic theory

Identical frictionless spheres of diameter σ, mass density ρ_p, volume fraction ν, restitution e:

$$p = \rho_p f_1 T \text{ and } s = \rho_p \sigma T^{1/2} f_2 \dot{\gamma},$$

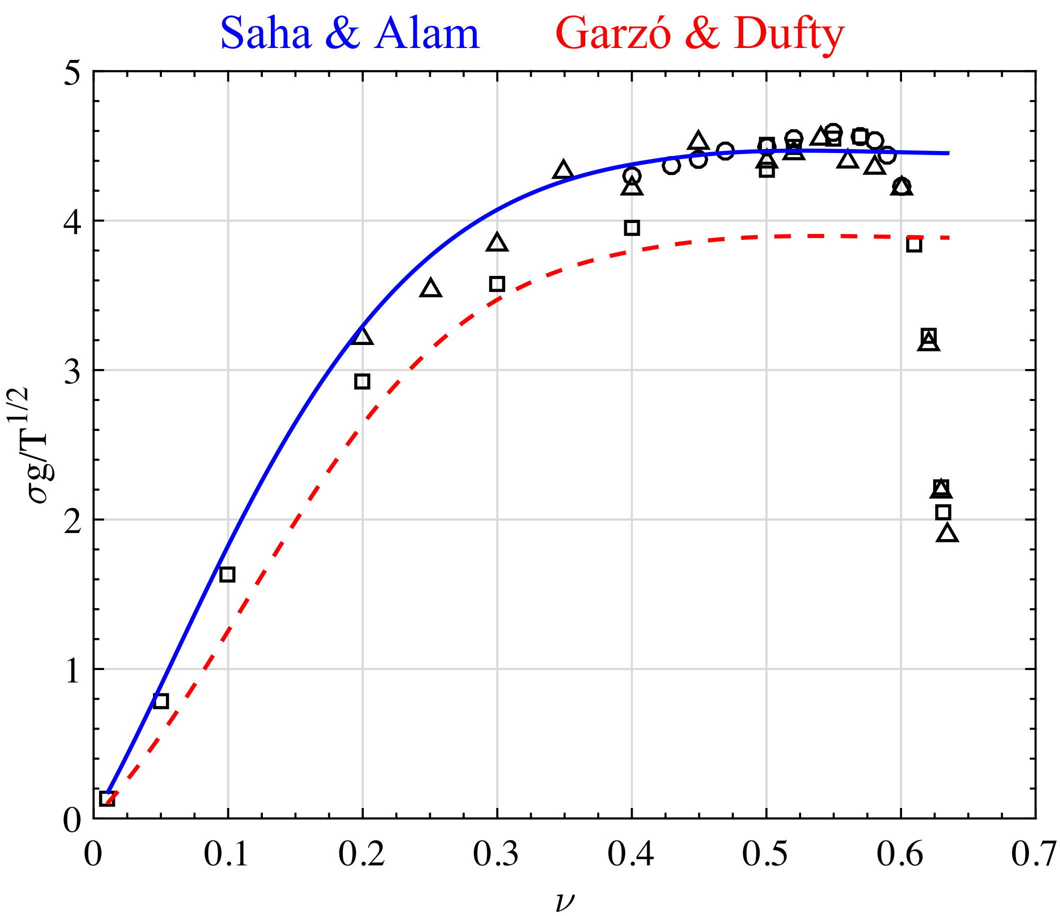
where f_1 and f_2 functions of ν, e, and, perhaps, $\sigma\dot{\gamma} / T^{1/2}$.

Fluidity: $g = \dot{\gamma}\dfrac{p}{s} = \dfrac{T^{1/2}}{\sigma}\dfrac{f_1}{f_2}$

Dimensionless fluidity: $\dfrac{\sigma g}{T^{1/2}} = \dfrac{f_1}{f_2}$

Dimensionless shear rate: $R \equiv \dfrac{\sigma\dot{\gamma}}{8T^{1/2}}$

Kinetic Theories



Saha & Alam: J. Fluid Mech. 795, 549 (2016)

$$f_1 = 2(1+e)\nu^2\chi_0\left(1+\frac{224}{105}R^2+\frac{112}{105\sqrt{\pi}}\eta R\right)$$

$$f_2 = \frac{4(1+e)}{5\sqrt{\pi}}\nu^2\chi_0\left(1+\sqrt{\pi}\frac{\eta}{8R}\right)$$

$\chi_0 = \chi_0(\nu)$: radial distribution function at contact

$\eta = \eta(\nu, R)$: in-plane anisotropy of the velocity fluctuations

Second velocity moment

$$\langle C_i C_j \rangle = T \begin{bmatrix} 1+\lambda^2-\eta & 0 & 0 \\ 0 & 1+\lambda^2+\eta & 0 \\ 0 & 0 & 1-2\lambda^2 \end{bmatrix}$$

λ^2: out-of-plane anisotropy

Dense flows: $\nu > 0.49$

Balance of in-plane deviator

$$\dot{\gamma}\left(\Theta_{xx}+\Theta_{yy}\right)=\Gamma_{xx}-\Gamma_{yy}$$

$$\Theta_{xx}+\Theta_{yy}=\frac{4(1+e)\rho_p \nu^2 \chi_0 T}{\pi^{1/2}}$$

$$\Gamma_{xx}-\Gamma_{yy}=\frac{24(42)(1+e)\rho_p \nu^2 \chi_0 T^{3/2}}{105\sigma\pi^{1/2}}\left[(1-e)\left(2\sqrt{\pi}R+\eta\right)+2\eta+2R\sqrt{\pi}\right]$$

Balance of trace

$$\dot{\gamma}\left(\Theta_{xx}-\Theta_{yy}\right)=A_{xx}+A_{yy}+A_{zz}$$

$$\Theta_{xx}-\Theta_{yy}=-\frac{8(1+e)\rho_p \nu^2 \chi_0 T}{35\pi^{1/2}}\left(56R+7\sqrt{\pi}\eta\right)$$

$$A_{xx}+A_{yy}+A_{zz}=-\frac{4(1-e^2)\rho_p \nu^2 \chi_0 T^{3/2}}{35\sigma\pi^{1/2}}\left(210+672R^2+21\eta^2+2\sqrt{\pi}\eta R\right)$$

Solutions

$$\eta = \frac{2\sqrt{\pi}}{3(3-e)}\left[5-3(2-e)\right]R$$

$$16R\left(\sqrt{\pi}\eta + 8R\right) = 3(1-e)\left(10 + 32R^2 + \eta^2 + 8\sqrt{\pi}\eta R\right)$$

So $\eta \propto R,\ R = R(e),$ and

$$\frac{\sigma g}{T^{1/2}} = \frac{4\left(105\sqrt{\pi} + 224\sqrt{\pi}R^2 + 112\eta R\right)}{21\left(8 + \sqrt{\pi}\eta / R\right)}$$

is independent of ν (i.e., flat).

What's missing?

Velocity Correlations

Correlation length: L

$$\frac{L}{\sigma} = f_0 \frac{\sigma \dot{\gamma}}{T^{1/2}}$$

Competition between the ordering influence of shearing and the randomizing influence of collisions

$$f_0 = \left[\frac{2J}{15\left(1-e^2\right)} \right]^{1/2} \left[1 + \frac{1}{10}\left(\frac{\nu - 0.49}{0.64 - \nu} \right) \right]^{3/2}$$

$$J = \frac{1+e}{2} + \frac{\pi}{4} \frac{\left(1+e\right)^2 \left(3e-1\right)}{24 - 6(1-e)^2 - 5\left(1-e^2\right)}$$

$$\frac{L}{\sigma} = 8 f_0 R$$

Replace σ by L in the collisional productions of second moment.

Collisional Production

$$\Gamma_{xx} - \Gamma_{yy} = \frac{24(42)(1+e)\rho_p \nu^2 \chi_0 T^{3/2}}{105 L \pi^{1/2}} \left[(1-e)\left(2\sqrt{\pi}R + \eta\right) + 2\eta + 2R\sqrt{\pi} \right]$$

$$A_{xx} + A_{yy} + A_{zz} = -\frac{4(1-e^2)\rho_p \nu^2 \chi_0 T^{3/2}}{35 L \pi^{1/2}} \left(210 + 672R^2 + 21\eta^2 + 2\sqrt{\pi}\eta R\right)$$

where

$$L = 8 f_0 R \sigma$$

New solutions to the balance equations

$$\eta = \frac{2\sqrt{\pi}}{3(3-e)} \left[40 f_0 R - 3(2-e) \right] R$$

and

$$128 f_0 R^2 \left(\sqrt{\pi}\eta + 8R\right) = 3(1-e)\left(10 + 32R^2 + \eta^2 + 8\sqrt{\pi}\eta R\right)$$

At lowest order,

$$R \doteq \frac{1}{f_0^{1/2}} \left\{ \frac{9(3-e)^2(1-e)}{128\pi\left[8(3-e)-5(1-e)\right]} \right\}^{1/4}$$

and

$$\frac{\eta}{R} = \frac{80\sqrt{\pi}}{3(3-e)} \left\{ \frac{9(3-e)^2(1-e)}{128\pi\left[8(3-e)-5(1-e)\right]}^{1/4} \right\} f_0^{1/2}$$

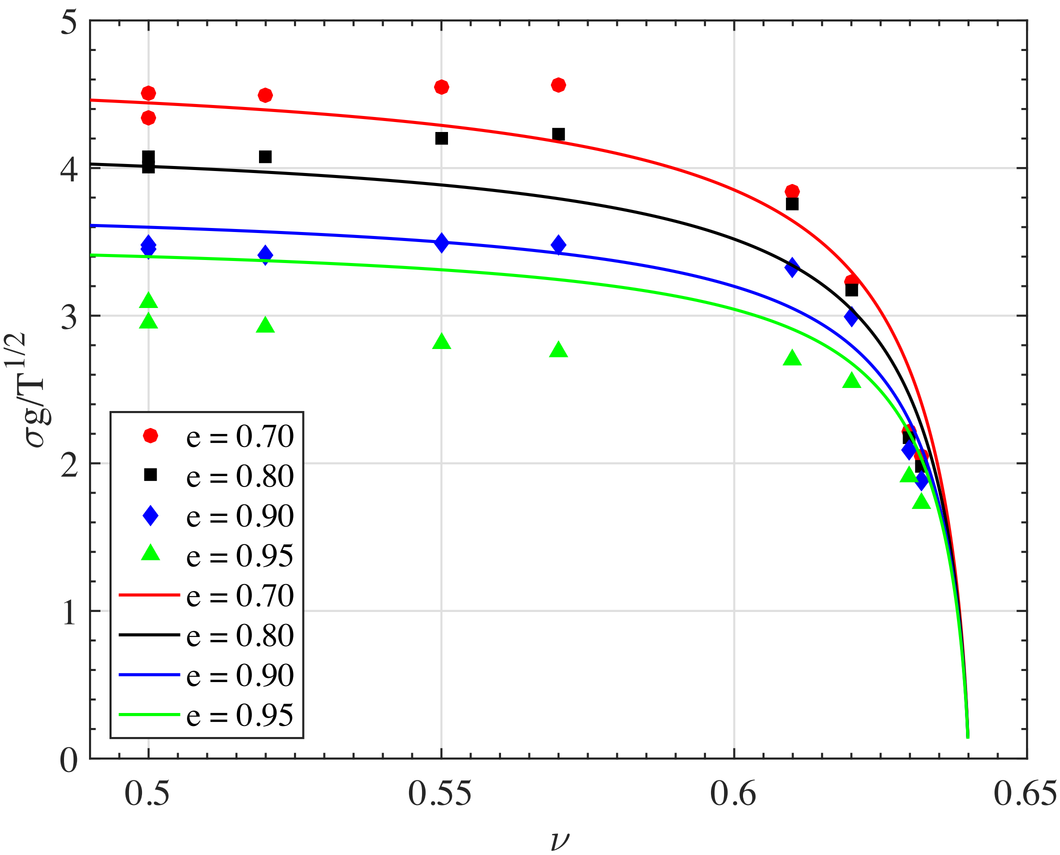
As $\nu \to 0.64$,

$f_0 \to \infty,\ R \to 0,$ and $\eta/R \to \infty;$

so,

$$\frac{\sigma g}{T^{1/2}} = \frac{4\left(105\sqrt{\pi} + 224\sqrt{\pi}R^2 + 112\eta R\right)}{21\left(8 + \sqrt{\pi}\eta / R\right)} \to 0.$$

Discrete Numerical Simulations



Inclusion of the correlation length in the collisional production terms provides the theory with the capacity to reproduce the observed behavior of the dimensionless fluidity as random close packing is approached.