Shear modulus of granular materials near jamming transition

Michio Otsuki (Osaka Univ.),

Collaborators : Hisao Hayakawa (Kyoto Univ.), Kazuhisa Inata (Shimane Univ.)

Outline

- **Introduction : Jamming transition**

- Shear modulus of granular materials

    1. Finite strain

       MO and H. Hayakawa, PRE 90, 042202 (2014)

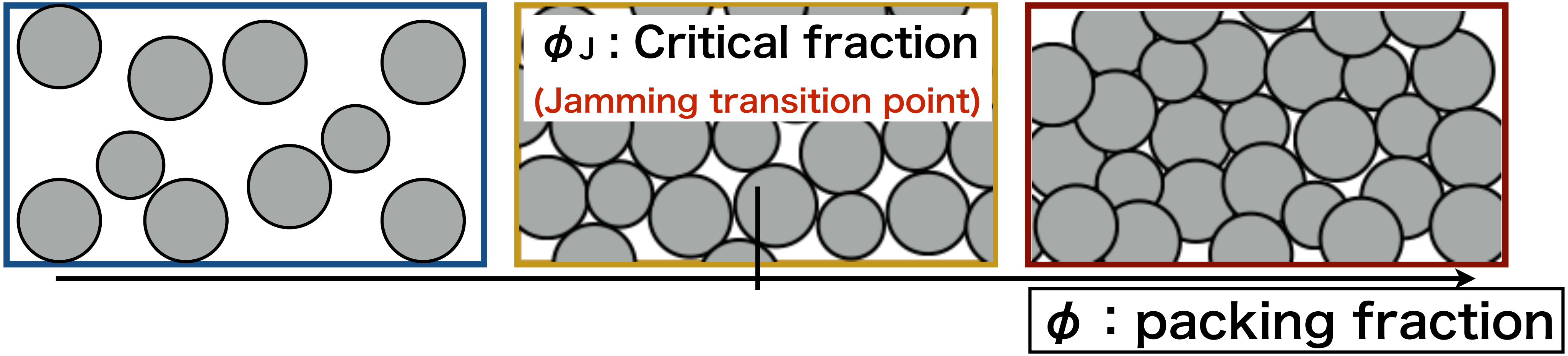
    2. Frictional grains

       MO and H. Hayakawa, PRE 95, 062902 (2017)

    3. Non-spherical grains (Preliminary results)

- Summary

Jamming transition

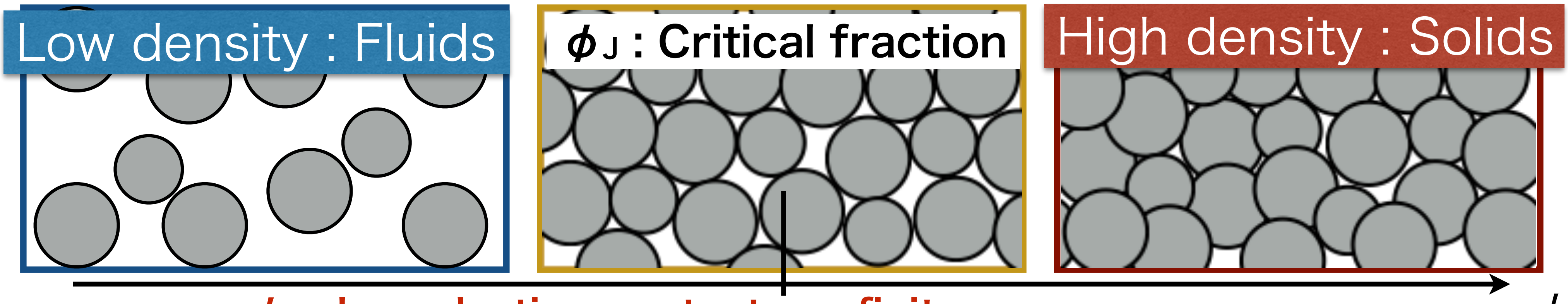


$\phi < \phi_J$ Granular materials flow like fluids.

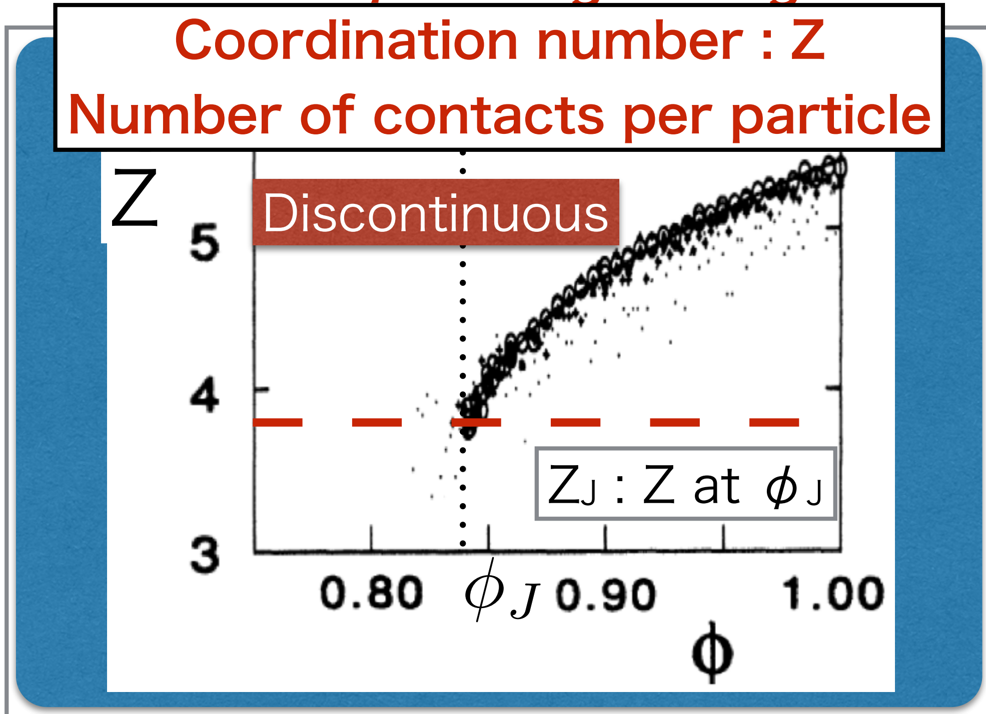
$\phi > \phi_J$ Granular materials have rigidity like solids.

Critical behavior near ϕ_J (frictionless)

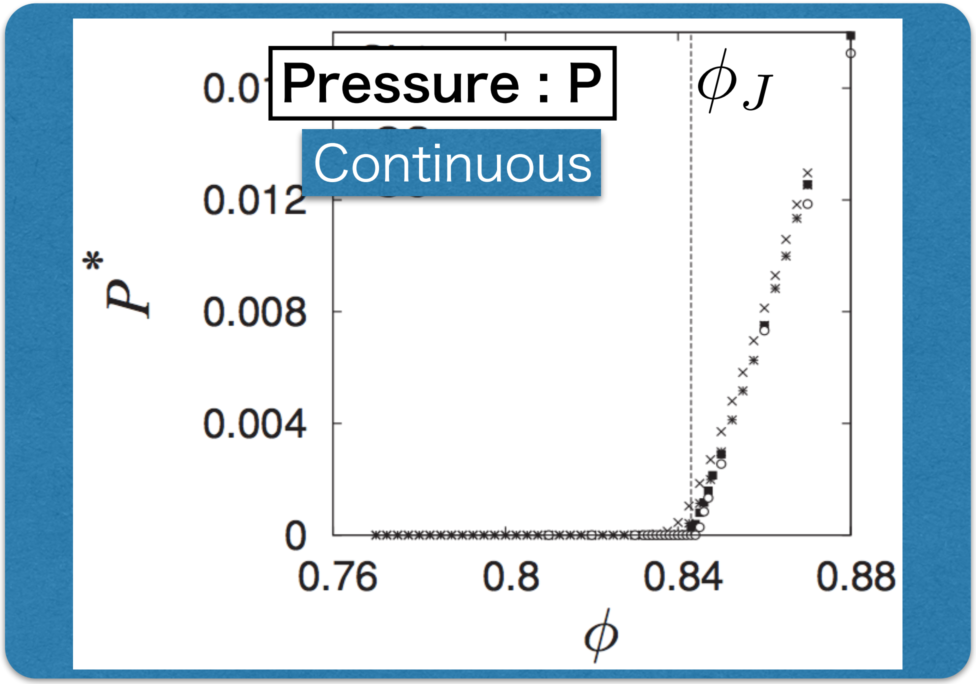
C. O'Hern, et al., Phys. Rev. Lett. 88, 075507 (2002)



ϕ_J : Long-lasting contacts or finite pressure appear

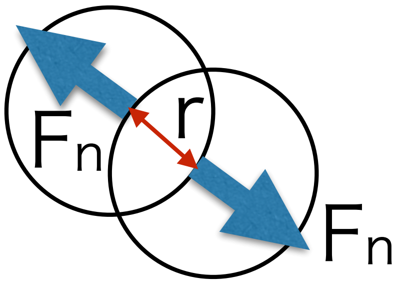


$$Z = \begin{cases} 0, & \phi < \phi_J \\ Z_J + a(\phi - \phi_J)^{1/2}, & \phi > \phi_J \end{cases}$$



$$P = \begin{cases} 0, & \phi < \phi_J \\ b(\phi - \phi_J), & \phi > \phi_J \end{cases}$$

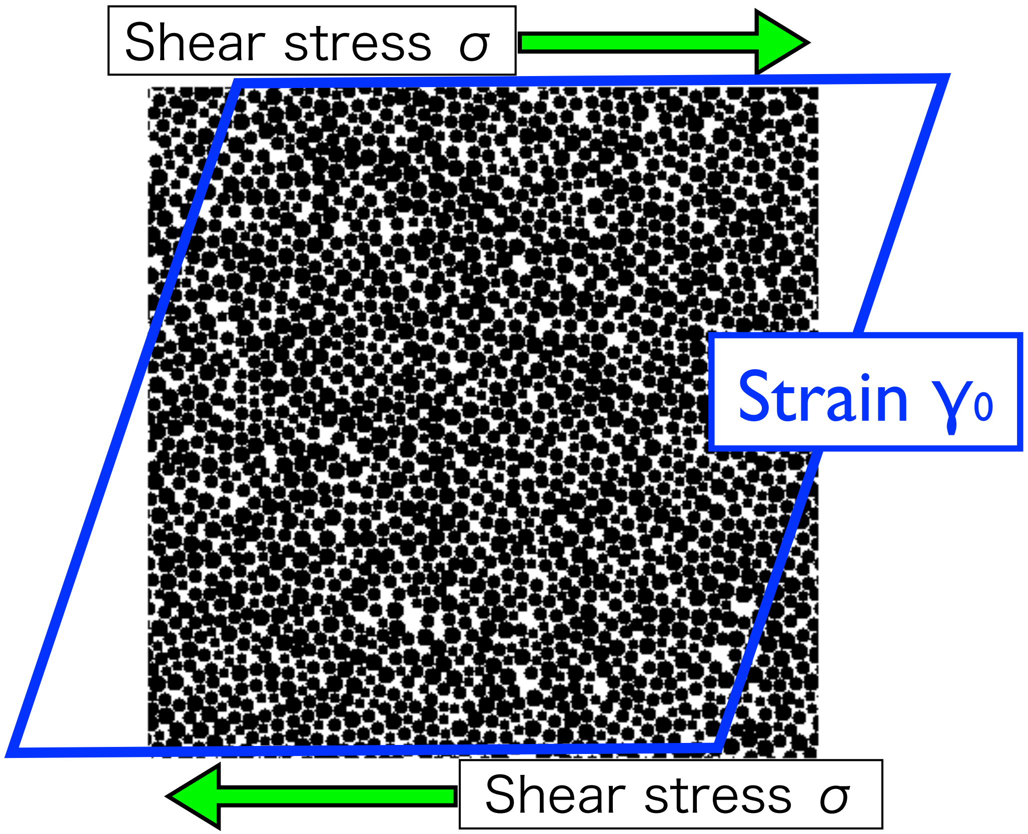
Shear modulus : frictionless grains



Elastic interaction force

$$F_n \propto r$$

compression length : r

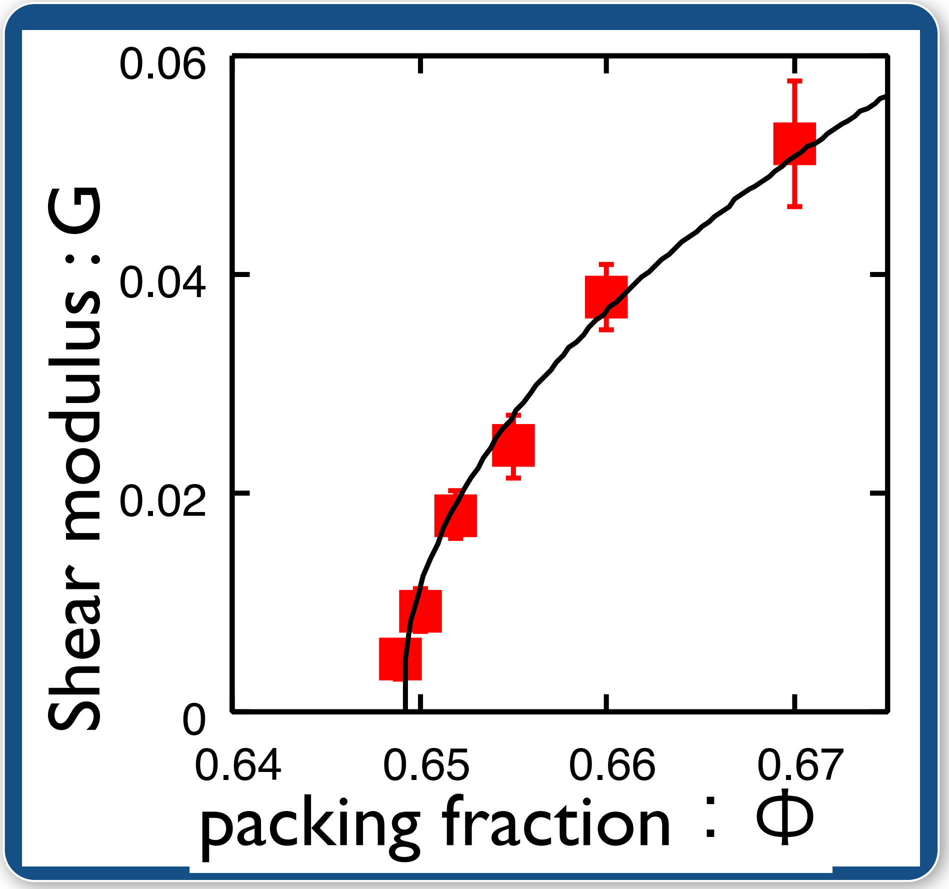


Shear modulus : $G = \sigma / \gamma_0$

$$G \propto (\phi - \phi_J)^{1/2}$$

Linear response

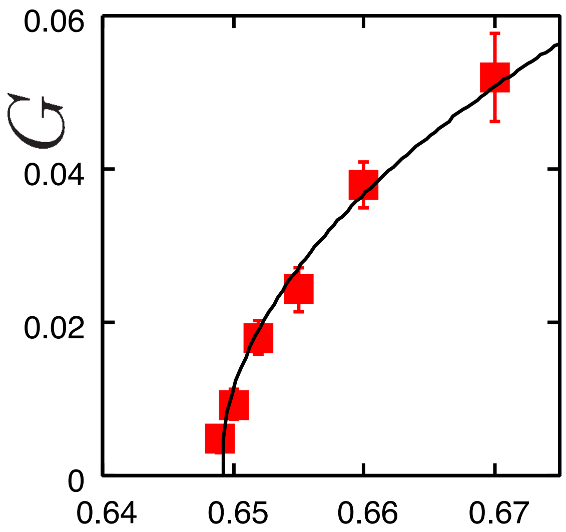
C. S. O'Hern et al., PRE 68, 011306 (2003)



[extensively small strain $\gamma_0 = 10^{-5}$]

Problem

C. S. O'Hern et al., PRE 68, 011306 (2003)



$$G \propto (\phi - \phi_J)^{1/2}$$

- The shear strain is too small (10^{-5})
- Particles are frictionless
- The shape of grains is disc or sphere.

Experiment :

The shear strain is finite.

We cannot ignore the friction between grains.

The shape of grains is not sphere.

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- Introduction : Jamming transition

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    1. **Finite strain**

       MO and H. Hayakawa, PRE 90, 042202 (2014)

    2. Frictional grains

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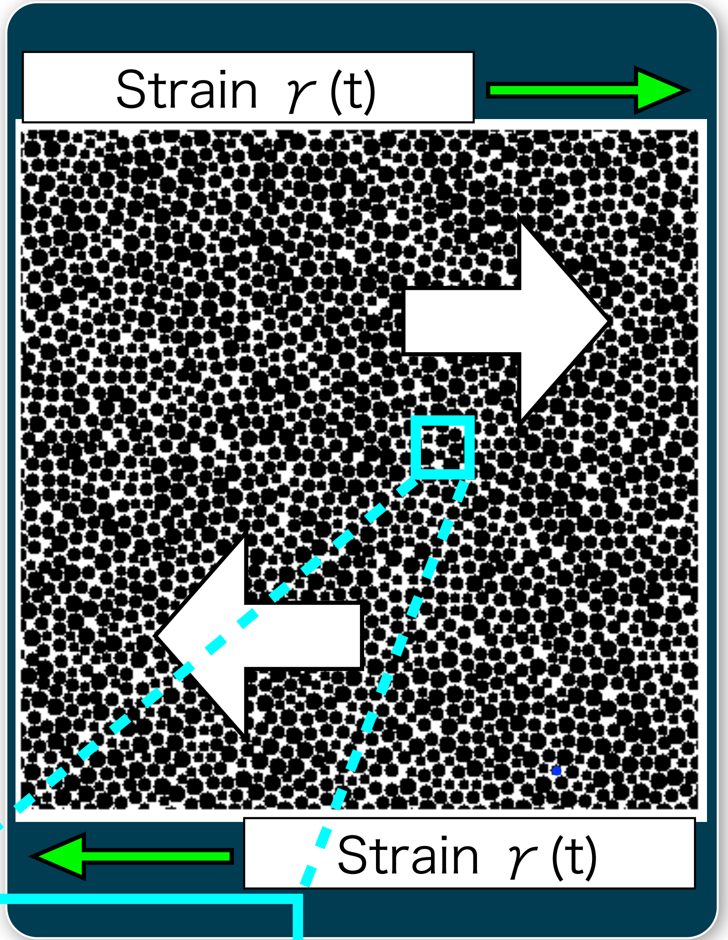
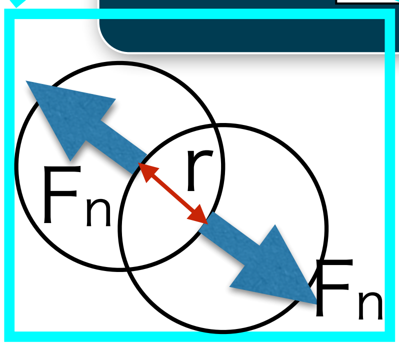
    3. Non-spherical grains (Preliminary results)

- Summary

Model of 3D frictionless particles





Normal interaction force

$$F_n = kr - \eta \dot{r}$$

compression length : r

- Oscillatory shear strain

$$\gamma(t) = \gamma_0 (1 - \cos \omega t)$$

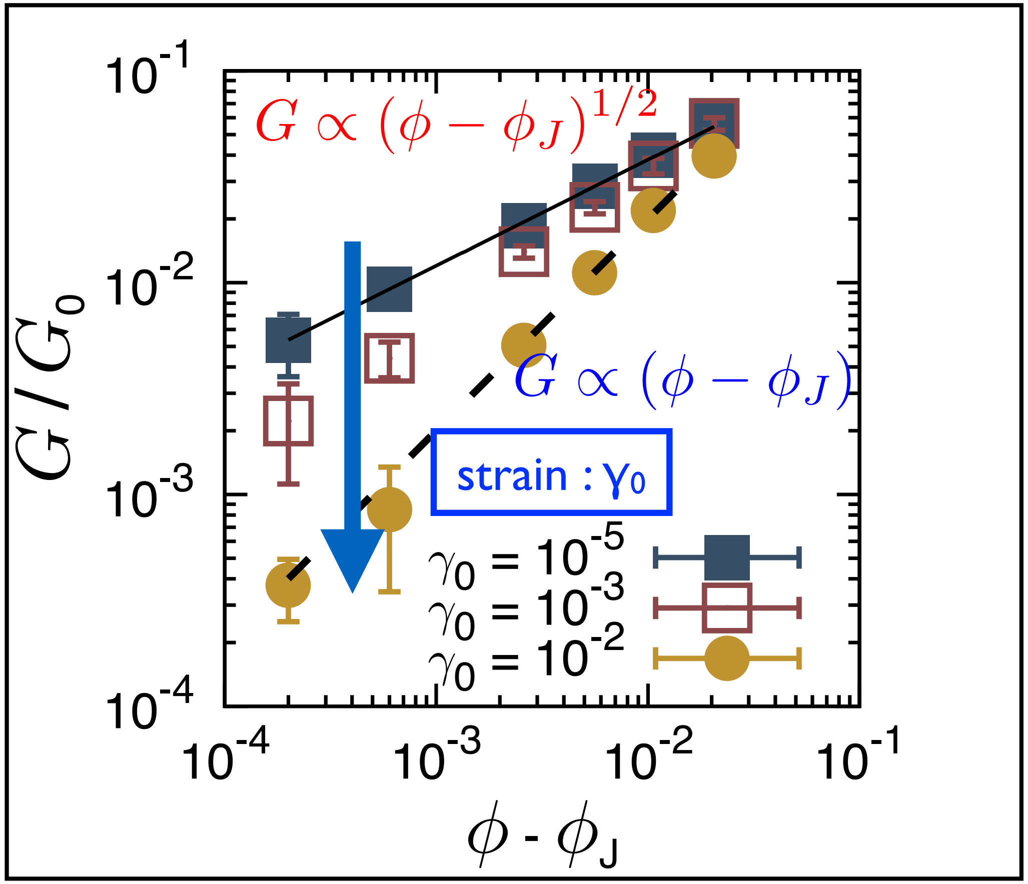
- Frequency : ω **Quasi-static limit : $\omega \to 0$**
- **Strain amplitude : γ_0**
- Shear stress : $\sigma(t)$

Shear storage modulus:

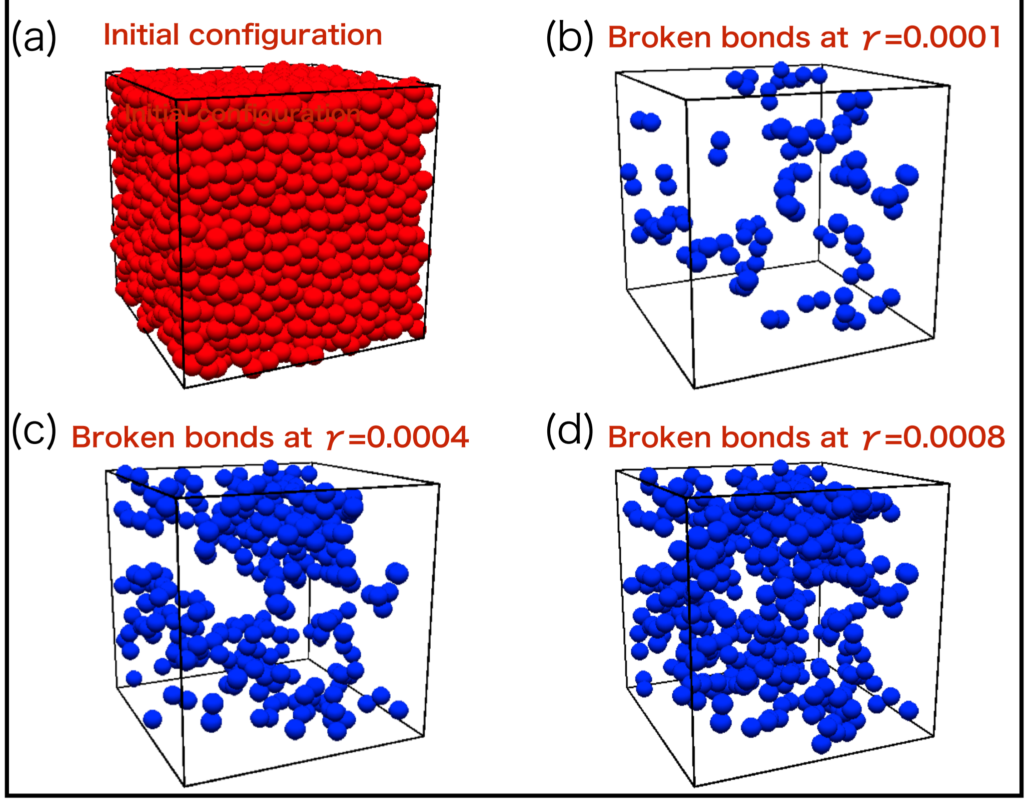
$$G = -\frac{\omega}{\pi} \int_0^{2\pi/\omega} dt \frac{\sigma(t) \cos(\omega t)}{\gamma_0}$$

We investigate the dependence of G on γ_0 and ϕ.

Problem : Shear modulus under finite strain



Avalanche-like bond breakage



Infinitesimal strain : $G \propto (\phi - \phi_J)^{1/2}$

Finite strain : $G \propto \gamma_0^{-c}(\phi - \phi_J)$

Origin : slip avalanche (correlated bond breakage)

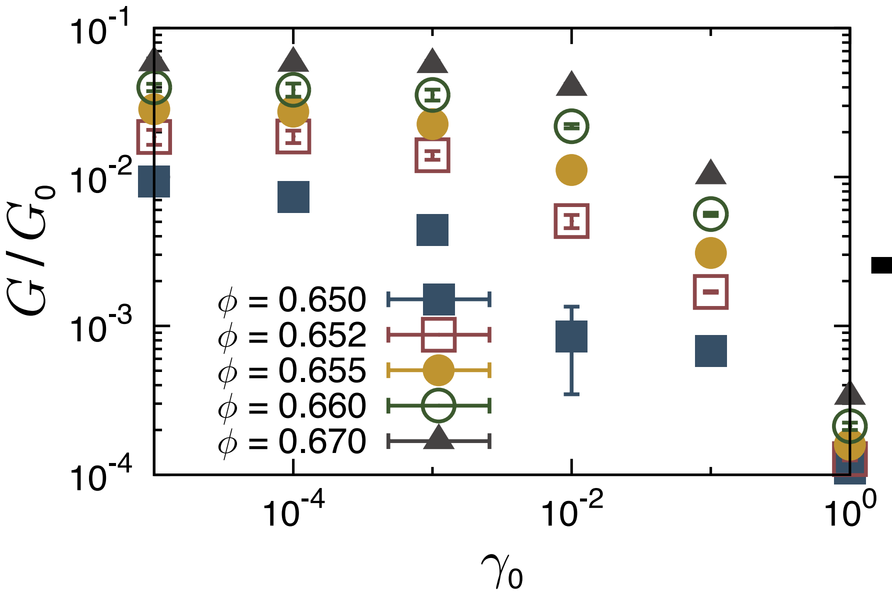
Critical scaling of G

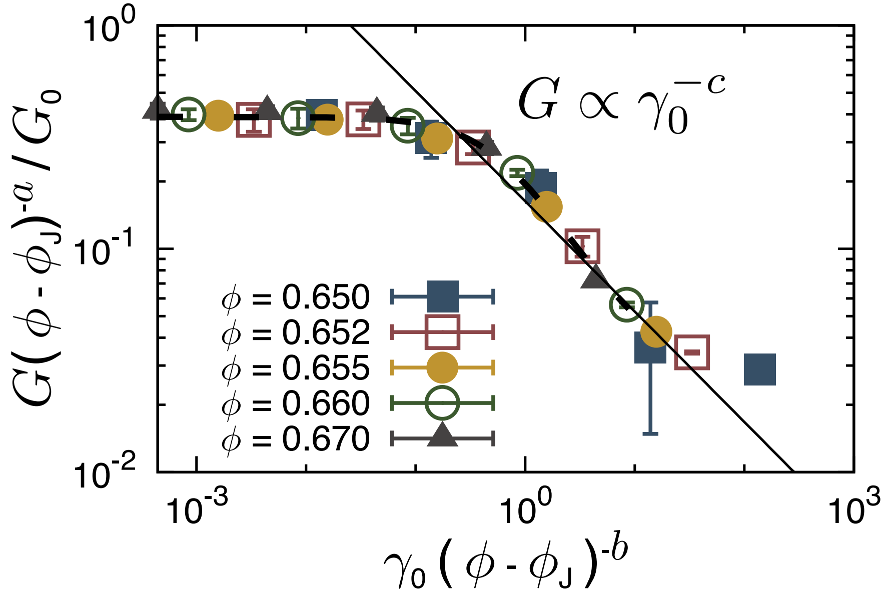
$$G(\gamma_0, \phi) = (\phi - \phi_J)^a \mathcal{G}(\gamma_0(\phi - \phi_J)^{-b})$$

$$\lim_{x \to 0} \mathcal{G}(x) = \text{const.} \qquad \lim_{x \to \infty} \mathcal{G}(x) \propto x^{-c}$$

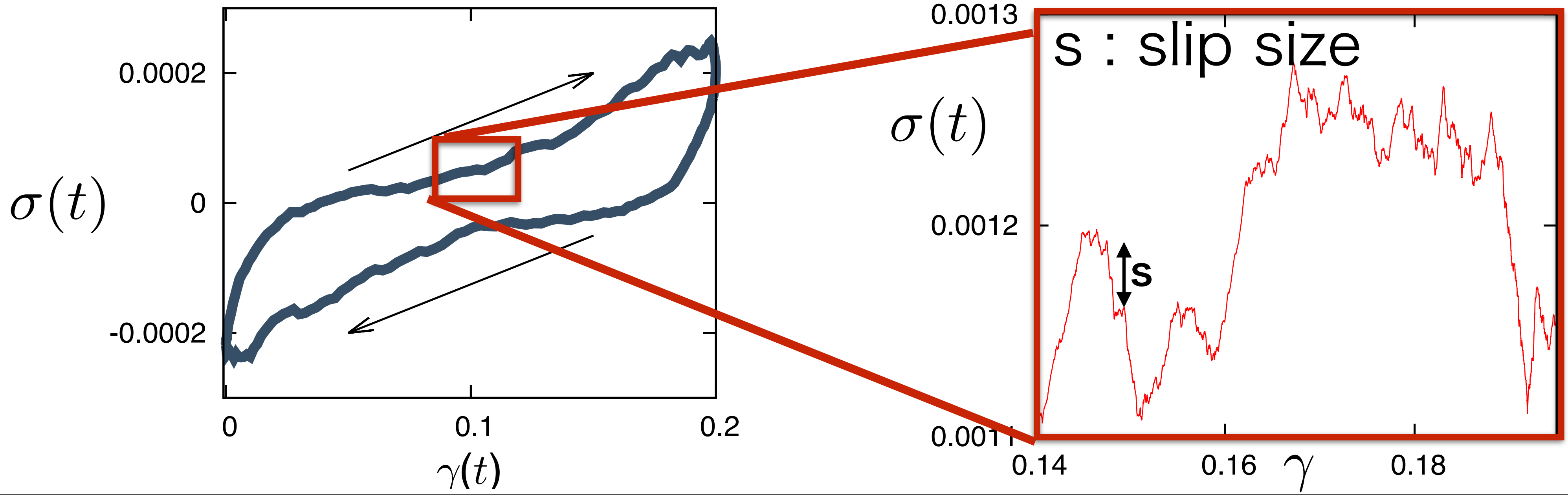
G : shear modulus,
ϕ : volume fraction,
γ_0 : strain amplitude

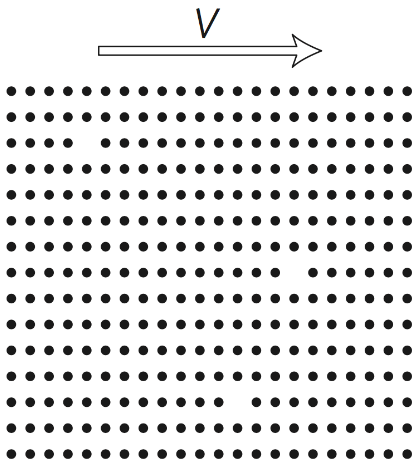
critical exponents **a =1/2, b = 1, c = ?**





Slip avalanches



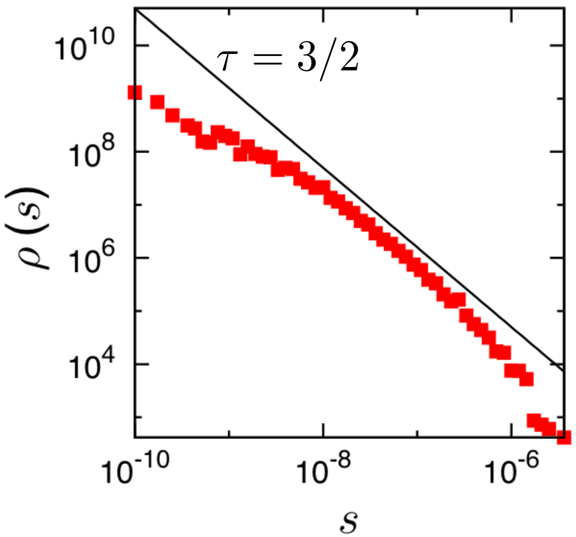


Analysis of a mean field lattice model

K. Dahmen, Y. Ben-Zion, and J. T. Uhl, Nat. Phys. 7, 554 (2011).

Distribution function : $\rho(s) \propto s^{-\tau}$

Exponent : $\tau = 3/2$



Elastic-plastic model

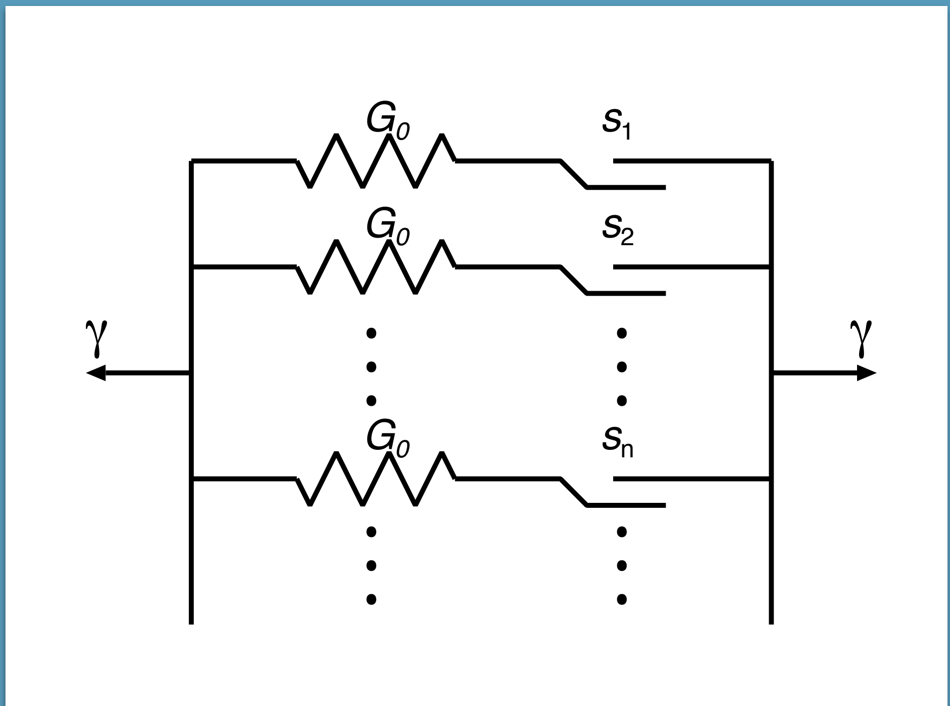
V. A. Lubarda, D. Sumarac, and D. Krajcinovic, Eur. J. Mech., A/Solids 12, 445 (1993).

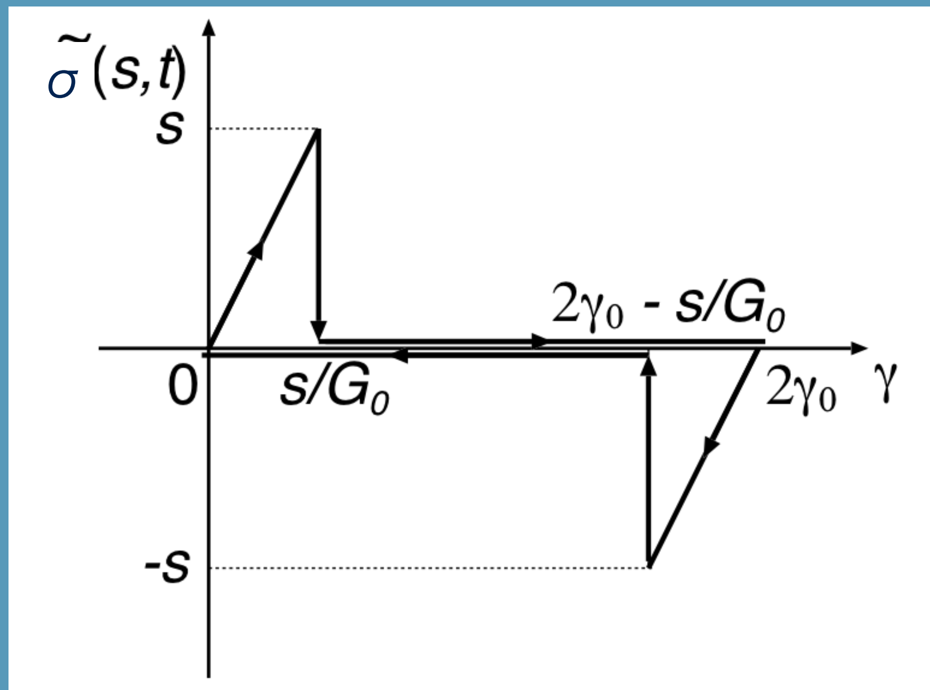
$$\sigma(t) = \int_0^\infty ds\ \rho(s)\tilde{\sigma}(s,t)$$

$\tilde{\sigma}$: stress of element

s_n : yield stress = stress drop

$\rho(s)$: size distribution





each elements have different yield stress

Phenomenological result

$$G(\gamma_0,\phi) = \frac{\omega}{\pi}\int_0^{2\pi/\omega} dt \frac{\sigma(t)\cos(\omega t)}{\gamma_0}$$

$$\sigma(t) = \int_0^{\infty} ds\ \rho(s)\tilde{\sigma}(s,t)$$

$$\rho(s) \propto s^{-\tau}$$

→ $G \propto \gamma_0^{-(\tau-1)}$ $\tau = 3/2$

Dahmen et al., (2011)

$G \propto \gamma_0^{-c}$ critical scaling

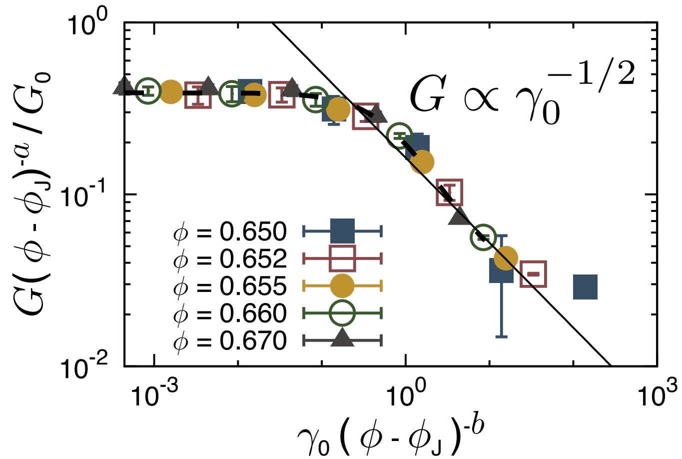
$$c = \tau - 1 = 1/2$$

$$G(\gamma_0,\phi) = (\phi-\phi_S)^a \mathcal{G}(\gamma_0(\phi-\phi_S)^{-b})$$ a = 1/2, b = 1, c = 1/2

→ $G \propto (\phi-\phi_J)^{1/2}$ Small strain

$G \propto \gamma_0^{-1/2}(\phi-\phi_J)$ Finite strain

The similar results are obtained in experiments
C . Coulais, et al., PRL. 113, 198001 (2014)



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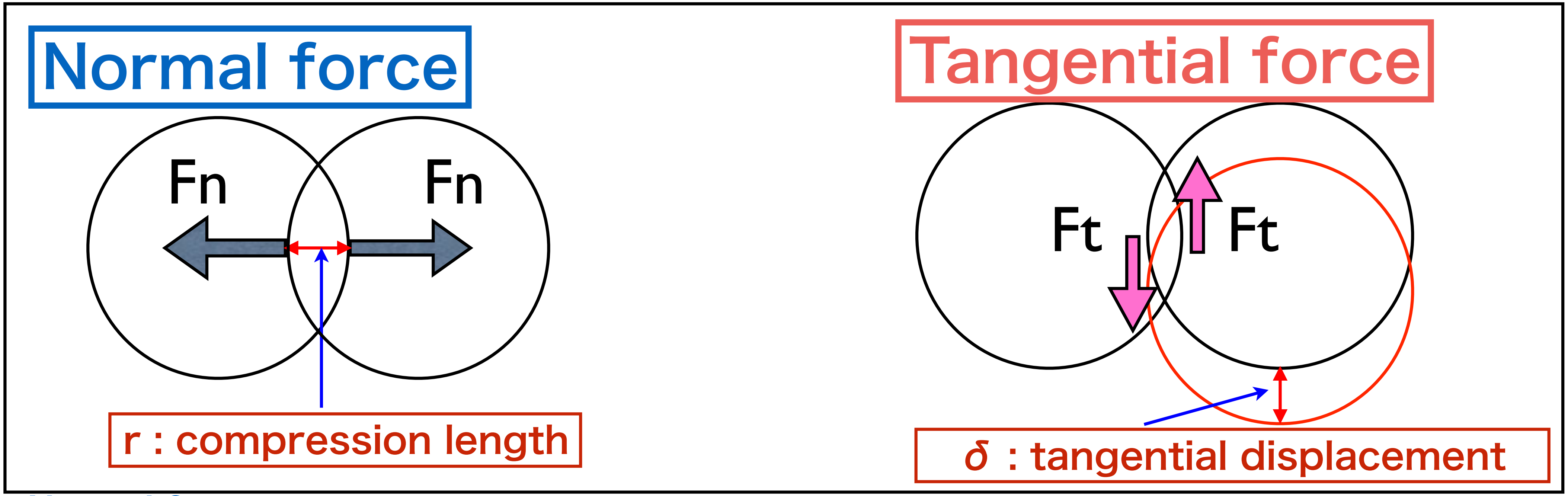
- Summary

2D model of frictional grains



Normal force

$$F_n = kr - \eta\dot{r}$$

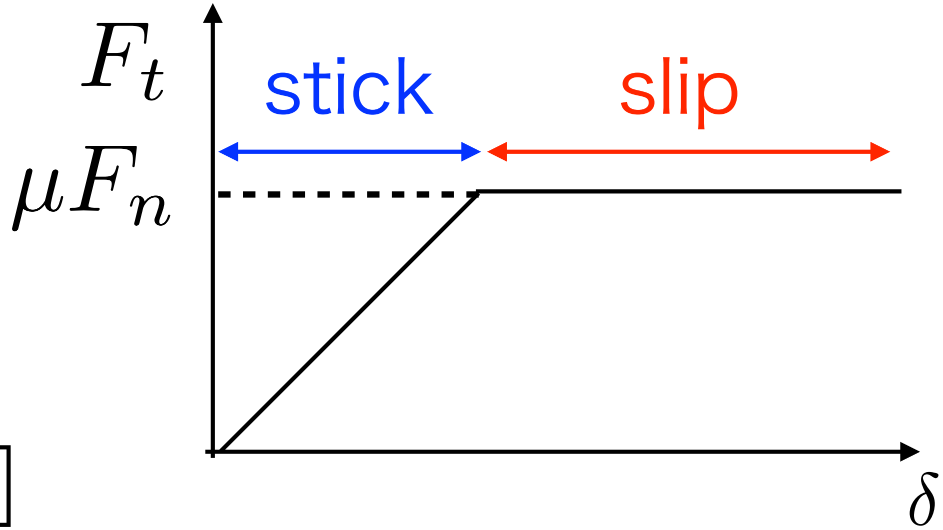
Tangential force

$$|F_t| = k_t\delta, \quad F_t < \mu F_n$$ (stick)

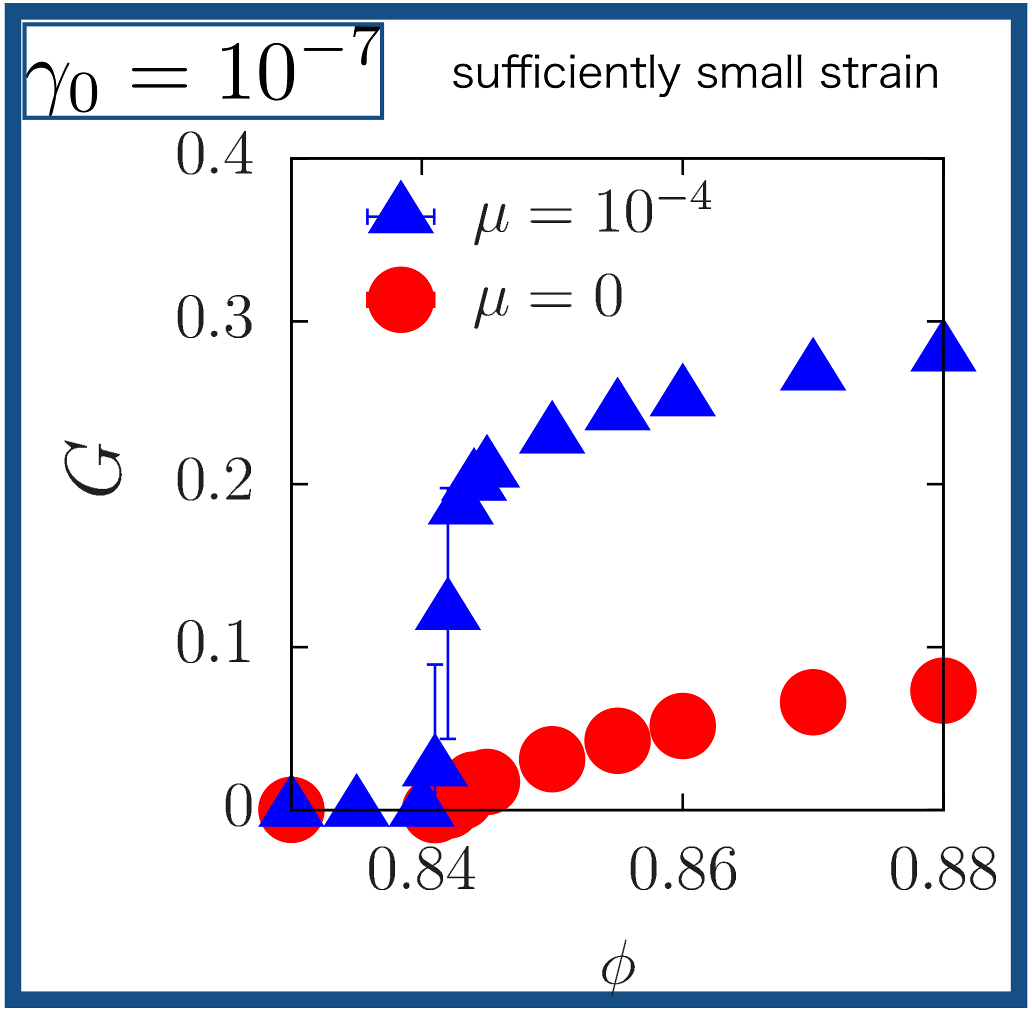
$$|F_t| = \mu F_n, \quad \text{otherwise}$$ (slip)

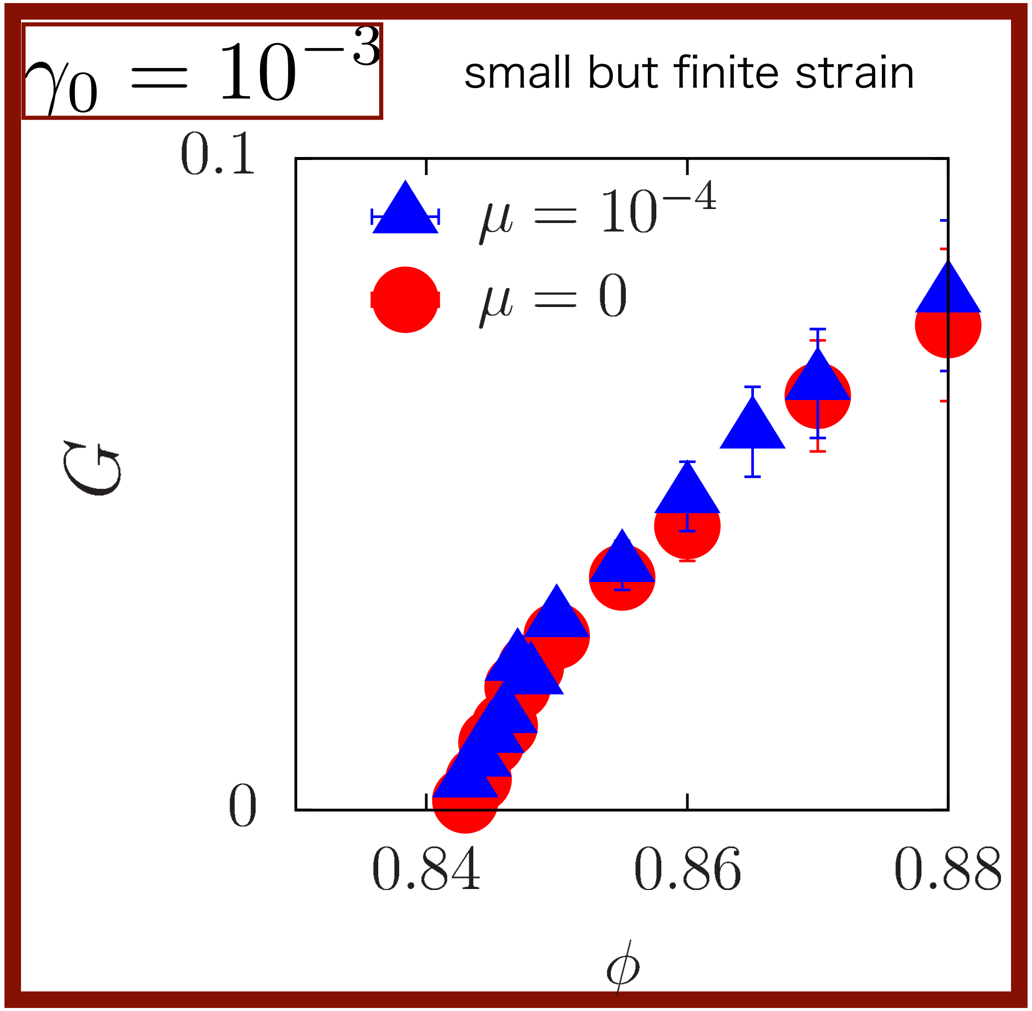
μ : friction coefficient

$\mu = 0$: frictionless, $\mu > 0$: frictional



μ-dependence of shear modulus

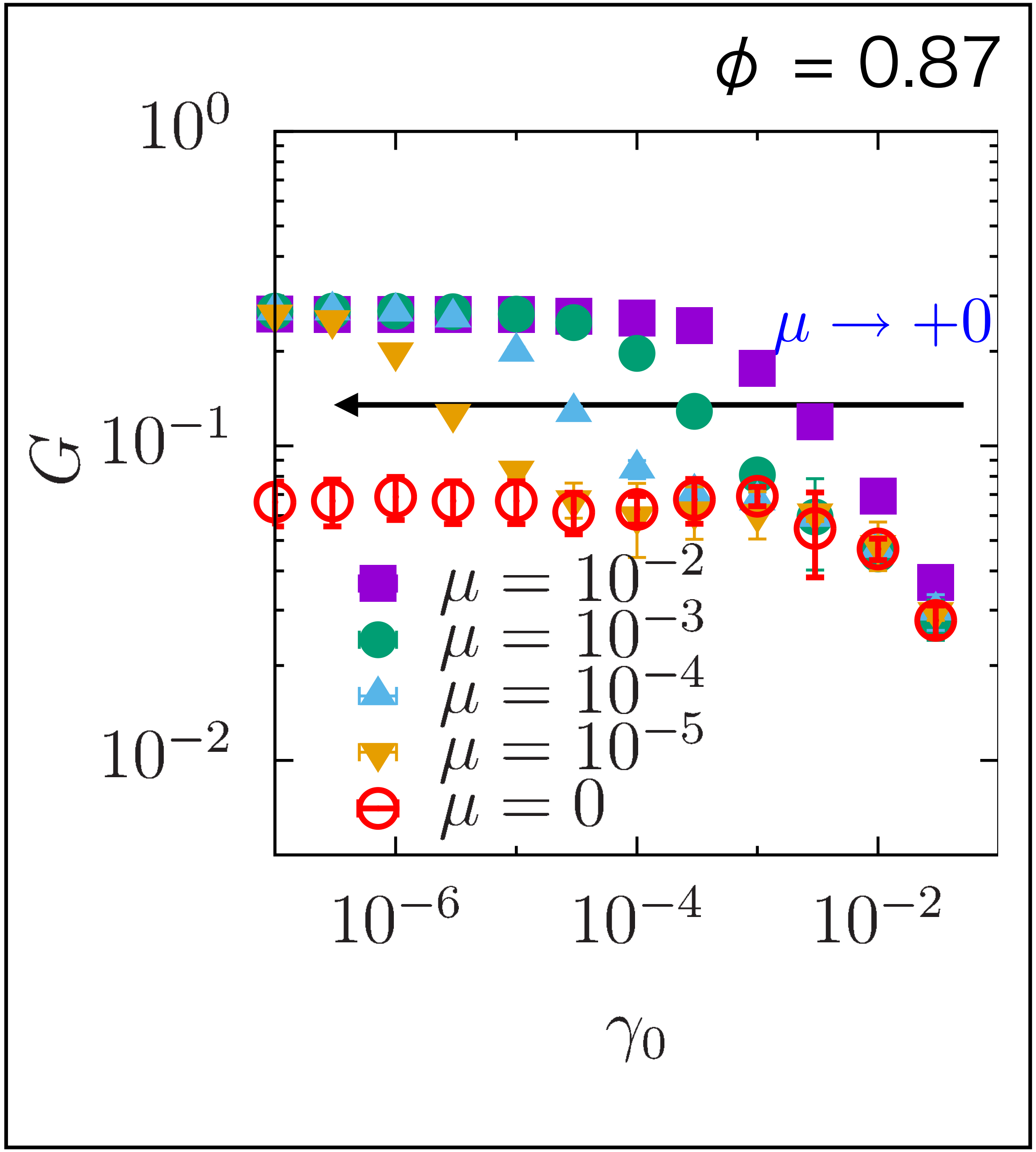




Sufficiently small γ_0 : G for $\mu=10^{-4}$ is different from that for $\mu=0$.

Relatively large γ_0 : The difference between $\mu=0$ and 10^{-4} is small.

γ_0-dependence of shear modulus



γ_0-dependence :

- For small γ_0, G is constant.

Linear elasticity

- As γ_0 increases, G decreases.

Nonlinear elasticity

Small γ_0 region :

- G for $\mu > 0$ does not depend on μ.
- G for $\mu > 0$ differs from G for $\mu = 0$.

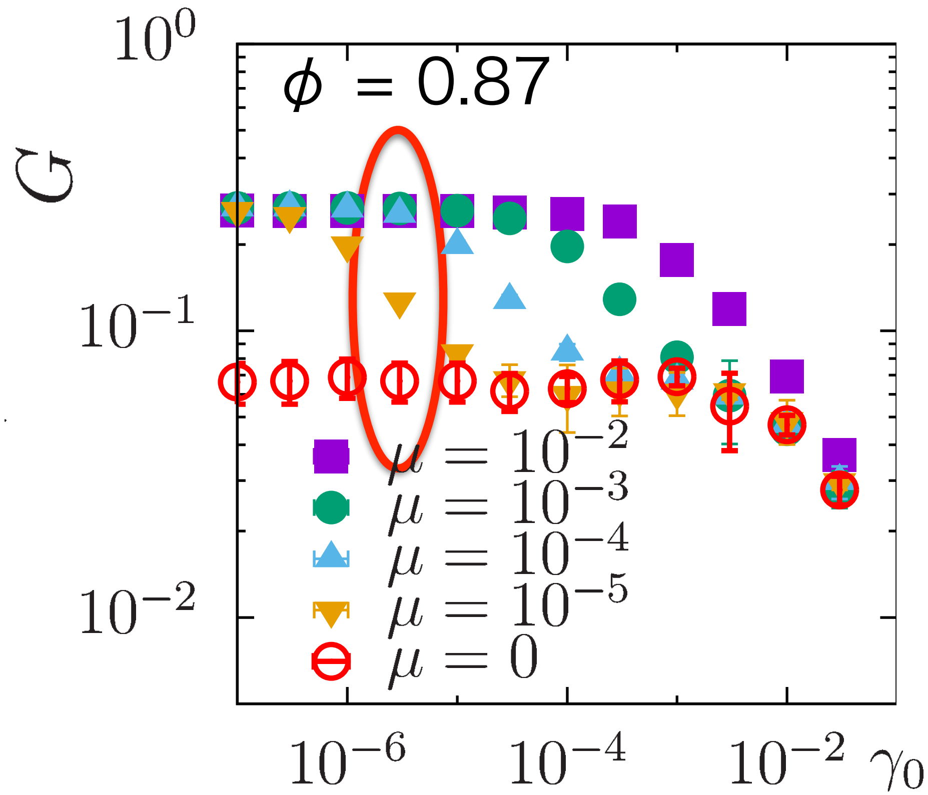
Large γ_0 region :

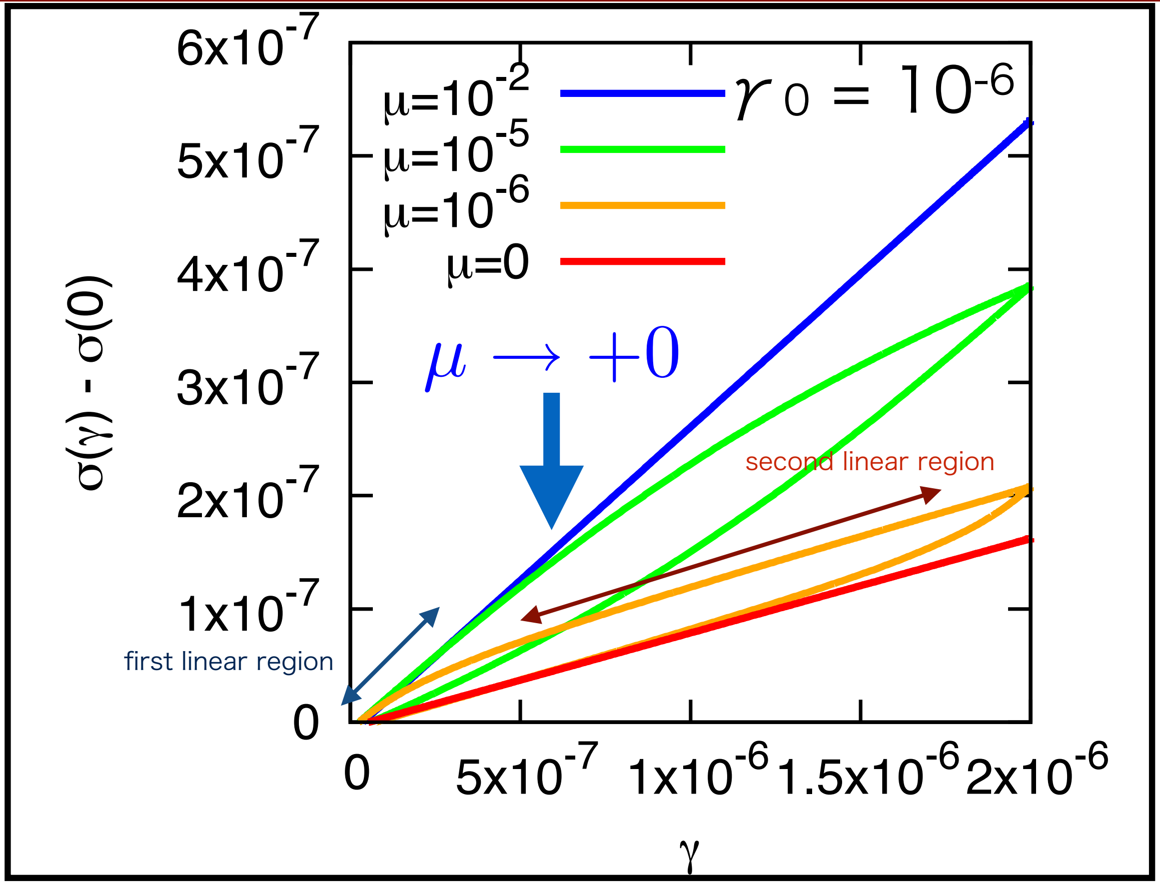
- G for $\mu > 0$ is the same with that for $\mu = 0$.

Sufficiently small μ :

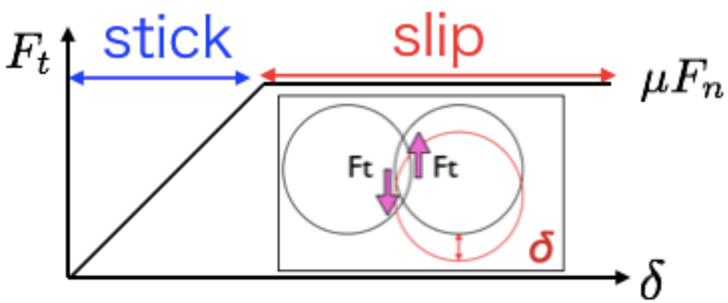
- G has two plateaus.

μ-dependence of σ-γ relation





Tangential friction force between grains



μ = 0 : Linear relation with smaller gradient

μ = 10^{-2} : Linear relation with larger gradient (F_t is the origin.)

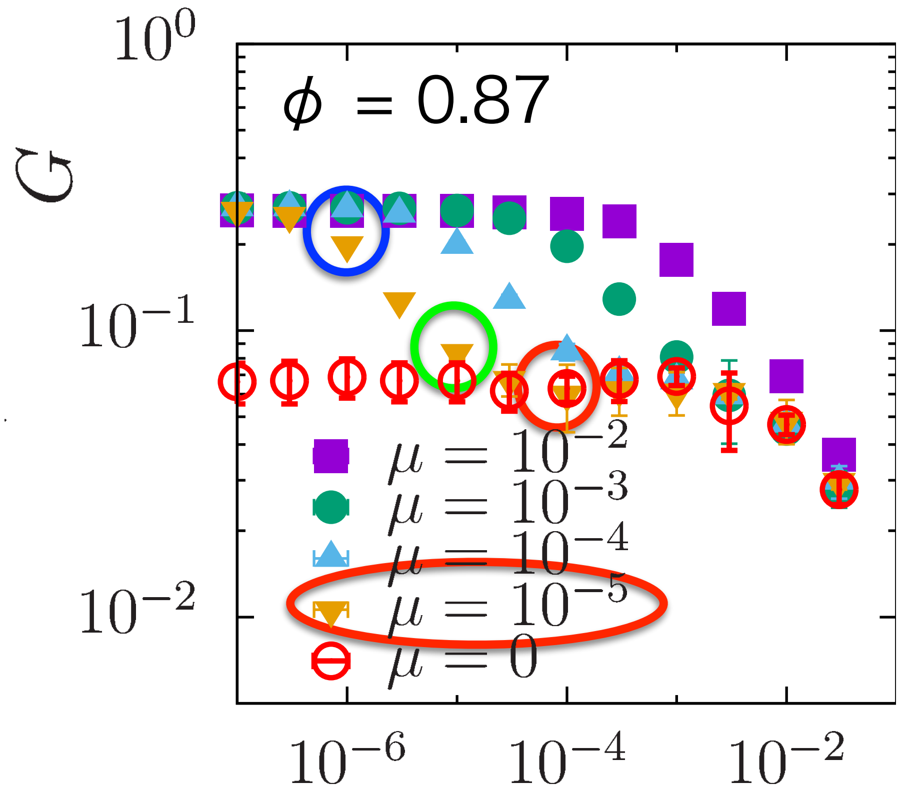
μ = 10^{-5} : Nonlinear relation with a loop

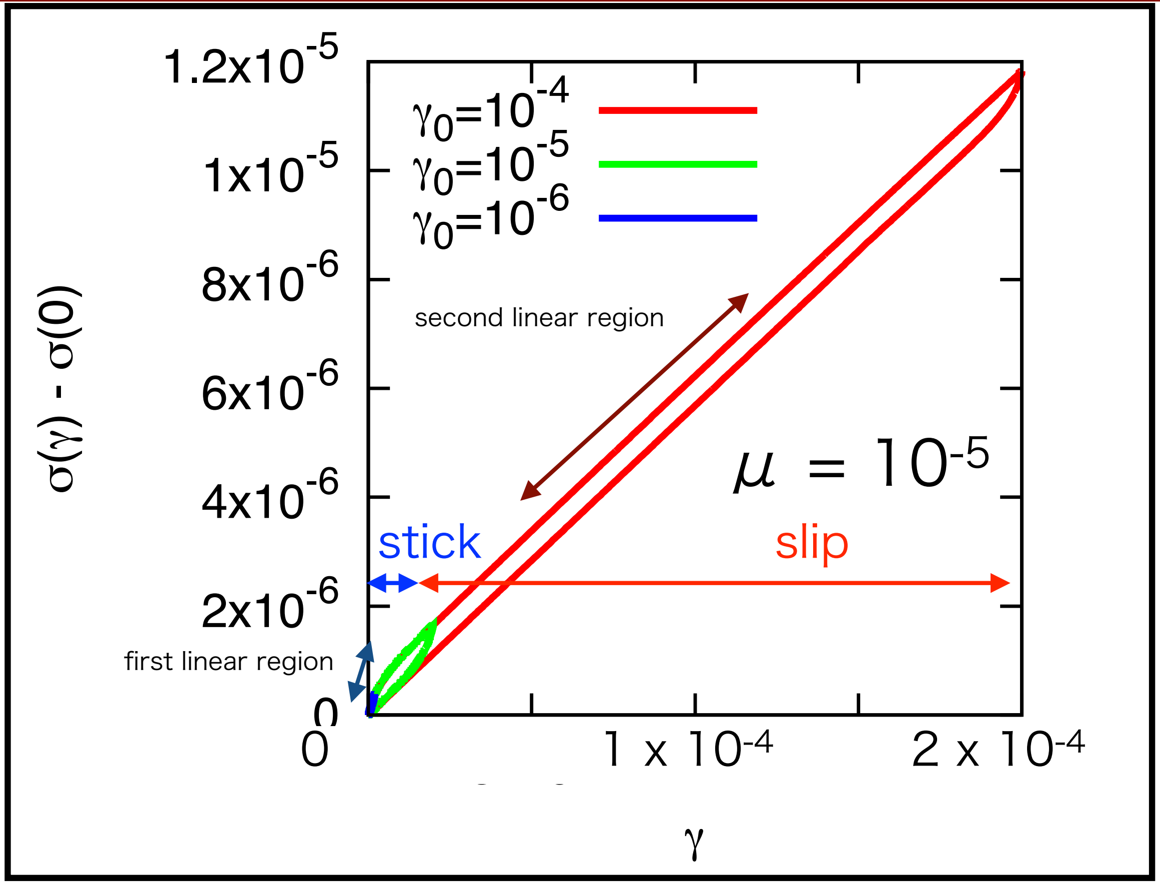
μ = 10^{-6} : Second linear region (the gradient is identical with μ=0.)

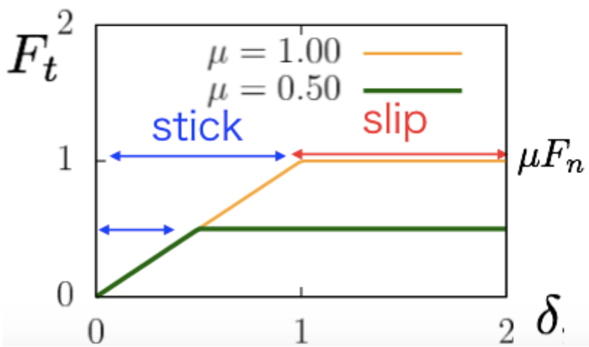
First linear region : the gradient is independent of μ → G is independent of μ

Second linear region : Transition of F_t to "slip phase" → The convergence of G

γ_0-dependence of σ-γ relation







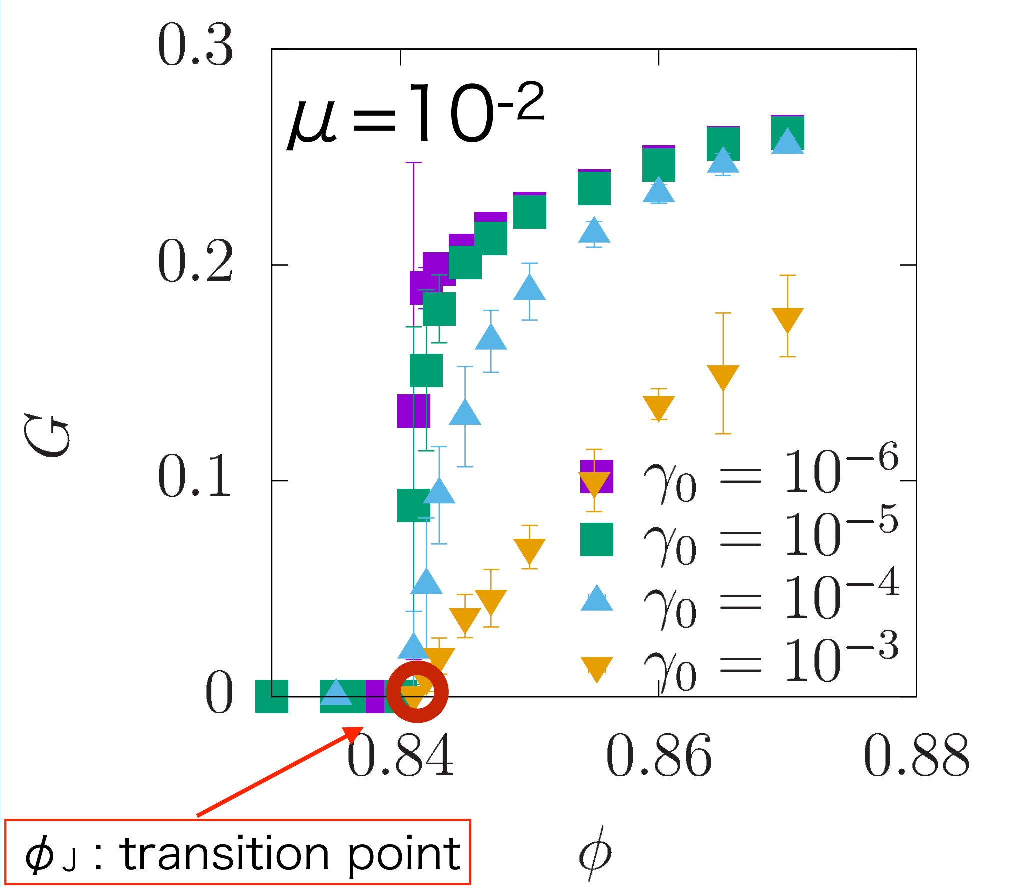
$\gamma_0 = 10^{-4}$: Linear relation with larger gradient

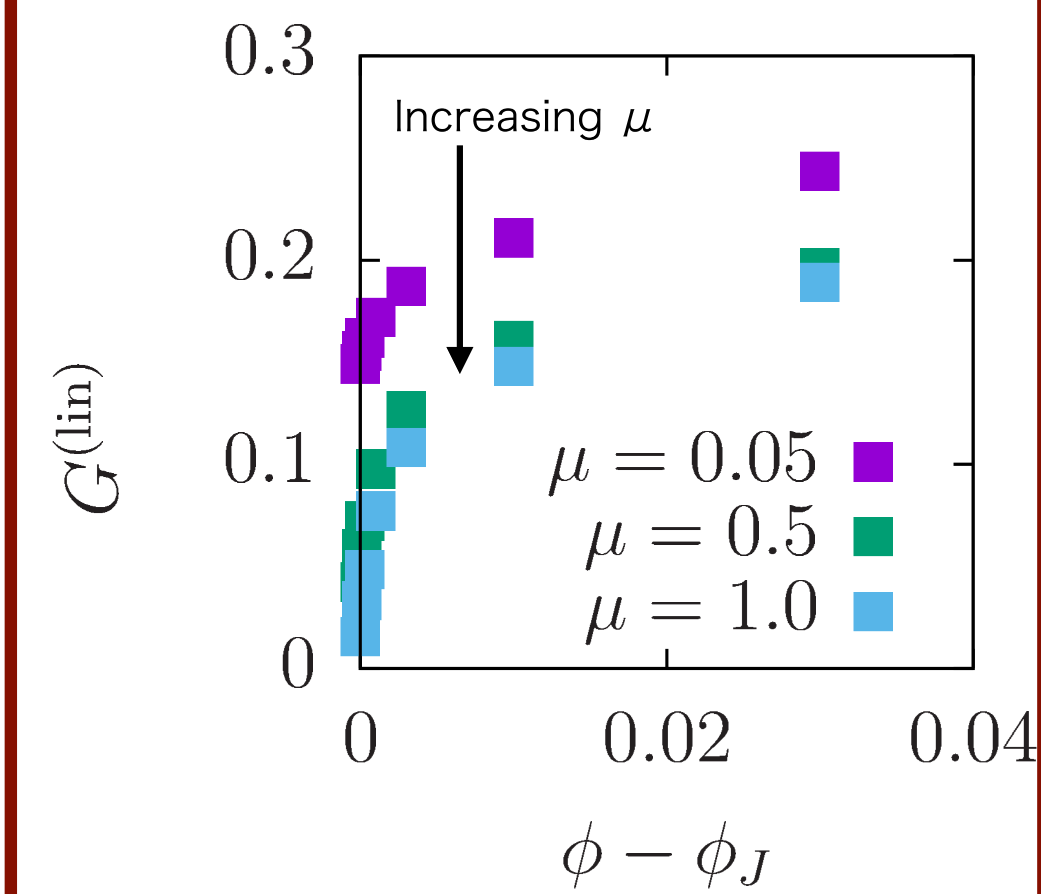
$\gamma_0 = 10^{-5}$: Nonlinear relation with a loop

$\gamma_0 = 10^{-6}$: Second linear region (the gradient is the same as that for μ=0)

Second linear region : Transition of F_t to "slip phase" → Second plateau in G

Shear modulus near ϕ_J



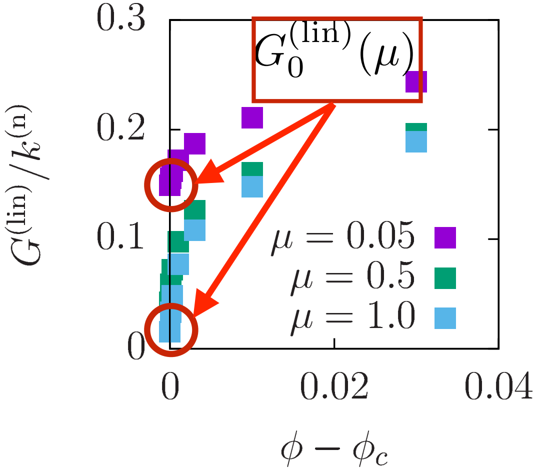


Small γ_0 : Discontinuous transition, Large γ_0 : Continuous transition

G in the linear response regime : $G^{(\mathrm{lin})}(\mu, \phi) \equiv \lim_{\gamma_0 \to 0} G(\gamma_0, \mu, \phi)$

$G^{(\mathrm{lin})}$ decreases as μ increases.

Scaling of $G^{(\mathrm{lin})}$



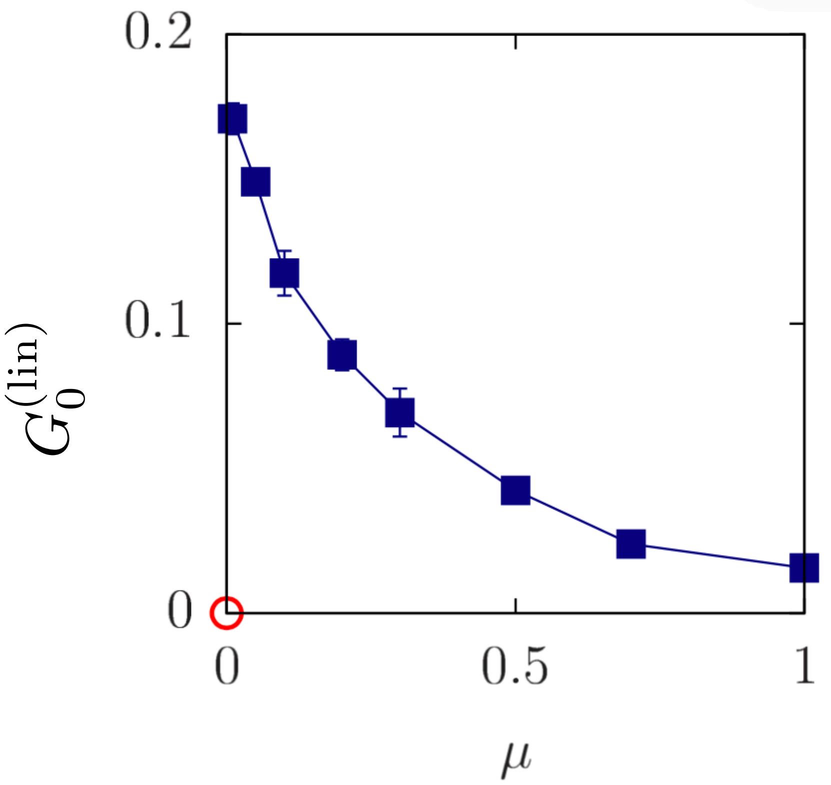
Shear modulus at ϕ_J: $G_0^{(\mathrm{lin})}(\mu) \equiv \lim_{\phi \to +\phi_J} G^{(\mathrm{lin})}(\mu, \phi)$

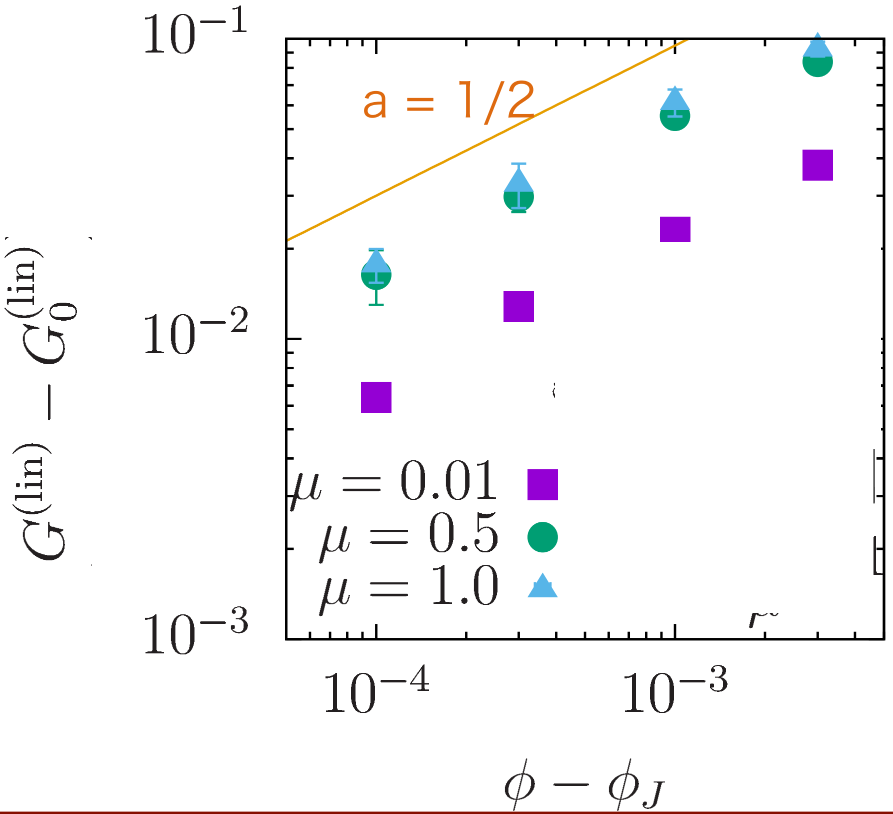
$G_0^{(\mathrm{lin})}$ decreases as μ increases.

Somai, et al., PRE (2007)

(Origin : μ-dependence of coordination number at ϕ_J.)

$$G^{(\mathrm{lin})}(\mu, \phi) - G_0^{(\mathrm{lin})}(\mu) \propto \{\phi - \phi_J(\mu)\}^a \qquad a \simeq 1/2$$



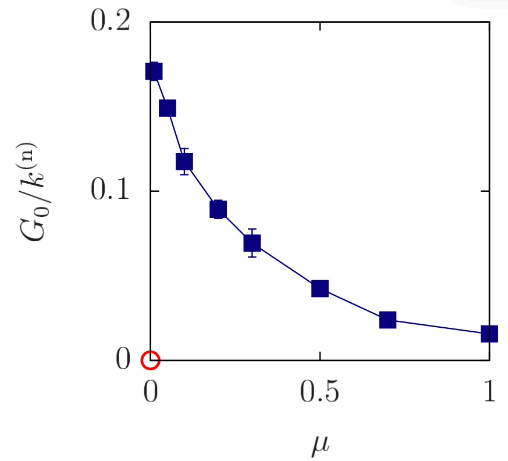


Shear modulus at ϕ_J

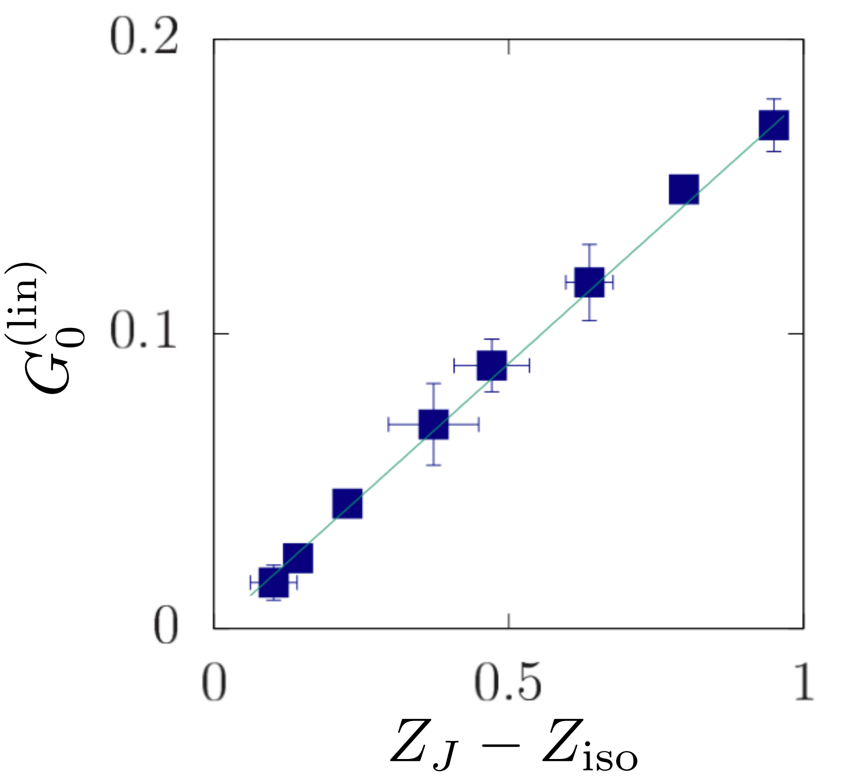
Z_J : Z at $\phi = \phi_J$

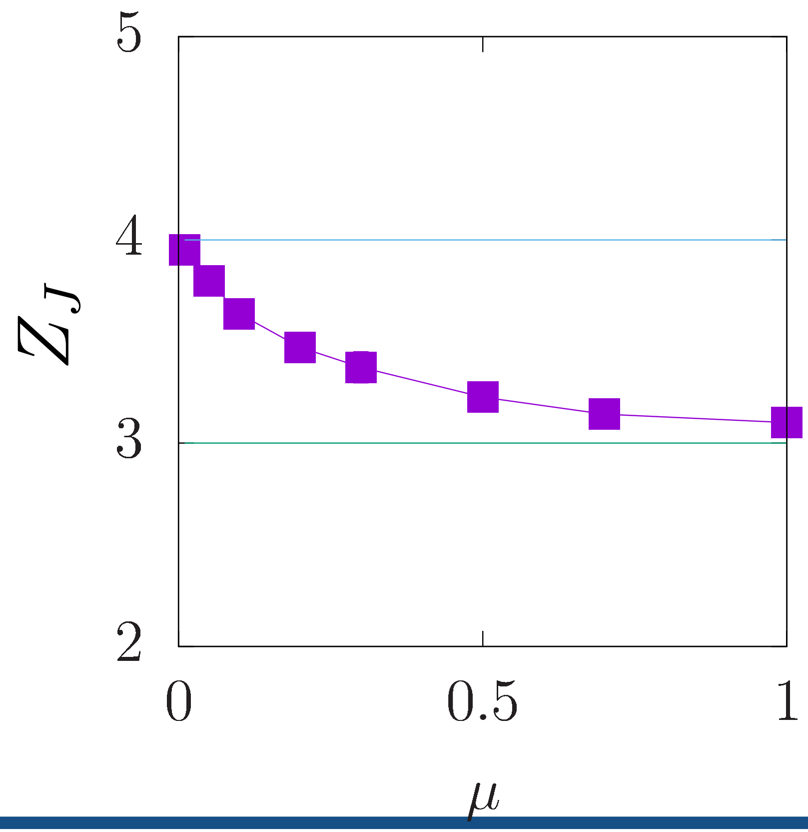
Z_{iso} : minimum Z where the system has rigidity.

$Z_{iso} = 4 \ (\mu = 0)$ $Z_{iso} = 3 \ (\mu > 0)$



$G_0 \propto Z_J - Z_{iso}$ $Z_J > Z_{iso}(\mu > 0)$





Scaling law of G

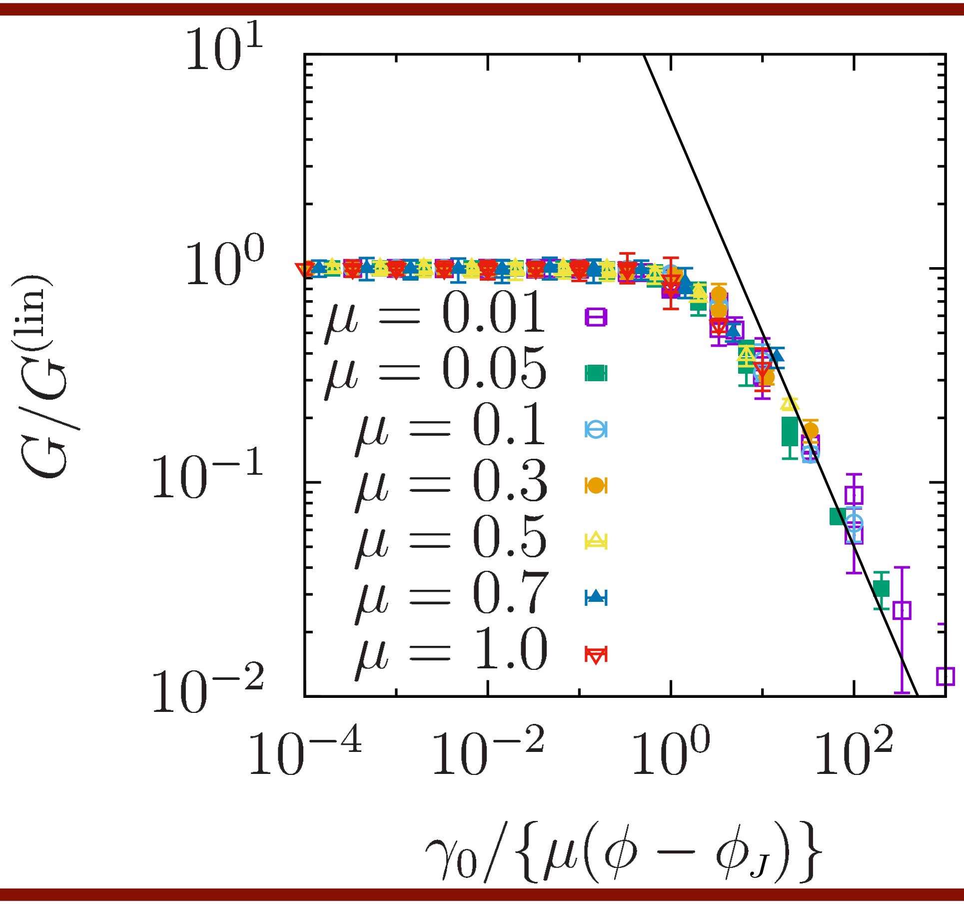
$$G(\gamma_0, \mu, \phi) = G^{(\mathrm{lin})}(\mu, \phi)\mathcal{F}\left(\frac{\gamma_0}{\mu\{\phi - \phi_J(\mu)\}}\right)$$

$$\lim_{x\to 0} \mathcal{F}(x) = 1$$

$$\lim_{x\to\infty} \mathcal{F}(x) \propto x^{-1}$$

$$G^{(\mathrm{lin})}(\mu, \phi) - G_0^{(\mathrm{lin})}(\mu) \propto \{\phi - \phi_J(\mu)\}^a \qquad a \simeq 1/2$$

Linear repulsive force



Shear modulus : G
Friction coefficient : μ
Strain amplitude : γ_0
Packing fraction : ϕ
Transition point : ϕ_J

G in the linear response regime : $G^{(\mathrm{lin})}$
Scaling function : F

Origin of scaling law

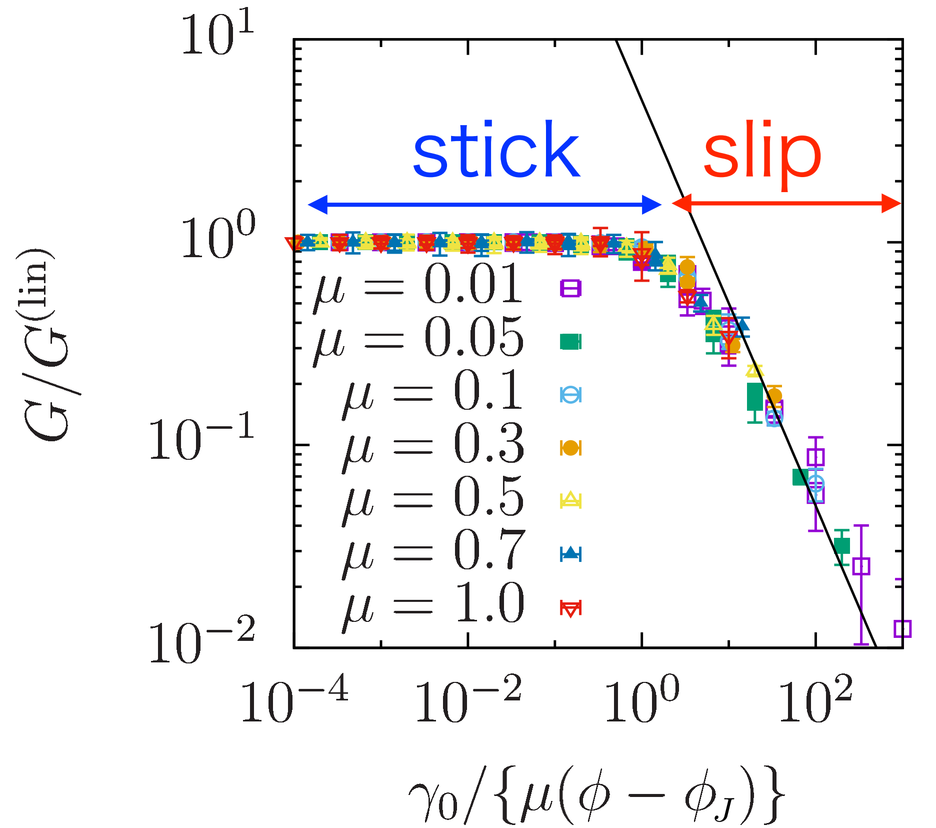
$$G(\gamma_0,\mu,\phi) = G^{(\mathrm{lin})}(\mu,\phi)\mathcal{F}\left(\frac{\gamma_0}{\mu\{\phi-\phi_J(\mu)\}}\right)$$

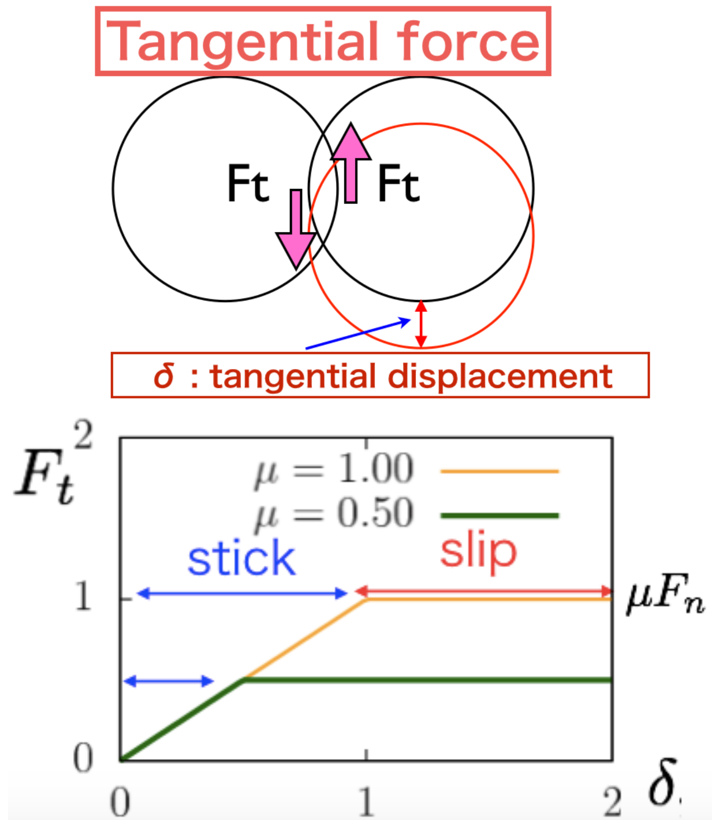
$$G^{(\mathrm{lin})}(\mu,\phi) - G_0(\mu) \propto \{\phi-\phi_J(\mu)\}^{1/2}$$

$$\lim_{x\to 0}\mathcal{F}(x) = 1$$

$$\lim_{x\to\infty}\mathcal{F}(x) \propto x^{-1}$$

Linear repulsive force





Continuous-discontinuous transition

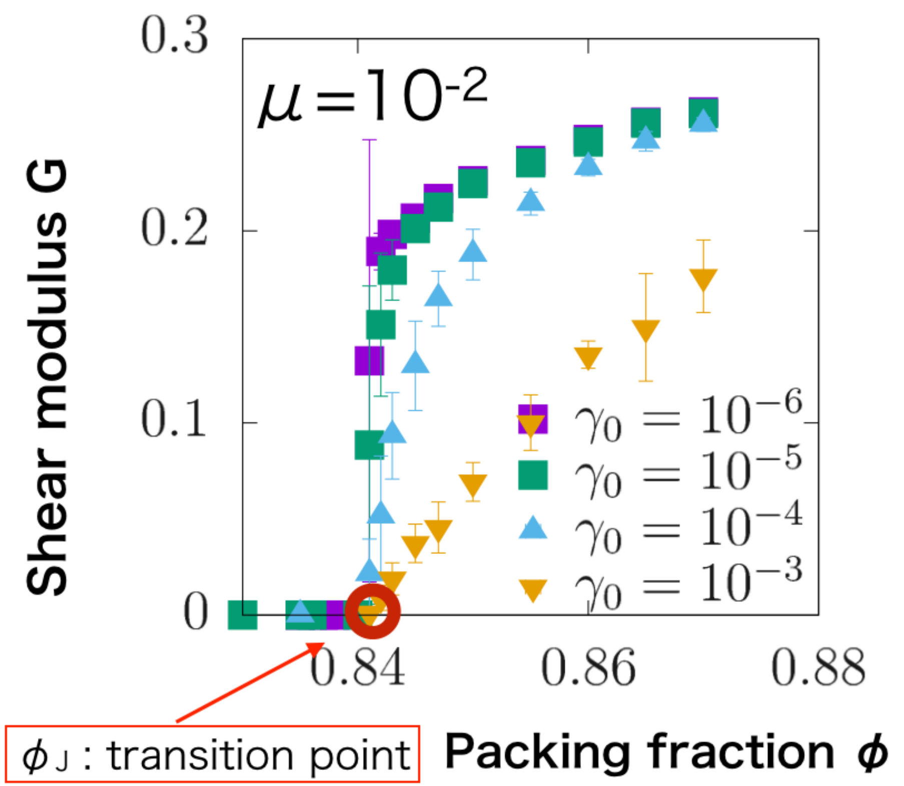
$$G(\gamma_0,\mu,\phi) = G^{(\mathrm{lin})}(\mu,\phi)\mathcal{F}\left(\frac{\gamma_0}{\mu\{\phi-\phi_J(\mu)\}}\right)$$

$$\lim_{x\to 0}\mathcal{F}(x) = 1$$

$$G^{(\mathrm{lin})}(\mu,\phi) - G_0(\mu) \propto \{\phi-\phi_J(\mu)\}^{1/2}$$

$$\lim_{x\to\infty}\mathcal{F}(x) \propto x^{-1}$$

Linear repulsive force



Discontinuous transition

$$\lim_{\gamma_0\to 0} G(\gamma_0,\mu,\phi) = G_0^{(\mathrm{lin})} + A(\phi-\phi_J)^{1/2}$$

Continuous transition

$$\lim_{\phi\to+\phi_J} G(\gamma_0,\mu,\phi) \propto G_0^{(\mathrm{lin})}\frac{\mu(\phi-\phi_J)}{\gamma_0}$$

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  1. Finite strain

     MO and H. Hayakawa, PRE 90, 042202 (2014)

  2. Frictional grains

     MO and H. Hayakawa, PRE 95, 062902 (2017)

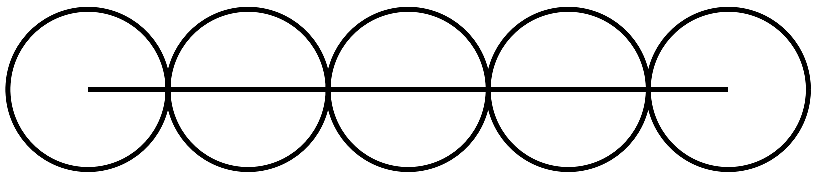
  3. **Non-spherical grains (Preliminary results)**

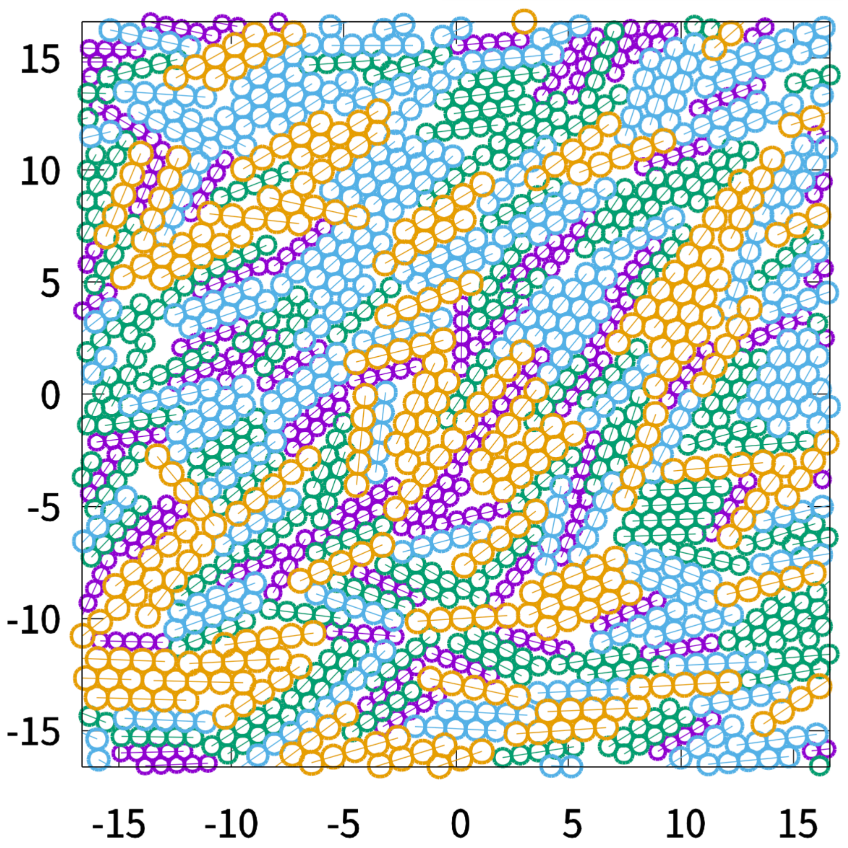
- Summary

Model of frictionless glued discs

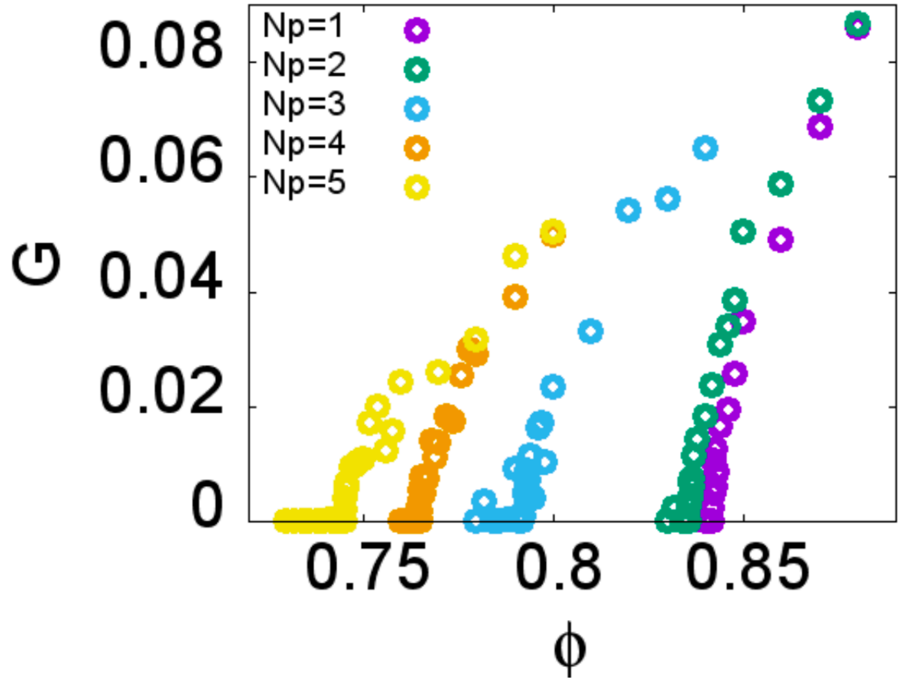
Collaboration with Kazuhisa Inata

Rigidly connected N_p discs





Linear elasticity



We expected discontinuous transition due to "effective" friction, but G exhibit continuous transition.

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- **Summary**

Summary

- Topic： Shear modulus of granular materials.
- Due to slip avalanches, G exhibits different critical behaviors.
- With small friction, linear elasticity changes drastically.
- Frictionless glued discs exhibits only a continuous transition.