Effusion of stochastic processes in one dimension

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with

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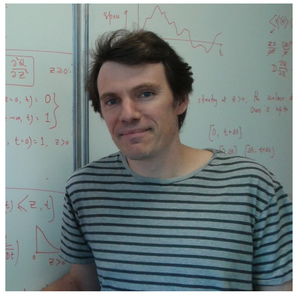


Plan of talk

1. Back ground - effusion problems in one dimension and link with elastically colliding stochastic processes

2. Two time statistics and physics meets football - the bivariate Poisson distribution

3. Role of initial configurational randomness

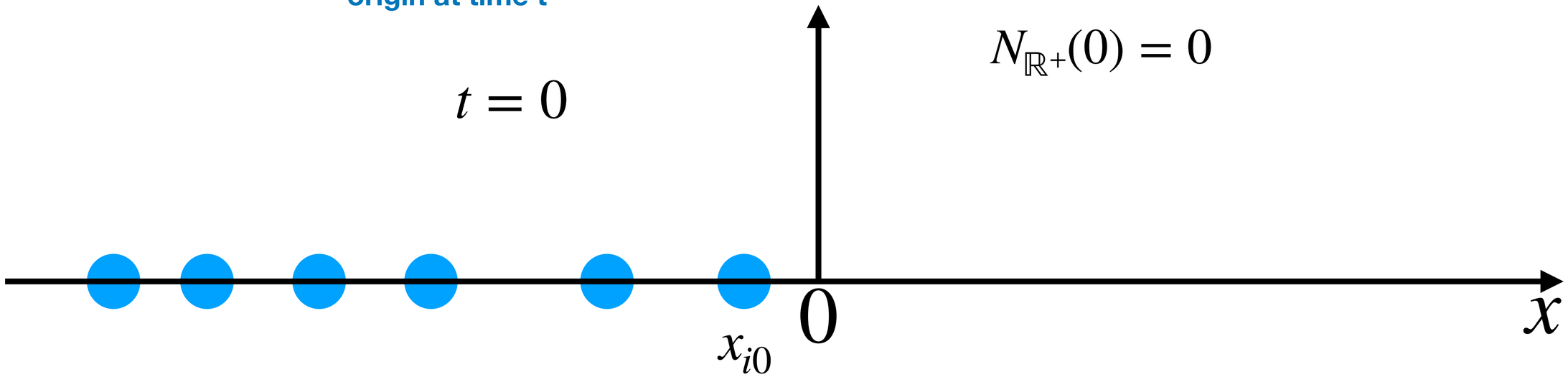
General effusion problem

$N_{\mathbb{R}+}(t)$

Number of particles to right of origin at time t

$t = 0$

$N_{\mathbb{R}+}(0) = 0$



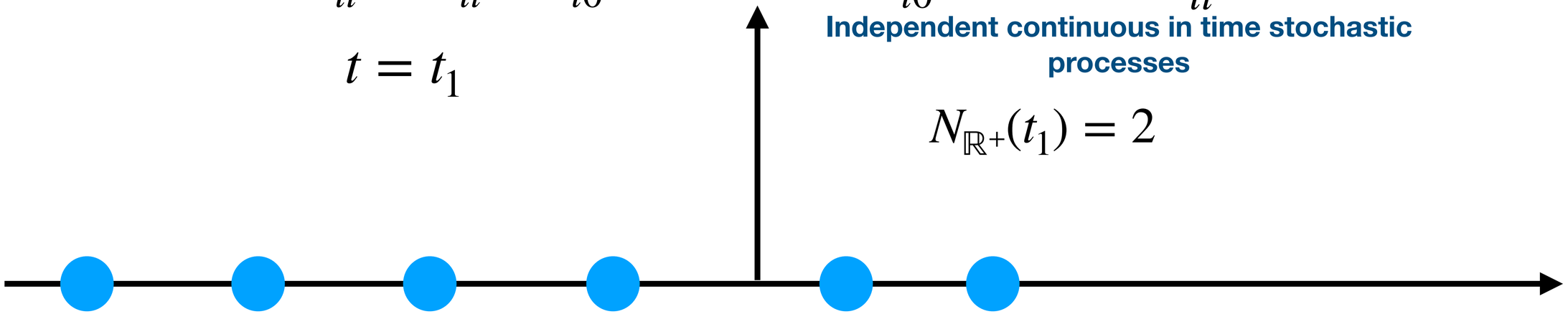
$X_{it} = Y_{it} + x_{i0}$

$Y_{i0} = 0 \qquad Y_{it}$

Independent continuous in time stochastic processes

$t = t_1$

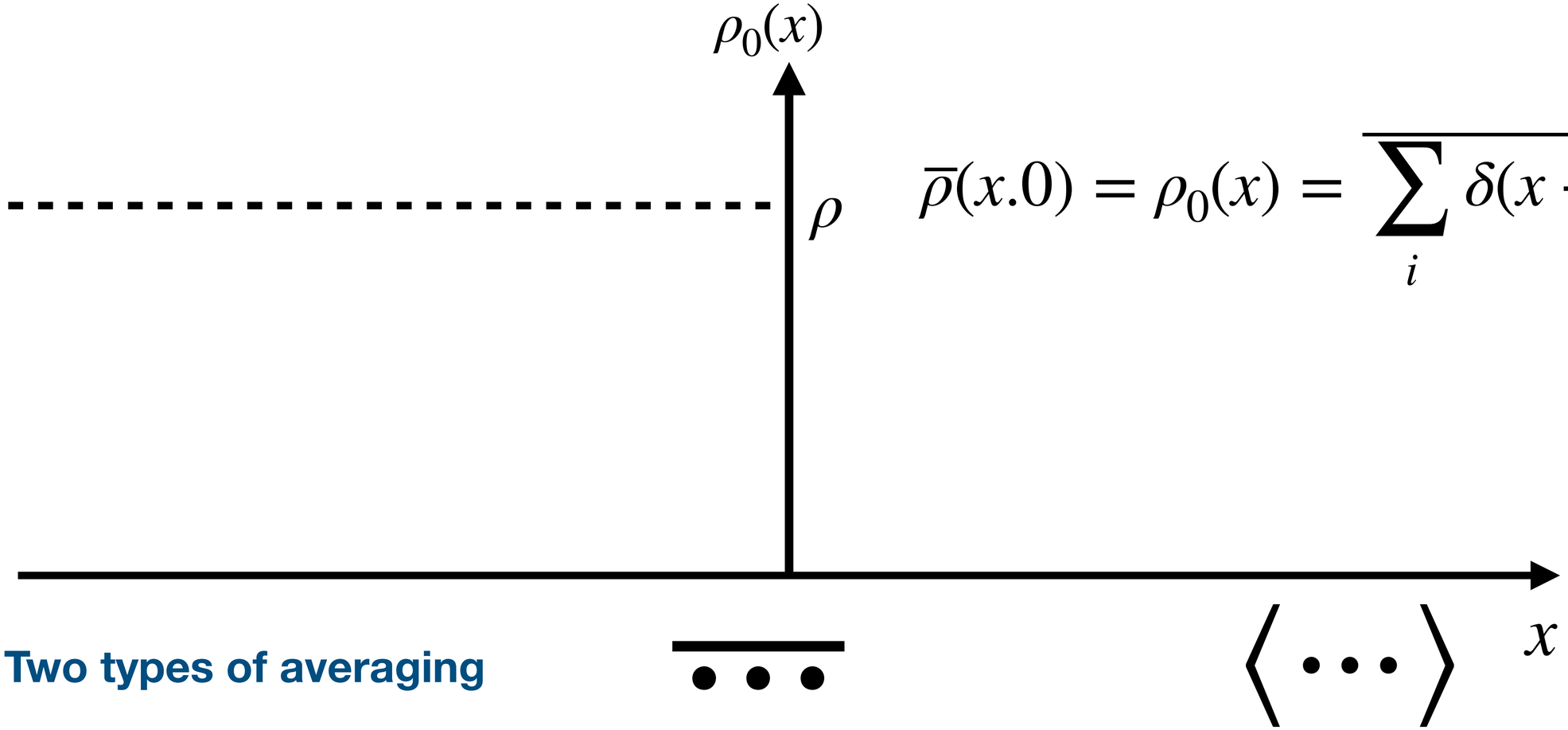
$N_{\mathbb{R}+}(t_1) = 2$



Coarse grained picture -step like initial density profile

B. Derrida and A. Gerschenfeld, J. Stat. Phys. 137, 978 (2009) - macroscopic fluctuation theory (BM)

Banerjee, S. N. Majumdar, A. Rosso, and G. Schehr, Phys. Rev. E 101, 052101 (2020) - direct calculation



$$\overline{\rho}(x.0) = \rho_0(x) = \overline{\sum_i \delta(x - x_{i0})}$$

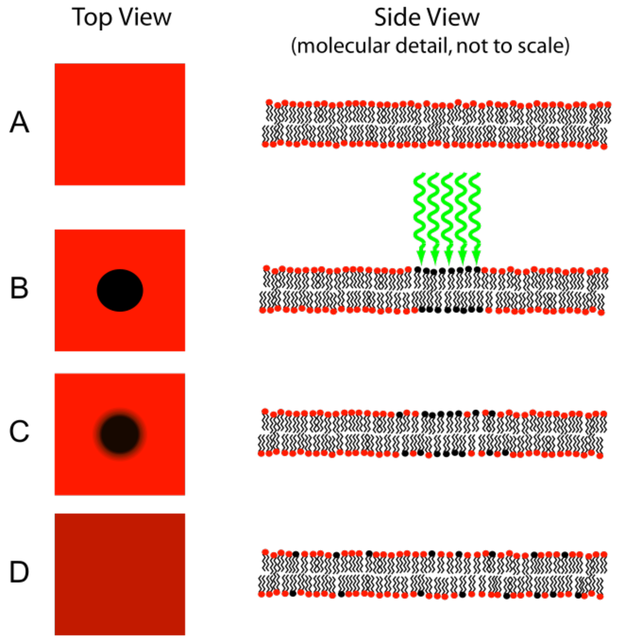
Two types of averaging

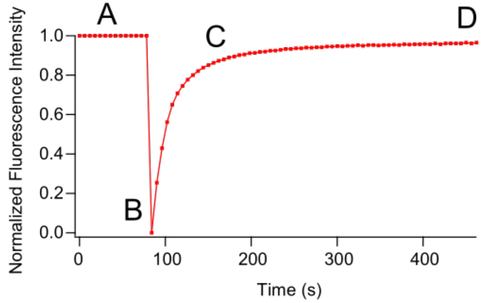
$\overline{\bullet\ \bullet\ \bullet}$ $\qquad$ $\langle \bullet\ \bullet\ \bullet \rangle$

initial conditions x_{i0} $\qquad$ **stochastic trajectories** Y_{it}

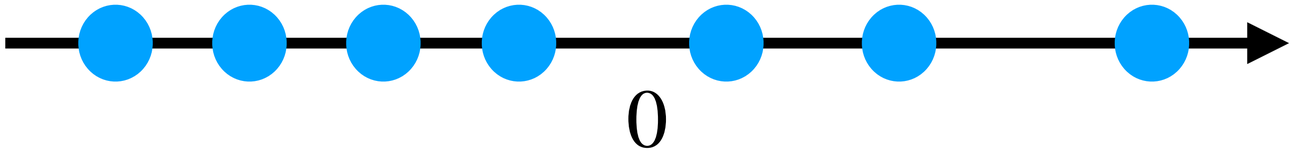
What are the statistics of $N_{\mathbb{R}+}(t)$?

Toy model for FRAP- Fluorescence Recover After Photobleaching

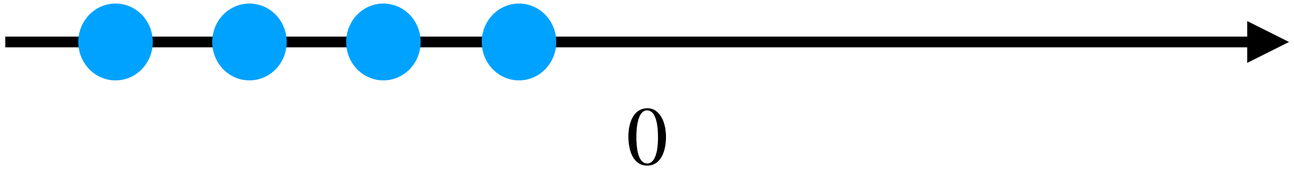




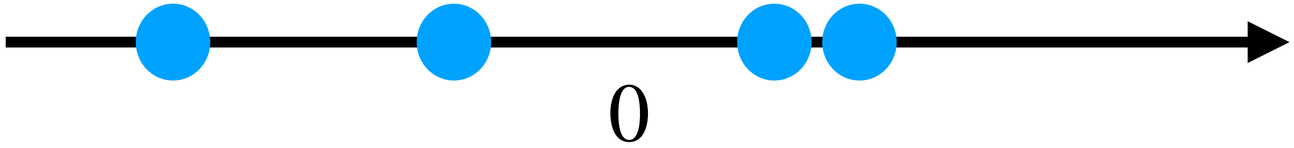
Source Wikipedia



Bleach particles to right of 0



Leave the system to evolve



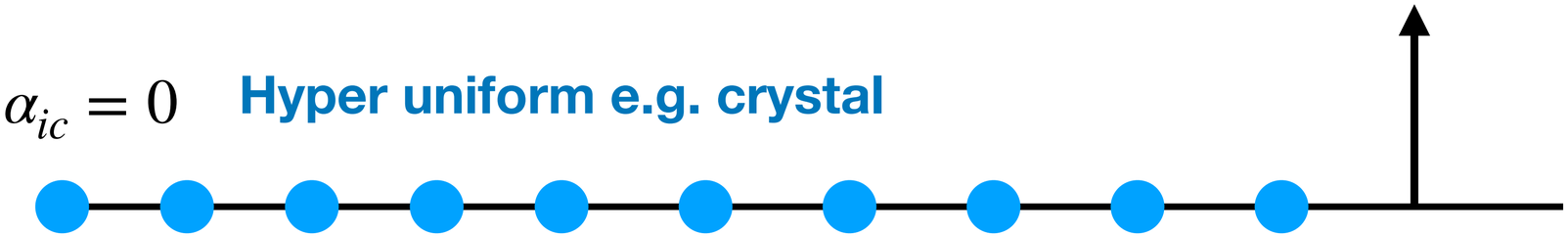
Role of initial conditions

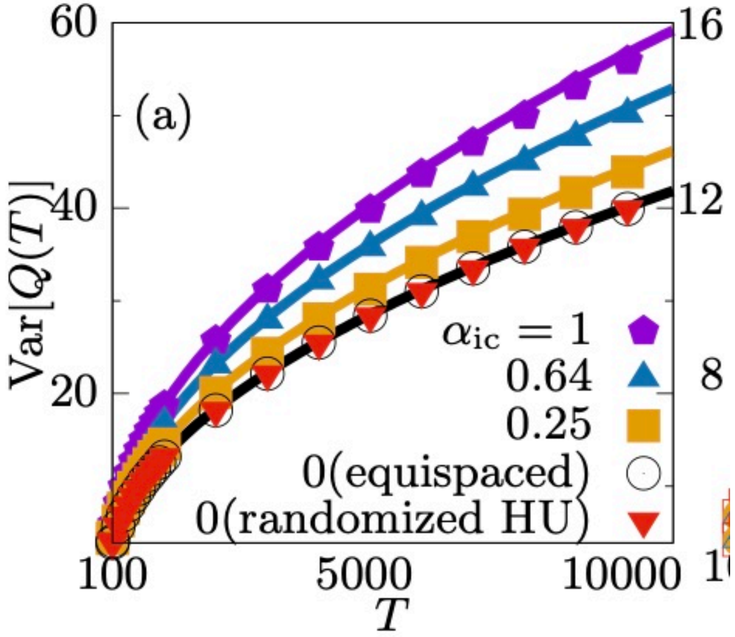
T. Banerjee, R. L. Jack, and M. E. Cates
Phys. Rev. E 106, (2022),

N. Leibovich and E. Barkai,
Phys. Rev. E 88, 032107 (2013)

$\alpha_{ic} = 1$ **Poisson -ideal gas**

$\alpha_{ic} = 0$ **Hyper uniform e.g. crystal**





$$\overline{\langle N^2_{\mathbb{R}^+}(t)\rangle} - \overline{\langle N_{\mathbb{R}^+}(t)\rangle}^2$$ **depends at all times on initial conditions**

Fano-factor
Generalised compressibility

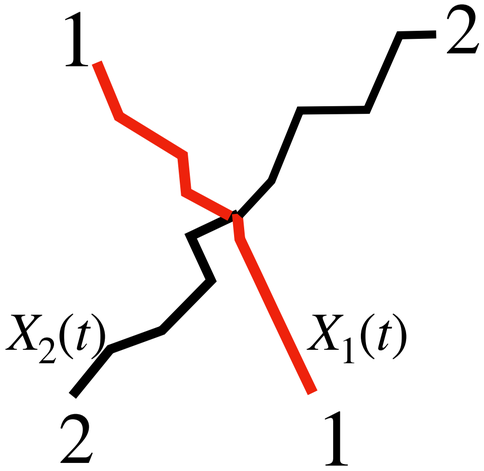
$$\alpha_{ic} = \lim_{\ell\to\infty} \frac{\mathrm{Var}\ n(\ell)}{\overline{n(\ell)}} = \lim_{q\to 0} S(q)$$

Structure factor

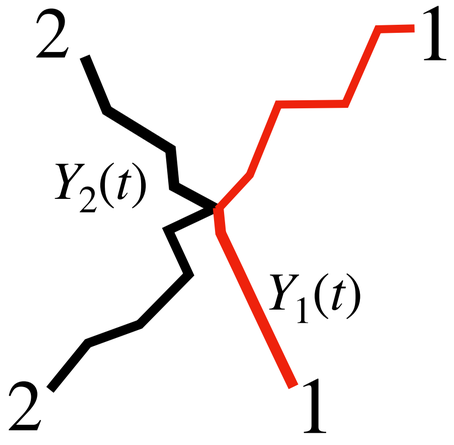
Link with elastic collision process

T. E. Harris, J. Appl. Probab. 2, 323 (1965)

Particles do not interact but change labels when they cross



Non-interacting processes



Elastically colliding processes
ECP

$$Y_1(t) = \max(X_1(t), X_2(t))$$

$$Y_2(t) = \min(X_1(t), X_2(t))$$

No interaction seen if particle numbers are not observed (not a SEP)

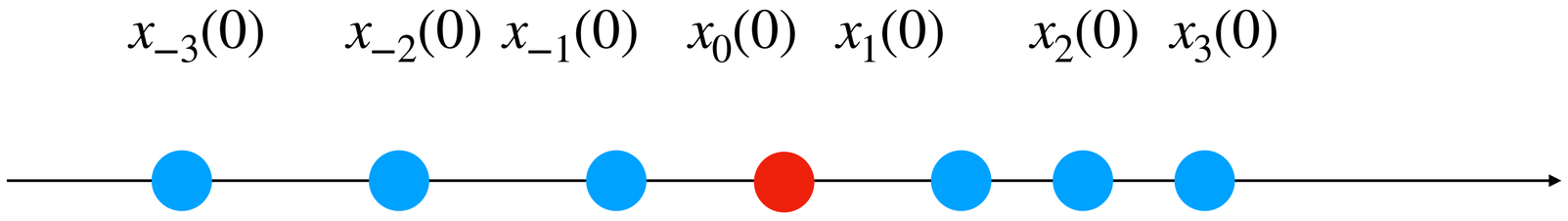
Can use macroscopic fluctuation theory write tracer in terms of density field

Derrida, Sadhu, Krapivsky Mallick ..

ECP tracer problem

D. Durr, S. Goldstein, and J. L. Lebowitz, Commun. Pure Appl. Math., 573 (1985)

$t = 0$



$Y(0) = 0$

Tracer closest to the origin at t=0, Poisson distribution density ρ along all of $\mathbb{R}$

$$\int_{-\infty}^{Y(t)} \rho(x,t)\,dx = \int_{-\infty}^{0} \rho(x,0)\,dx$$

Central Limit Theorem

$$Y(t)\,\bar{\rho} \simeq N_+(t) - N_-(t)$$

$$Y(t) \simeq \frac{N_+(t) - N_-(t)}{\bar{\rho}}$$

Single particle problem

Let A be s subset of $\mathbb{R}^n$ and $I_A(x)$ be its indicator function

Generating function $$G_2(z_1, z_2, x_0, t_1, t_2) = \langle z_1^{I_A(X_{t_1})} z_2^{I_A(X_{t_2})} \rangle$$

Write generating function in seemingly perverse way

$$G_2(z_1, z_2, x_0, t_1, t_2) = \langle 1 + (z_1 - 1) I_A(X_{t_1}) I_{A^c}(X_{t_2}) + (z_2 - 1) I_A(X_{t_2}) I_{A^c}(X_{t_1}) + (z_1 z_2 - 1) I_A(X_{t_1}) I_A(X_{t_2}) \rangle$$

$$\langle I_A(X_{t_2}) I_{A^c}(X_{t_1}) \rangle = \int_A dx_2 \int_{A^c} dx_1 p(x_2, t_2; x_1, t_1 \,|\, x_0) \qquad \langle I_{A^c}(X_{t_2}) I_A(X_{t_1}) \rangle = \int_{A^c} dx_2 \int_A dx_1 p(x_2, t_2; x_1, t_1 \,|\, x_0)$$

$$\langle I_A(X_{t_2}) I_A(X_{t_1}) \rangle = \int_A dx_2 \int_A dx_1 p(x_2, t_2; x_1, t_1 \,|\, x_0)$$

$$p(x_2, t_2; x_1, t_1 \,|\, x_0)$$ **joint density of X_{t_1} and X_{t_2} given that $X_0 = x_0$**

Annealed generating function

N **particles on** $A^c = [-L,0]$ **independently and uniformly distributed**

Annealed generating function

$$g_{a2}(z_1, z_2, t_1, t_2) = [\frac{1}{L}\int_{-L}^{0} dx_0 G(z_1, z_2, x_0, t_1, t_2)]^N$$

Take limit $L \to \infty$ **with** $\rho = N/L$ **fixed**

$$g_{a2}(z_1, z_2, t_1, t_2) = \exp(\lambda_{10}(t_1, t_2)(z_2 - 1) + \lambda_{01}(t_1, t_2)(z_2 - 1) + \lambda_{11}(t_2, t_1)(z_2 z_1 - 1))$$

$$\lambda_{01}(t_1, t_2) = \rho \int_{-\infty}^{0} dx_0 \int_A dx_2 \int_{A^c} dx_1 p(x_2, t_2; x_1, t_1 \,|\, x_0)$$

$$\lambda_{10}(t_1, t_2) = \rho \int_{-\infty}^{0} dx_0 \int_{A^c} dx_2 \int_A dx_2 p(x_2, t_2; x_1, t_1 \,|\, x_0$$

$$\lambda_{11}(t_1, t_2) = \rho \int_{-\infty}^{0} dx_0 \int_A dx_2 \int_A dx_2 p(x_2, t_2; x_1, t_1 \,|\, x_0)$$

Bivariate Poisson Distribution

$$g_2(z_1, z_2) = \langle z_1^{N_1} z_2^{N_2} \rangle = \exp(\lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_{12}(z_2 z_1 - 1))$$

Generating function

$$P(n_2, n_1) = \exp(-\lambda_1 - \lambda_2 - \lambda_{12}) \frac{\lambda_1^{n_1} \lambda_2^{n_2}}{n_1! n_2!} \sum_{n=0}^{\min(n_1, n_2)} \binom{n_2}{n} \binom{n_1}{n} n! \left(\frac{\lambda_{12}}{\lambda_1 \lambda_2} \right)^n$$

Joint pdf

$$N_1 \equiv M_1 + M_{12},\ N_1 \equiv M_2 + M_{12}$$

$M_1,\ M_2,\ M_{12}$ **independent Poisson random variables with parameters** $\lambda_1,\ \lambda_2,\ \lambda_{12}$

BV Poisson - mostly in statistics

Modelling sports scores

Groll A., Kneib T., Mayr Andreas and Schauberger G.,
"On the dependency of soccer score – a sparse bivariate Poisson model
for the UEFA European football championship 2016,
Journal of Quantitative Analysis in Sports, De Gruyter, vol. 14(2), 65 (2018)

	Round of 16	Quarter Finals	Semi Finals	Final	European Champion
Spain	95.4	72.9	52.3	35.1	21.8
Germany	99.3	79.5	51.3	34.4	21.0
France	97.5	71.9	48.2	25.8	13.8
England	95.2	69.4	43.4	23.9	12.9
Belgium	93.9	58.7	32.8	18.7	9.5
Portugal	92.5	52.3	27.4	12.6	5.5
Italy	87.7	47.6	23.8	11.4	4.8
Croatia	73.2	35.3	16.8	7.3	2.7
Poland	86.0	42.2	15.6	5.5	1.6
Austria	79.1	34.0	13.4	4.4	1.3
Switzerland	77.9	35.8	13.3	4.3	1.2
Turkey	56.1	21.2	8.3	2.8	0.8
Wales	65.6	27.4	9.6	2.8	0.8
Russia	62.3	25.1	8.6	2.5	0.6
Ukraine	71.0	25.8	7.7	2.0	0.4
Iceland	61.7	20.0	6.2	1.5	0.3
Czech Rep.	42.5	13.6	4.6	1.3	0.3
Slovakia	44.5	13.6	3.6	0.8	0.2
Sweden	42.9	11.2	3.3	0.8	0.1
Ireland	41.7	10.6	3.1	0.7	0.1
Romania	45.6	12.3	2.8	0.5	0.1
Albania	41.3	10.4	2.2	0.4	0.1
Hungary	37.1	8.1	1.8	0.3	0.0
Nor. Ireland	10.0	1.0	0.1	0.0	0.0

Score in final
Portugal 1- France 0

Motivation for 2024?

Annealed cumulants

Average

$$\overline{\langle N_A(t_1)\rangle} = \mu(t_1)$$

Single time variance

$$\overline{\langle N_A^2(t_1)\rangle}_c = \overline{\langle N_A^2(t_1)\rangle} - \overline{\langle N_A(t_1)\rangle}\ \overline{\langle N_A(t_1)\rangle} = \mu(t_1)$$

Two time correlation function

$$\overline{\langle N_A(t_1)N_A(t_2)\rangle}_c = \overline{\langle N_A(t_1)N_A(t_2)\rangle} - \overline{\langle N_A(t_1)\rangle}\ \overline{\langle N_A(t_2)\rangle} = \lambda_{11}(t_1, t_2)$$

Conditional hole probabilities

$$P(N_A(t) = 0) = \exp(-\mu(t))$$

$$P(N_A(t_2) = n \,|\, N_A(t_1) = 0) = \exp(-\mu(t_2) + \lambda_{11}(t_1, t_2))\frac{(\mu(t_2) - \lambda_{11}(t_2, t_2))^n}{n!}$$

$$P(N_A(t_1) = n \,|\, N_A(t_2) = 0) = \exp(-\mu(t_1) + \lambda_{11}(t_1, t_2))\frac{(\mu(t_1) - \lambda_{11}(t_2, t_1))^n}{n!}$$

Quenched generating function

$$g_{2q}(z_1, z_2, t_1, t_2) = \exp(\overline{\ln[\langle z_1^{N_A(t_1)} z_2^{N_A(t_2)}]\rangle]})$$

due to its additive form $\ln[\langle z_1^{N_A(t_1)} z_2^{N_A(t_2)}\rangle]$ is self averaging

$$g_{2q}(z_1, z_2, t_1, t_2) =$$

$$\exp\left(\rho \int_{-\infty}^{0} dx_0 \ln[1 + (z_1 - 1)\int_{A^c} dx_2 \int_A dx_1 p(x_2, t_2; x_1, t_1 | x_0) + (z_2 - 1)\int_A dx_2 \int_{A^c} dx_1 p(x_2, t_2; x_1, t_1 | x_0) + (z_2 z_1 - 1)\int_A dx_2 \int_A dx_1 p(x_2, t_2; x_1, t_1 | x_0)\right).$$

Average - same as annealed case

$$\overline{\langle N_A(t_1)\rangle} = \mu(t_1)$$

Quenched two point function

annealed result

$$\overline{\langle N_A(t_1) N_A(t_2)\rangle} - \overline{\langle N_A(t_1)\rangle\langle N_A(t_2)\rangle} = \lambda_{11}(t_1, t_2) - \lambda_{11}^*(t_1, t_2)$$

Smaller fluctuations with respect to annealed case

$$\lambda_{11}^*(t_1, t_2) = \rho \int_{-\infty}^{0} dx_0 \int_A dx_1 p(x_1, t_1 | x_0) \int_A dx_2 p(x_2, t_2 | x_0)$$

Results for Gaussian processes

Double half line problem

$$A = \mathbb{R}^+$$

$$X(t) = x_0 + Y(t)$$

$Y(t)$ **zero mean Gaussian process**

$$\langle Y(t) \rangle = 0 \qquad \langle Y(t)Y(t') \rangle = c(t, t')$$

Average

$$\langle \theta(X_{t_1}) \rangle = \langle \theta(Y_{t_1} + x_0) \rangle \quad = \frac{1}{\sqrt{2\pi c(t,t)}} \int_{-x_0}^{\infty} dy \exp(-\frac{y^2}{2c(t,t)}) = \frac{1}{2}\mathrm{erfc}(\frac{-x_0}{\sqrt{2c(t,t)}})$$

$$\mu(t) = \frac{\rho}{2} \int_0^{\infty} dx_0 \ \mathrm{erfc}(\frac{x_0}{\sqrt{2c(t,t)}}) = \rho \frac{\sqrt{c(t,t)}}{\sqrt{2\pi}}$$

Two point functions -(Mathematica knows the integrals)

$$\langle \theta(X_{t_2})\theta(X_{t_1})\rangle = \langle \theta(Y_{t_2}+x_0)\theta(Y_{t_1}+x_0)\rangle$$

Case $x_0 = 0$ has been extensively studied, only results for 2 and 3 point functions are known

D. Slepian, Bell Syst. Tech J. 41, 463 (1962)

$$\langle \theta(Y_{t_1})\theta(Y_{t_2})\rangle = \frac{1}{2\pi}\sin^{-1}\left(\frac{c(t_2,t_1)}{\sqrt{c(t_2,t_2)c(t_2,t_1)}}\right) + \frac{1}{4}$$

$$\langle \theta(Y_{t_1})\theta(Y_{t_2})\theta(Y_{t_3})\rangle = \frac{1}{2}\langle \theta(Y_{t_1})\theta(Y_{t_2})\rangle + \frac{1}{2}\langle \theta(Y_{t_1})\theta(Y_{t_3})\rangle + \frac{1}{2}\langle \theta(Y_{t_2})\theta(Y_{t_3})\rangle - \frac{1}{4}$$

Follows from neat symmetry argument $\theta(-Y) = 1 - \theta(Y)$

$$\langle \theta(Y_{t_1})\theta(Y_{t_2})\theta(Y_{t_3})\rangle = \langle 1 - \theta(Y_{t_1}) - \theta(Y_{t_2}) - \theta(Y_{t_3}) + \theta(Y_{t_1})\theta(Y_{t_2}) + \theta(Y_{t_1})\theta(Y_{t_3}) + \theta(Y_{t_2})\theta(Y_{t_3}) - \theta(Y_{t_1})\theta(Y_{t_2})\theta(Y_{t_3})\rangle$$

Two point functions continued

$$\theta(x) = -i\int_{-\infty}^{\infty}\frac{dk}{2\pi k}\exp(ikx)$$

Fourier representation for Heaviside function

$$J(x_0,t_1,t_2) = \langle\theta(Y_{t_2}+x_0)\theta(Y_{t_1}+x_0)\rangle = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dkdk'}{4\pi^2kk'}\exp(-\frac{1}{2}[k^2c(t_2,t_2)+2kk'c(t_2,t_1)+k'^2c(t_1,t_1)]+i(k+k')x_0])$$

$$\frac{\partial J(x_0,t_1,t_2)}{\partial c(t_2,t_1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_2dk_1}{4\pi^2}\exp(-\frac{1}{2}[k_2^2c(t_2,t_2)+2k_2k_1c(t_2,t_1)+k_1^2c(t_1,t_1)]+i(k_2+k_1)x_0])$$

Do the Gaussian integral

$$= \frac{1}{2\pi\sqrt{c(t_2,t_2)c(t_1,t_1)-c^2(t_2,t_1)}}\exp(-\frac{x_0^2}{2[c(t_2,t_2)c(t_1,t_1)-c^2(t_2,t_1)]}[c(t_2,t_2)+c(t_1,t_1)-2c(t_2,t_1)])$$

Two point functions continued - a good exam question !

$$\int_{-\infty}^{0} dx_0 \frac{\partial J(x_0, t_1, t_2)}{\partial c(t_2, t_1)} = \frac{1}{2\sqrt{2\pi}\sqrt{c(t_2, t_2) + c(t_1, t_1) - 2c(t_2, t_1)}}$$

Integrate wrt $c(t_2, t_1)$

$$\int_{-\infty}^{0} dx_0 J(x_0, t_1, t_2) = -\frac{\sqrt{c(t_2, t_2) + c(t_1, t_1) - 2c(t_2, t_1)}}{2\sqrt{2\pi}} + C(t_2) + C'(t_1)$$

Use result for $c(t, t)$

$$\int_{-\infty}^{0} dx_0 J(x_0, t, t) = \frac{\sqrt{c(t, t)}}{\sqrt{2\pi}}$$

$$\lambda_{11}(t_1, t_2) = \frac{\rho}{2\sqrt{2\pi}} \left[\sqrt{c(t_2, t_2)} + \sqrt{c(t_1, t_1)} - \sqrt{c(t_2, t_2) + c(t_1, t_1) - 2c(t_2, t_1)}\right].$$

Summary

Average

$$\mu(t) = \rho \frac{\sqrt{c(t,t)}}{\sqrt{2\pi}}$$

agrees with previous results for BM

Annealed two Point function

$$\lambda_{11}(t_1,t_2) = \frac{\rho}{2\sqrt{2\pi}} \left[\sqrt{c(t_2,t_2)} + \sqrt{c(t_1,t_1)} - \sqrt{c(t_2,t_2) + c(t_1,t_1) - 2c(t_2,t_1)} \right].$$

Quenched two Point function

$$\lambda_{q11}(t_1,t_2) = \frac{\rho}{2\sqrt{2\pi}} \left[\sqrt{c(t_1,t_1) + c(t_2,t_2)} - \sqrt{c(t_2,t_2) + c(t_1,t_1) - 2c(t_2,t_1)} \right]$$

$$\lambda_{11}(t,t) = \frac{\rho}{\sqrt{2\pi}} \sqrt{c(t,t)}$$

$$\lambda_{q11}(t,t) = \frac{\rho}{2\sqrt{\pi}} \sqrt{c(t,t)}$$

agrees with previous results for BM

T. Banerjee, R. L. Jack, and M. E. Cates
Phys. Rev. E 106, (2022),

N. Leibovich and E. Barkai,
Phys. Rev. E 88, 032107 (2013)

correlations in initial conditions never forgotten

Examples

Brownian motion $$c(t_2, t_1) = 2D\ \min(t_2, t_1)$$

Average $$\mu(t) = \rho \frac{\sqrt{Dt}}{\sqrt{\pi}}$$

Annealed two Point function $$\lambda_{11}(t_1, t_2) = \frac{\rho\sqrt{D}}{2\sqrt{\pi}}(\sqrt{t_2} + \sqrt{t_1} - \sqrt{t_2 - t_1})$$

Quenched two Point function $$\lambda_{q11}(t_1, t_2) = \frac{\rho\sqrt{D}}{2\sqrt{\pi}}(\sqrt{t_2 + t_1} - \sqrt{t_2 - t_1})$$

Correlation function for fractal Brownian motion $$c(t_2, t_1) = D_H(t_2^{2H} + t_1^{2H} - |t_2 - t_1|^{2H})$$

$$H = \frac{1}{4}$$ **but $N_{\mathbb{R}^+}(t)$ is not Gaussian**

S. Marbach, J. Chem. Phys. 154, 171101 (2021)

Under damped thermalised Brownian motion

Langevin equation $$m\frac{dV_t}{dt} = -\gamma V_t + \sqrt{2\gamma T}\eta(t)$$

Correlation function $$c(t_2, t_1) = c(t_1, t_1) + \frac{k_B T \tau^2}{m}(1 + \exp(-\frac{t_2}{\tau}) - \exp(-\frac{t_1}{\tau}) - \exp(-\frac{t_2 - t_1}{\tau}))$$

$$c(t, t) = \frac{2k_B T \tau^2}{m}(\frac{t}{\tau} - 1 + \exp(-\frac{t}{\tau}))$$

$\tau = m/\gamma$ **relaxation time separating the ballistic short time regime from the diffusive long time regime**

For $t_1,\ t_2,\ (t_2 - t_1) \gg \tau$ recover Brownian behavior

Short times $$c(t_2, t_1) = \frac{k_B T}{m} t_1 t_2$$ **ballistic regime - Jepsen gas**

Particles can't go back

$$\mu(t) = \rho\sqrt{\frac{k_B T}{2\pi m}}t \qquad \lambda_{11}(t_1, t_2) = \mu(t_1) \qquad \lambda_{q11}(t_1, t_2) = \frac{1}{2}\rho\sqrt{\frac{k_B T}{2\pi m}}\left[\sqrt{t_2^2 + t_1^2} - t_2 + t_1\right]$$

The current at the origin

$$j(0,t) = \frac{dN_{\mathbb{R}^+}}{dt}$$

By definition - even when no Fokker-Planck equation to identify a current

Fractional Brownian motion

Average current

$$\overline{\langle j(0,t)\rangle} = \rho H \frac{\sqrt{D_H} t^{H-1}}{\sqrt{\pi}}$$

Annealed current fluctuations

$$\overline{\langle j(0,t_1) j(0,t_2)\rangle} - \overline{\langle j(0,t_1)\rangle}\ \overline{\langle j(0,t_2)\rangle} = -\frac{\rho\sqrt{D_H} H(1-H)}{2\sqrt{\pi}(t_2-t_1)^{2-H}}$$

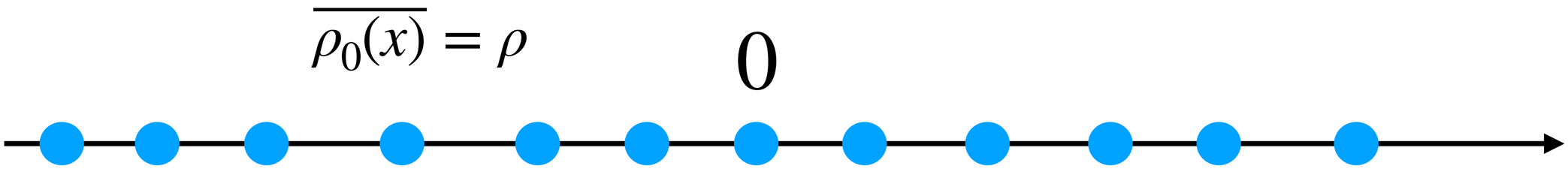
Quenched current fluctuations

$$\overline{\langle j(0,t_1)\rangle\langle j(0,t_2)\rangle} - \overline{\langle j(0,t_1)\rangle\langle j(0,t_2)\rangle} = -\frac{\rho\sqrt{D_H} H(1-H)}{2\sqrt{\pi}(t_2-t_1)^{2-H}} - \frac{\rho\sqrt{D_H} H^2}{2\sqrt{\pi}} \frac{t_1^{2H-1} t_2^{2H-1}}{(t_1^{2H}+t_2^{2H})^{\frac{3}{2}}}$$

Anti-correlated

Correlations in initial configuration

$$\overline{\rho_0(x)} = \rho$$



Translationally invariant with non-trivial correlation function of structure factor

$$\overline{\rho_0(x)\,\rho_0(y)} - \rho^2 = \rho \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, S(q)\, e^{-i\,q\,(x-y)}$$

Kill (photo-bleach) all particles to right of origin and evolve

Annealed correlation function

$$C(t_1, t_2) = \overline{\langle N_{\mathbb{R}^+}(t_2) N_{\mathbb{R}^+}(t_1)\rangle} - \overline{\langle N_{\mathbb{R}^+}(t_2)\rangle}\;\overline{\langle N_{\mathbb{R}^+}(t_1)\rangle} = C_{\text{noise}}(t_1, t_2) + C_{\text{ic}}(t_1, t_2)$$

$$C_{\text{noise}}(t_1, t_2) = \overline{\langle N_{\mathbb{R}^+}(t_2) N_{\mathbb{R}^+}(t_1)\rangle} - \overline{\langle N_{\mathbb{R}^+}(t_2)\rangle\langle N_{\mathbb{R}^+}(t_1)\rangle}$$

$$C_{\text{ic}}(t_1, t_2) = \overline{\langle N_{\mathbb{R}^+}(t_2)\rangle\langle N_{\mathbb{R}^+}(t_1)\rangle} - \overline{\langle N_{\mathbb{R}^+}(t_2)\rangle}\;\overline{\langle N_{\mathbb{R}^+}(t_1)\rangle}$$

$$C_{\text{noise}}(t_2, t_1) = \lambda_{q11}(t_1, t_2) = \frac{\rho}{2\sqrt{2\pi}} \left[\sqrt{c(t_1, t_1) + c(t_2, t_2)} - \sqrt{c(t_2, t_2) + c(t_1, t_1) - 2c(t_2, t_1)} \right]$$

Independent of disorder correlations - equivalent to quenched result

$$C_{\text{ic}}(t_1, t_2) = \frac{\rho}{8\pi} \int_0^\infty dxdy \int dq \; S(q) \; \exp(iq(x-y)) \text{erfc}\left(\frac{x}{\sqrt{2c(t_2, t_2)}}\right) \text{erfc}\left(\frac{y}{\sqrt{2c(t_1, t_1)}}\right)$$

At late times

$$C_{\text{ic}}(t_1, t_2) = \frac{\alpha_{ic}\rho(c(t_1, t_1)c(t_2, t_2))^{\frac{1}{4}}}{2\sqrt{2\pi}} \frac{1 + a(t_1, t_2)^2 - \sqrt{1 + a(t_1, t_2)^4}}{a(t_1, t_2)}$$

$$a(t_1, t_2) = \left(\frac{c(t_2, t_2)}{c(t_1, t_1)}\right)^{\frac{1}{4}}$$

Equal late times

$$C_{\text{ic}}(t, t) = \frac{\alpha_{ic}\rho(c(t, t))^{\frac{1}{2}}}{2\sqrt{\pi}} (\sqrt{2} - 1)$$

Perspectives

Very rich phenomenology for non-interacting but non-trivial Gaussian processes

Potentially provides insight to help solve equations arising in macroscopic fluctuation approach

Consequences for elastically colliding processes merit further investigation

3 point functions

Need to compute $\langle \prod_{i=1}^{n} \theta(x_0 + Y_{t_i}) \rangle$

Slepian tells us we can do this for n=1, 2, 3, when $x_0 = 0$

Can we do n= 3 for Brownian motion ? Use Markov property

$$\langle \prod_{i=1}^{n} \theta(x_0 + Y_{t_i}) \rangle = \langle \theta(t_1 - T_0) \prod_{i=1}^{n} \theta(Y_{t_i - T_0}) \rangle$$

T_0 **first passage time of** Y_t **started at** $Y_0 = 0$ **to** 0

$$\langle \prod_{i=1}^{n} \theta(x_0 + Y_{t_i}) \rangle = \int_0^{t_1} d\tau F(\tau, x_0) \langle \prod_{i=1}^{n} \theta(Y_{t_i - \tau}) \rangle$$

$F(t, x_0)$ **first passage time density to** 0 **from** x_0

For Brownian motion

$$F(\tau, x_0) = \frac{|x_0| \exp(-\frac{x_0^2}{4D\tau})}{\sqrt{4\pi D \tau^3}}$$

$$\int_{-\infty}^{0} dx_0 \langle \prod_{i=1}^{n} \theta(x_0 + Y_{t_i}) \rangle = \sqrt{\frac{D}{\pi}} \int_0^{t_1} d\tau \frac{1}{\sqrt{\tau}} \langle \prod_{i=1}^{n} \theta(Y_{t_i - \tau}) \rangle$$

Recovering 1+2 point for BM

$$\int_{-\infty}^{0} dx_0 \langle \theta(x_0 + Y_t) \rangle = \int_0^t \frac{\sqrt{D}}{\sqrt{\pi\tau}} \times \frac{1}{2} = \frac{\sqrt{Dt}}{\sqrt{\pi}}$$

$$\int_{-\infty}^{0} dx_0 \langle \theta(x_0 + Y_t)\theta(x_0 + Y_{t'}) \rangle = \frac{\sqrt{D}}{\sqrt{\pi}} \int_0^{t'} \frac{1}{\sqrt{\tau}} [\frac{1}{2\pi} \sin^{-1}(\frac{\sqrt{t' - \tau}}{\sqrt{t - \tau)}}) + \frac{1}{4}] = \frac{\sqrt{D}}{2\sqrt{\pi}} \left[\sqrt{t} + \sqrt{t'} - \sqrt{t - t'}\right]$$

Agrees with general Gaussian result

$$\int_{-\infty}^{0} dx_0 \langle \theta(x_0 + Y_{t_1})\theta(x_0 + Y_{t_2})\theta(x_0 + Y_{t_3}) = \int_{-\infty}^{0} dx_0 \langle \theta(t_1 - T_0)\langle \theta(Y_{t_1 - T_0})\theta(Y_{t_2 - T_0})\theta(Y_{t_3 - T_0})\rangle$$

Use neat symmetry argument

$$= \frac{1}{2}\int_{-\infty}^{0} dx_0 \langle \theta(t_1 - T_0)\langle \theta(Y_{t_1 - T_0})\theta(Y_{t_2 - T_0})\rangle + \langle \theta(Y_{t_1 - T_0})\theta(Y_{t_3 - T_0})\rangle + \langle \theta(Y_{t_2 - T_0})\theta(Y_{t_3 - T_0})\rangle - \frac{1}{2}\rangle$$

Have already computed these terms

New term fully 3 time Quantity

Final result

$$\int_{-\infty}^{0} dx_0 \langle \theta(x_0 + Y_{t_3})\theta(x_0 + Y_{t_2})\theta(x_0 + Y_{t_1})\rangle =$$

$$\frac{\sqrt{D}}{4\sqrt{\pi}}[\sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3} - \sqrt{t_2 - t_1} - \sqrt{t_3 - t_1} + 2\frac{\sqrt{t_1}}{\pi}\sin^{-1}(\frac{\sqrt{t_2 - t_1}}{\sqrt{t_3 - t_1}}) - \frac{2\sqrt{t_3 - t_2}\sin^{-1}(\sqrt{\frac{t_1}{t_2}}) - 2\sqrt{t_3}\tan^{-1}(\sqrt{\frac{t_1(t_3 - t_2)}{t_3(t_2 - t_1)}})}{\pi}]$$

Not pretty or neat ! -anyone see a pattern ?

Joint distribution is trivariate Poisson