


Theory of Mpemba-like phenomena: a few examples including quantum effect

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Refs: PRE**103**, 032901 (2021) and in preparation.

Contents

- Introduction
- Mpemba effect in inertial suspension (PRE**103**, 032901 (2021))
- Quantum Mpemba effect in Anderson model
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- Concluding remarks

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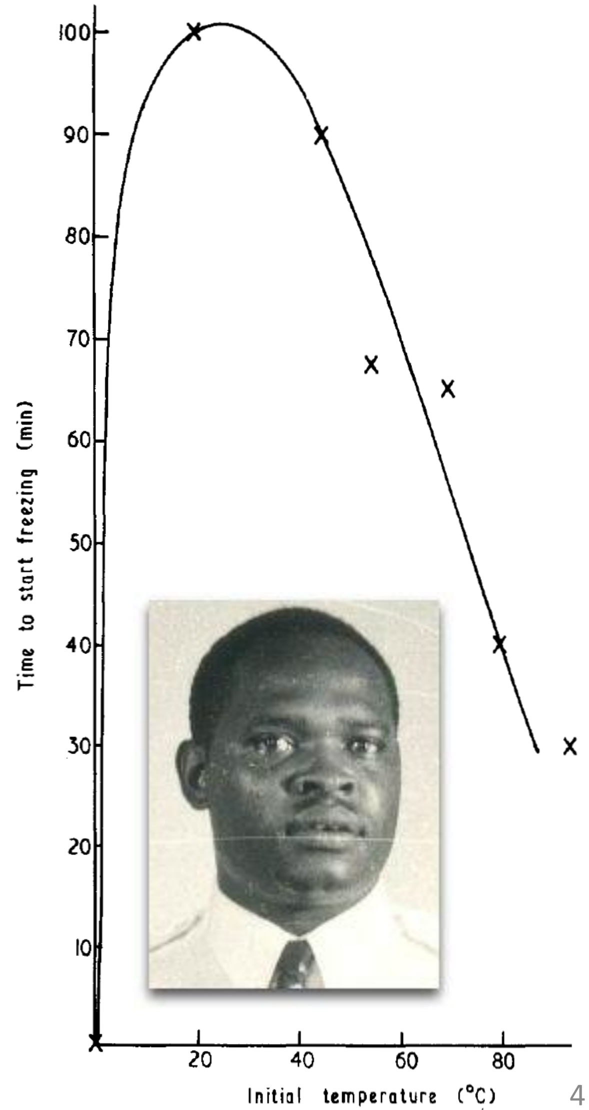
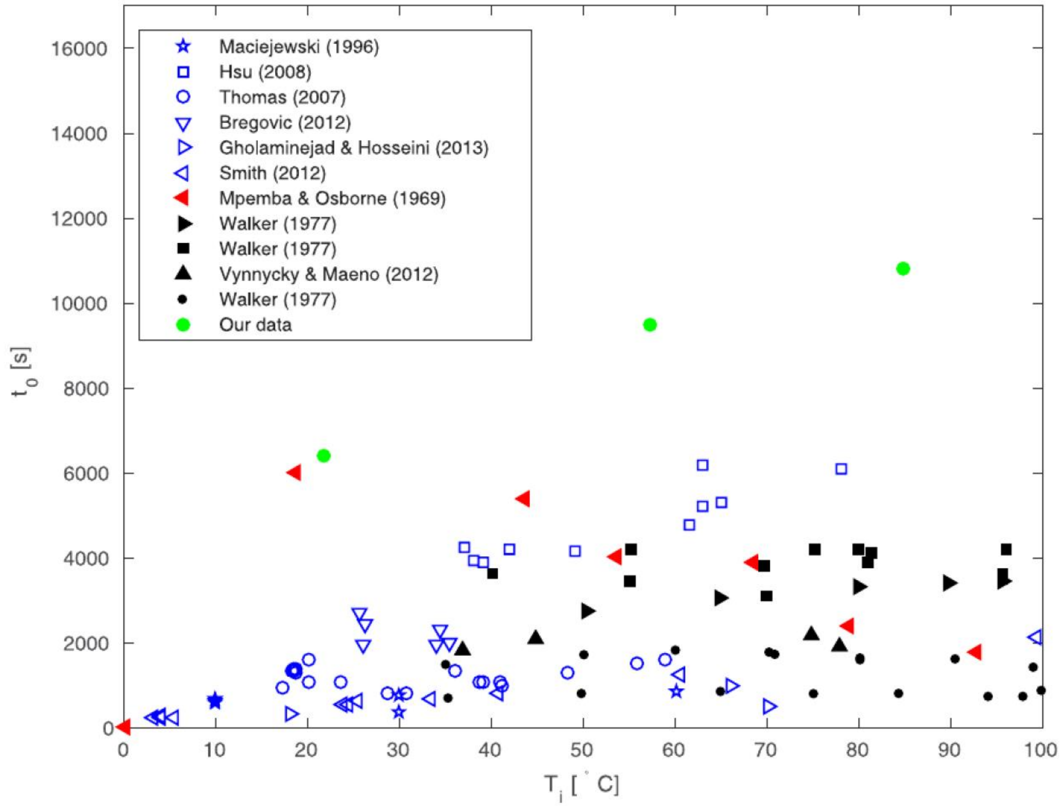
Introduction on Mpemba effect

- What is Mpemba effect?
  - Erasto B. Mpemba found that some hot suspensions (*ice cream mix*) can freeze faster than cold (1963).
  - With the help of D. G. Osborne he has published a scientific paper (1969).



Debates

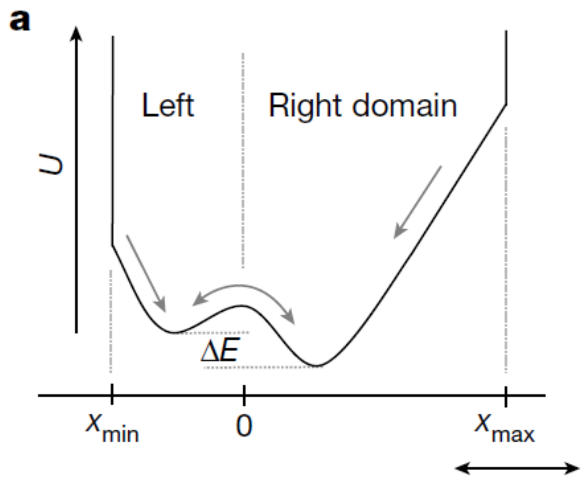
- Poor reproducibility
- The right figure is one counter example of Mpemba effect.
- However, people believe the existence of Mpemba-like phenomena.

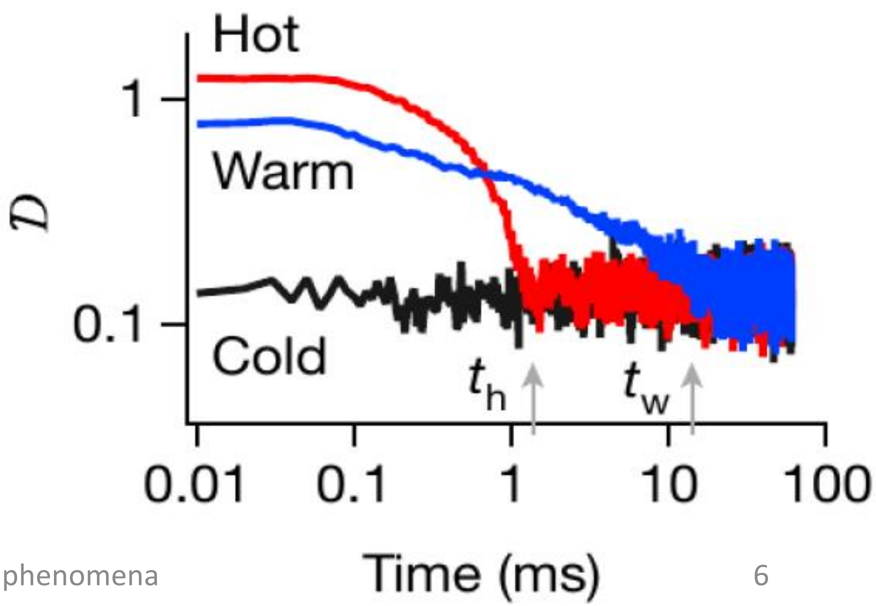


Burridge and Linden, Sci. Rep. **6**, 37665 (2016).

Experimental confirmation

- Kumar & Bechhoeffer, Nature **584**, 64 (2020).
- They have analyzed trapped colloids in a double well potential.
- They observed the distance between the distribution and equilibrium one.





Lu & Raz (PNAS2017)

- They have analyzed the master equation:

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \sum_j R_{ij}(T_b)p_j(t) \quad \text{for } i = 1, 2, \cdots, n.$$

- They are interested in the slowest relaxation mode:

$$\vec{p}(t) = \vec{\pi}(T_b) + e^{\lambda_2 t} a_2 \vec{v}_2 + \ldots \qquad \pi_i(T_b) = \frac{e^{-E_i/k_B T_b}}{\sum_i e^{-E_i/k_B T_b}}$$

- The condition for Markovian Mpemba effect:

$$|a_2^c| > |a_2^h|$$

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Illustration of Mpemba effect

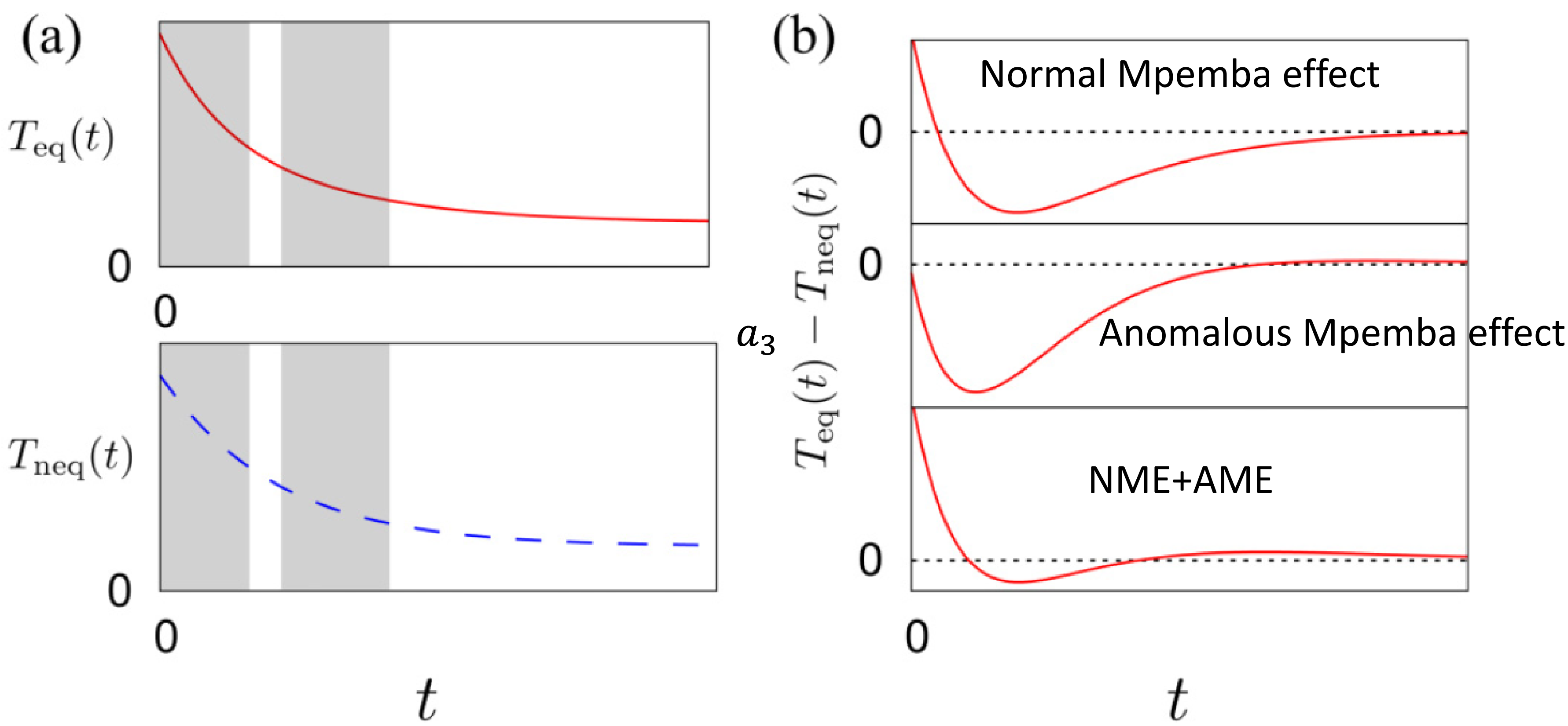
- We can write the energy equation (for an uniform system):

$\dot{\gamma}$: shear rate

$$\frac{\partial}{\partial t}T = \underbrace{-\frac{2\dot{\gamma}}{dn}P_{xy}}_{\gtrsim 0} + 2\zeta(T_{\rm ex} - T)$$

- If the system is at equilibrium, the viscous heating is absent (P_{xy}=0).
- If the system is in non-equilibrium, the heating term must exist.
- Then, the system at equilibrium must have faster cooling than that at non-equilibrium.

Essence of Mpemba effect



Langevin model

- Suspensions are influenced by collisions and dag as well as thermal activation:

Collision force

$$\frac{d\boldsymbol{p}_i}{dt} = -\zeta \boldsymbol{p}_i + \boldsymbol{F}_i^{(\mathrm{imp})} + m\boldsymbol{\xi}_i,$$

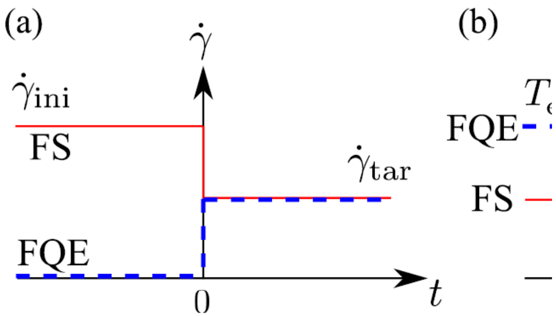
- where the noise satisfies

$$\langle \boldsymbol{\xi}_i(t) \rangle = 0, \quad \langle \xi_{i,\alpha}(t)\xi_{j,\beta}(t') \rangle = \frac{2\zeta T_{\mathrm{env}}}{m} \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$$

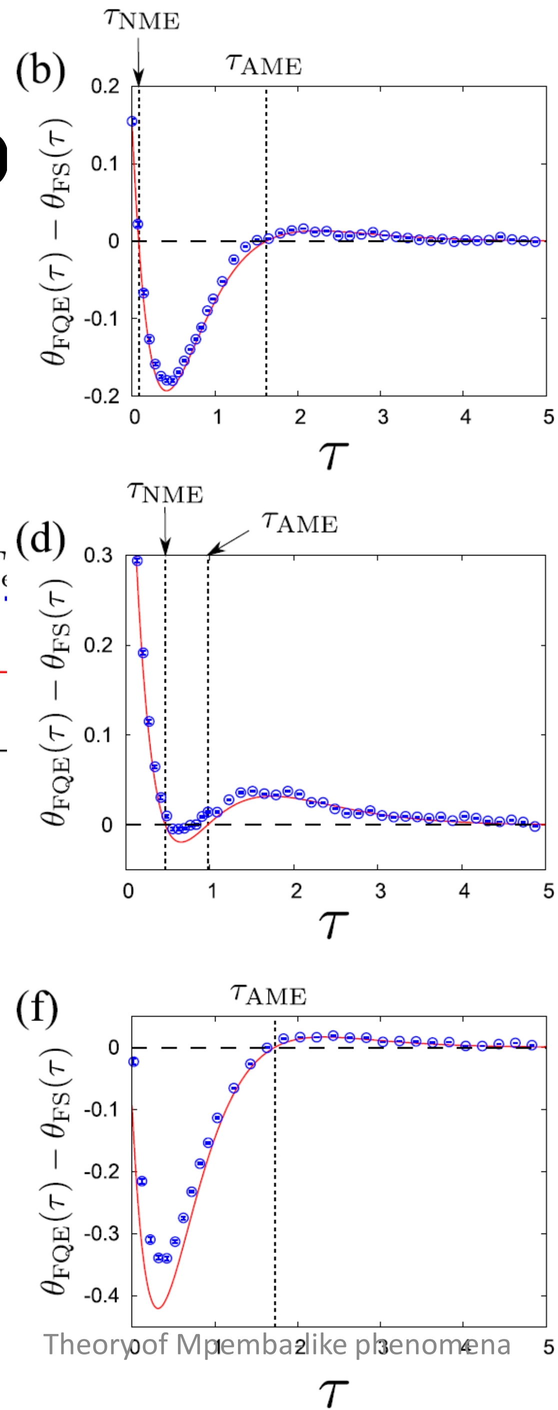
- Here, we have introduced the peculiar momentum $\boldsymbol{p}_i \equiv m(\boldsymbol{v}_i - \dot{\gamma} y \boldsymbol{e}_x) = m\boldsymbol{V}_i$
- Langevin ⇔ Bolzmann-Enskog+Fokker-Planck

Pro lts

- Protocol



- Typical results



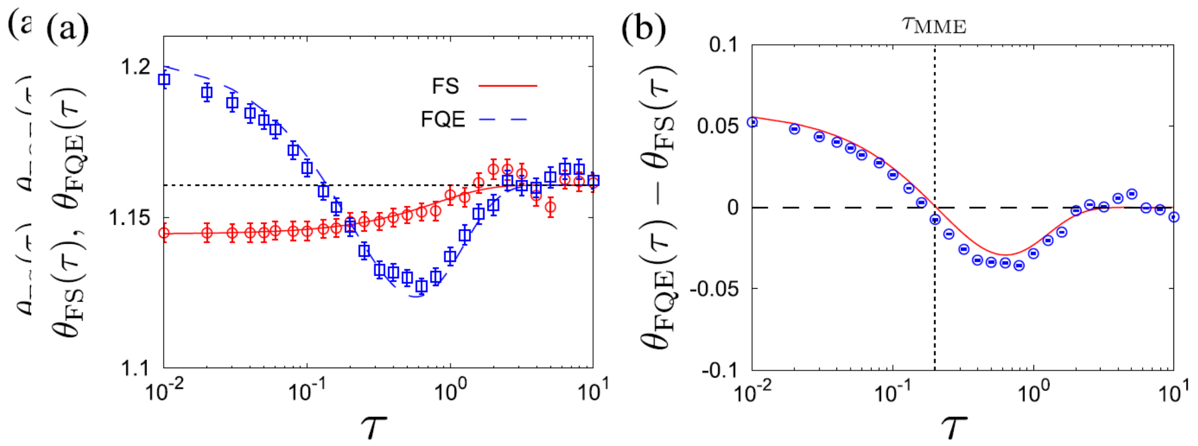
$$T = \frac{m}{3n}\int d\boldsymbol{V}\,\boldsymbol{V}^2 f(\boldsymbol{V})$$

$$\theta := T/T_{env}$$

$f(V)$:
velocity distribution function

Inverse Mpemba & Mixed Mpemba

- We can observe the inverse Mpemba effect in heating processes.



- We can observe the mixed Mpemba effect in which

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Brief summary and comments of Mpemba effect

- We illustrated how Mpemba effect can happen in experimental accessible situations.
- Mpemba effect can happen for sheared suspensions.
- Most important theoretical problem: What is the temperature?
  - We use a quench protocol, but the reservoir temperature does not evolve with time.

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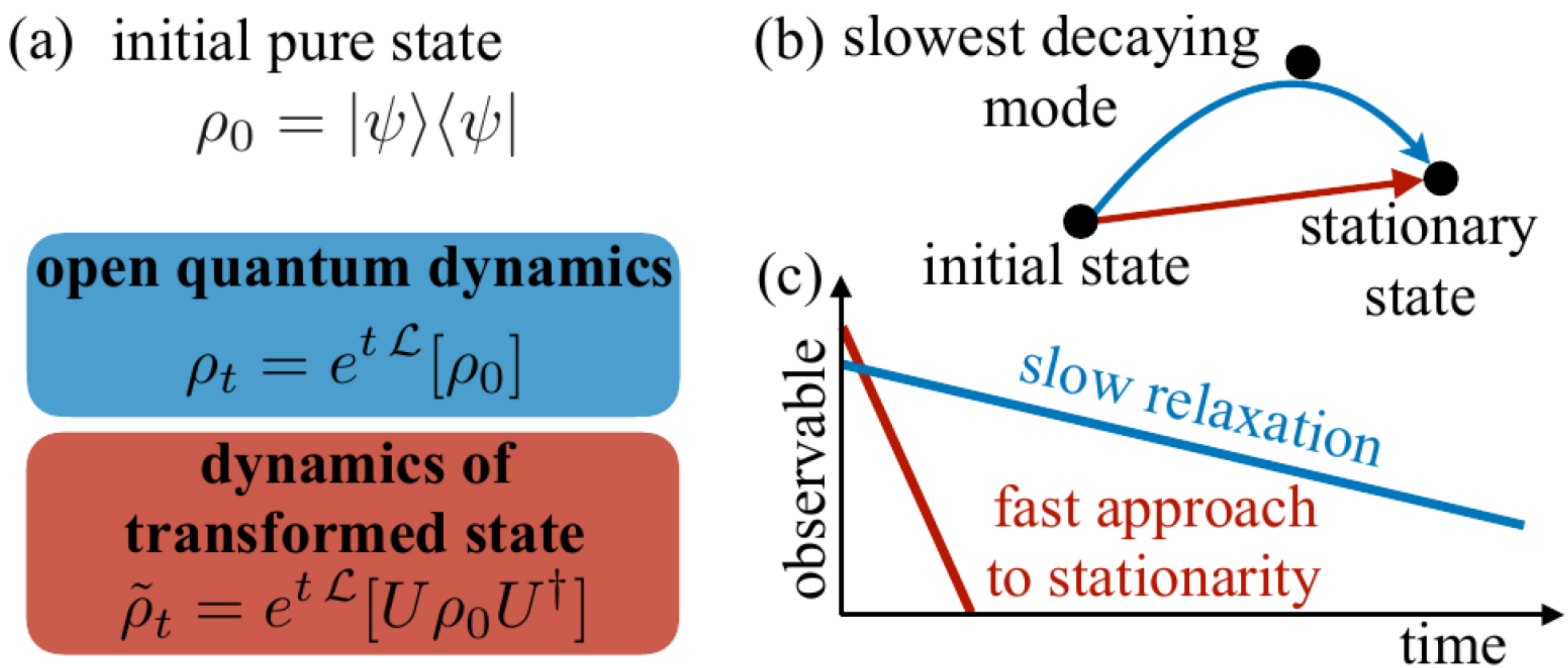
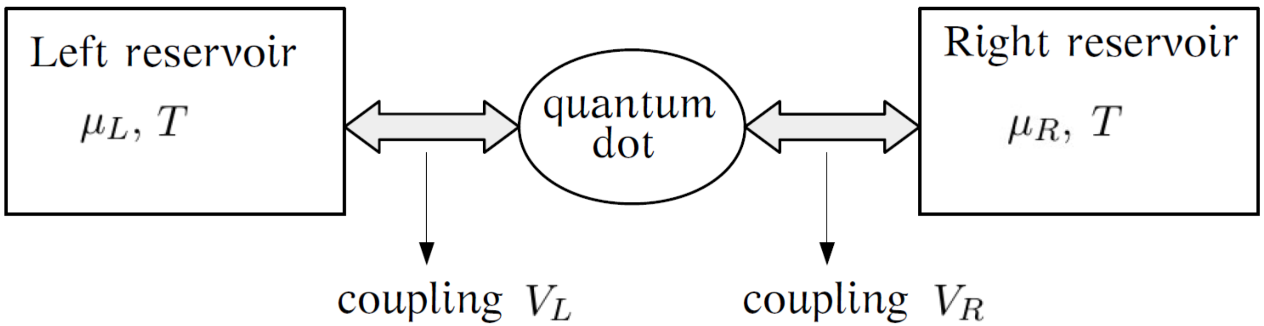
Earlier study on quantum Mpemba effect:

F. Carollo, A. Lasanta and Lesanovsky, *PRL 127, 060401 (2021)*.



Quench dynamics of Anderson model

<u>A single quantum dot connected to two reservoirs</u>



Total Hamiltonian:

$$\hat{H}^{tot} = \hat{H}^{s} + \hat{H}^{r} + \hat{H}^{int}$$

System Hamiltonian

Reservoir Hamiltonian

System-reservoirs interaction Hamiltonian

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$$\hat{H}^s = \sum_\sigma \epsilon_0 \hat{d}_\sigma^\dagger \hat{d}_\sigma + U\hat{n}_\uparrow \hat{n}_\downarrow$$

$$\hat{H}^r = \sum_{\gamma,k,\sigma} \epsilon_k \hat{a}_{\gamma,k,\sigma}^\dagger \hat{a}_{\gamma,k,\sigma}$$

$$\hat{H}^{int} = \sum_{\gamma,k,\sigma} V_\gamma \hat{d}_\sigma^\dagger \hat{a}_{\gamma,k,\sigma} + \text{h.c.}$$

ϵ_0: energy of electron in quantum dot

ϵ_k: energy of electron corresponding to wave number k in reservoirs

U: electron-electron interaction in quantum dot

V_L, V_R: coupling strength between quantum dot and reservoirs

$\hat{d}^\dagger, \hat{d}$: creation and annihilation operators in quantum dot

$\hat{a}^\dagger, \hat{a}$: creation and annihilation operators in reservoirs

$\hat{n}$: number operator $(= \hat{d}^\dagger \hat{d})$

γ: reservoir indices L, R $\quad$ σ: up-spin $(\uparrow)$ or down-spin $(\downarrow)$

Quantum Master equation:

The time evolution of the density matrix (column vector) is given by

$$\boxed{\frac{d}{dt}|\hat{\rho}(t)\rangle = \hat{K}|\hat{\rho}(t)\rangle}$$

with the following Lindbladian (or, rate matrix)

$$\hat{K} = \begin{pmatrix} -2f_-^{(1)} & f_+^{(1)} & f_+^{(1)} & 0 \\ f_-^{(1)} & -f_-^{(0)} - f_+^{(1)} & 0 & f_+^{(0)} \\ f_-^{(1)} & 0 & -f_-^{(0)} - f_+^{(1)} & f_+^{(0)} \\ 0 & f_-^{(0)} & f_-^{(0)} & -2f_+^{(0)} \end{pmatrix}$$

where

$$f_+^{(j)} := f_L^{(j)}(\mu_L, U) + f_R^{(j)}(\mu_R, U) \qquad \text{and} \qquad f_-^{(j)} = 2 - f_+^{(j)}$$

with the Fermi-Dirac distribution: $\boxed{f_\gamma^{(j)}(\mu_\gamma, U) = \frac{1}{1+e^{(\epsilon_0 + jU - \mu_\gamma)/T}}}$

Eigen values of the Lindbladian:

$$\boxed{\lambda_1 = 0, \qquad \lambda_2 = -2 + f_+^{(0)} - f_+^{(1)}, \qquad \lambda_3 = -2 - f_+^{(0)} + f_+^{(1)}, \qquad \lambda_4 = -4}$$

Steady state eigen mode: $\boxed{\lambda_1 = 0}$

Slowest relaxation mode: $\boxed{\lambda_2 = -2 + f_+^{(0)} - f_+^{(1)}}$

Steady state: $$|\hat{\rho}^{ss}(f_+^{(0)}, f_+^{(1)})\rangle\rangle = \frac{1}{4 + 2(f_+^{(0)} - f_+^{(1)})} \begin{pmatrix} f_+^{(0)} f_+^{(1)} \\ f_+^{(0)}(2 - f_+^{(1)}) \\ f_+^{(0)}(2 - f_+^{(1)}) \\ (2 - f_+^{(0)})(2 - f_+^{(1)}) \end{pmatrix}$$

singly occupied states occur with equal probabilities

I: $\mu_L \neq \mu_R$, II $\mu_L = \mu_R = \mu_R^I$

initial condition I before quench:

$$\boxed{\left|\hat{\rho}^I\left(0, f_+^{(0),I}, f_+^{(1),I}\right)\right\rangle}$$

initial condition II before quench:

$$\boxed{\left|\hat{\rho}^{II}\left(0, f_+^{(0),II}, f_+^{(1),II}\right)\right\rangle}$$

System I is a non-equilibrium and system II is an equilibrium one.

What we calculate:

Difference in density matrix elements starting from two initial conditions:

$$\Delta\rho_\alpha(t) = \rho_\alpha^I(t) - \rho_\alpha^{II}(t)$$

$$\text{where } \alpha = d, \uparrow, \downarrow, e$$

Order parameter:

$$\tilde{t}_\alpha = t_\alpha \text{ if } 0 < t_\alpha < \infty, \\ = 0 \text{ if no finite } t_\alpha \text{ exists}$$

If there exists a finite t_α $(0 < t_\alpha < \infty)$

such that $\Delta\rho_\alpha(t_\alpha) = 0$,

then the α-th component of density matrix

exhibits Mpemba-like effect

Difference in von-Neumann entropy starting from two initial conditions:

$$\Delta S_{vN} = \left(-\sum_{\alpha=1}^{4} \rho_\alpha^I(t) \ln \rho_\alpha^I(t)\right) - \left(-\sum_{\alpha=1}^{4} \rho_\alpha^{II}(t) \ln \rho_\alpha^{II}(t)\right)$$

Results

- a_2 is always zero.
- The difference of matrix elements

$$\Delta\rho_\alpha(t) = \rho_\alpha^{I}(t) - \rho_\alpha^{II}(t), \qquad \alpha = 1, 2, 3, 4$$

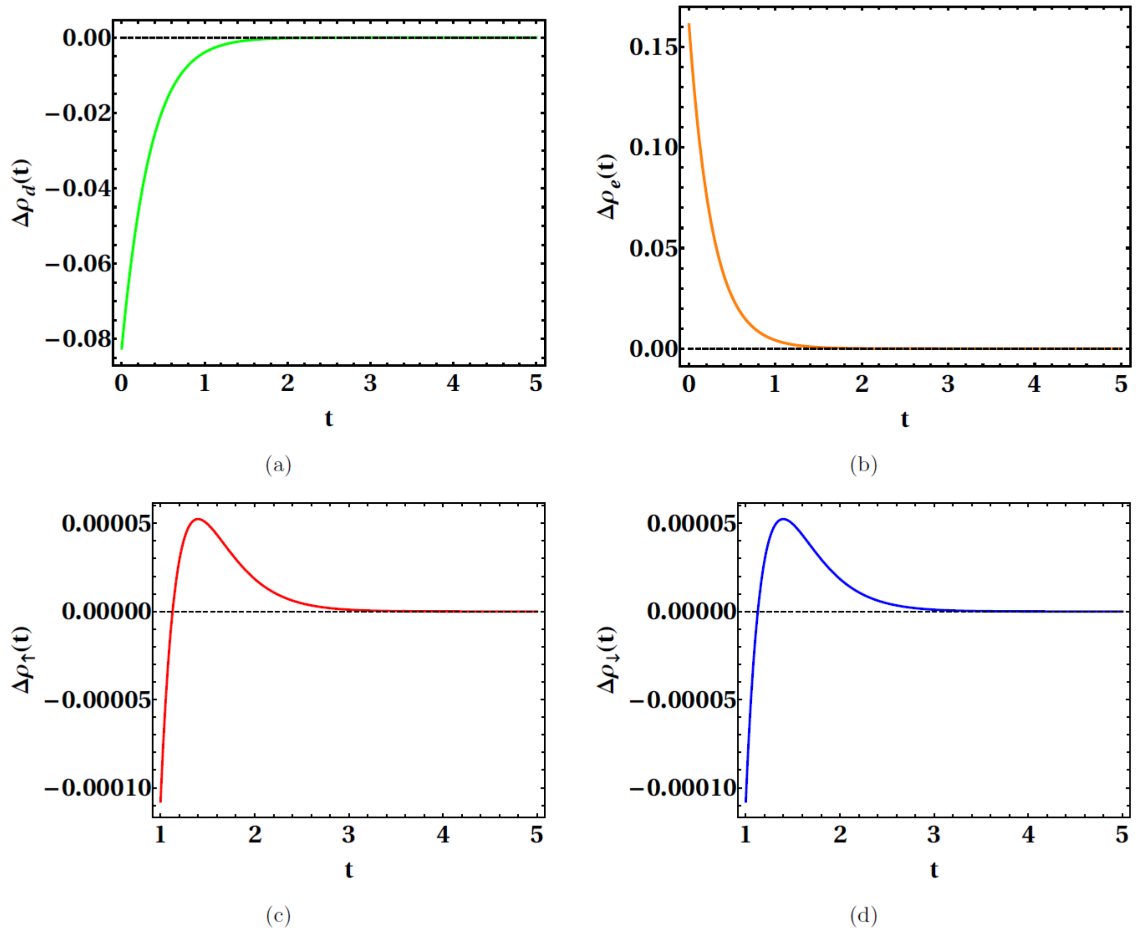
$$\Delta\rho_\alpha(t) = e^{\lambda_3 t}\hat{R}_{\alpha,4}\Delta a_4 \left[\frac{\hat{R}_{\alpha,3}}{\hat{R}_{\alpha,4}}\frac{\Delta a_3}{\Delta a_4} + e^{-(\lambda_3-\lambda_4)t}\right]$$

- Necessary condition for QMPE

$$S_\alpha < 0 \quad \& \quad |S_\alpha| < 1,$$

$$\text{where} \qquad S_\alpha = \frac{\hat{R}_{\alpha,3}}{\hat{R}_{\alpha,4}}\frac{\Delta a_3}{\Delta a_4}.$$

An example of set of time evolutions



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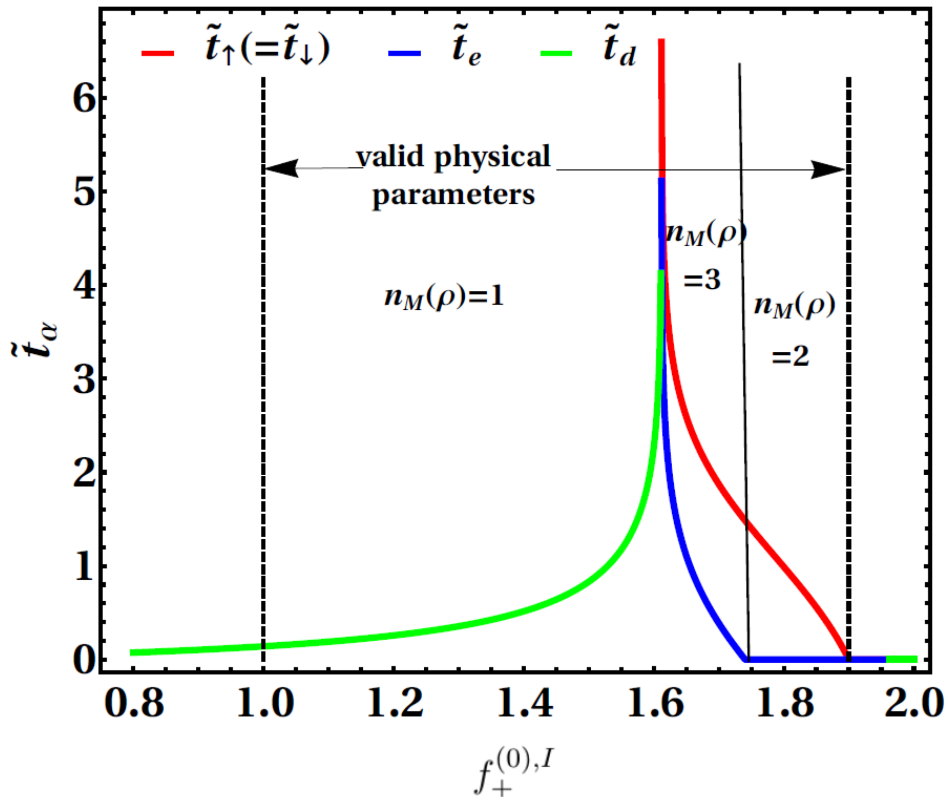
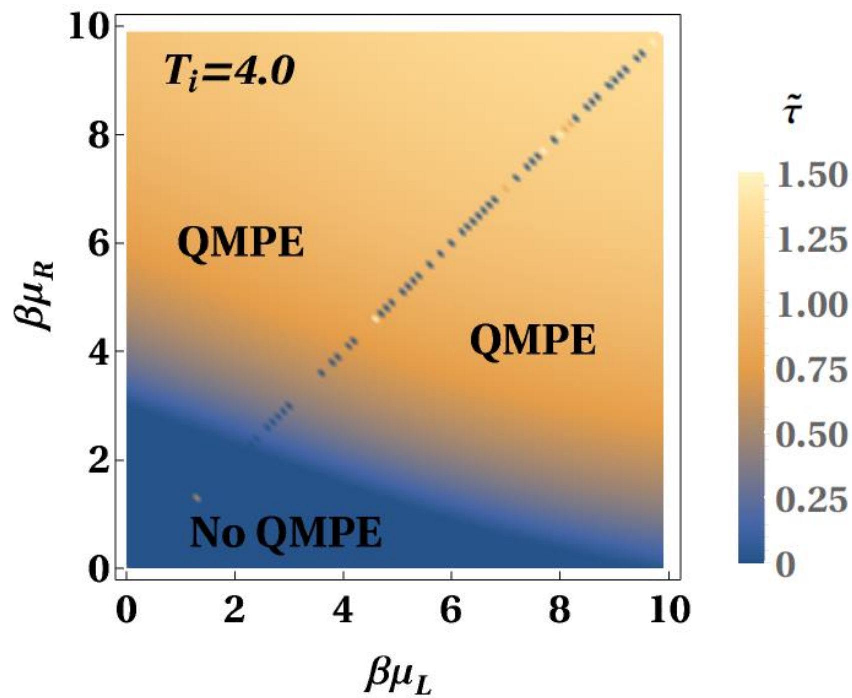
$$\frac{\mu}{T} = 5.0, \quad \frac{\epsilon_0}{T} = 2.80, \quad \frac{U}{T} = 3.58, \quad \frac{T_i}{T} = 1.18, \quad \frac{\mu^{II}}{T_i} = 4.32.$$

Mpemba effect

- The onset time exhibits the inversion:

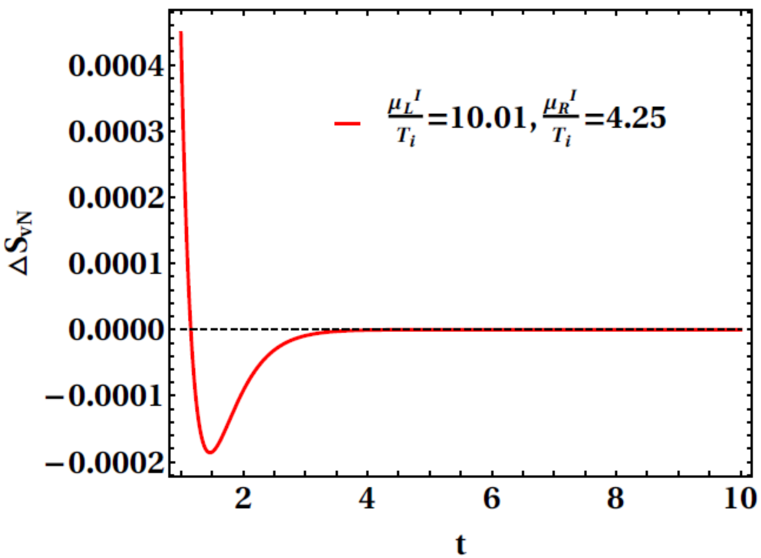


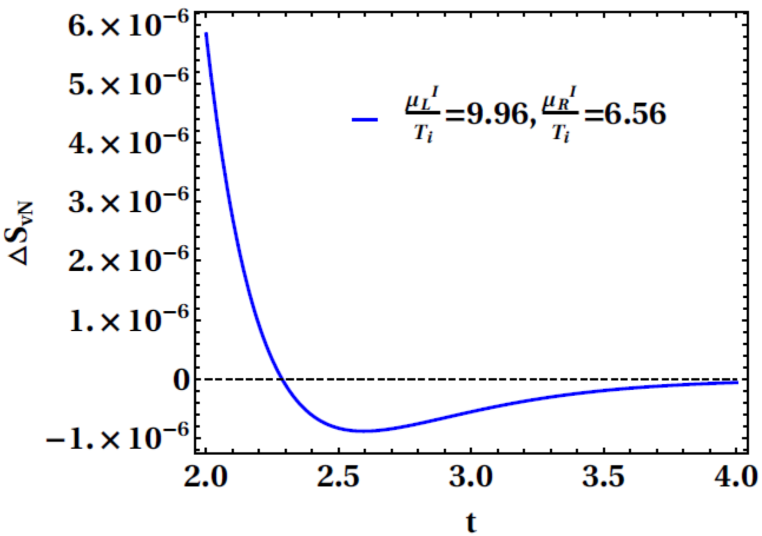


$n_M(\rho)$: number of density matrix elements that change sign or show Mpemba – like effect for a given set of parameter values.

von Neuman entropy

- KL divergence does not exhibit any Mpemba-like behavior within our calculation.
- Instead, von Neumann exhibits it.

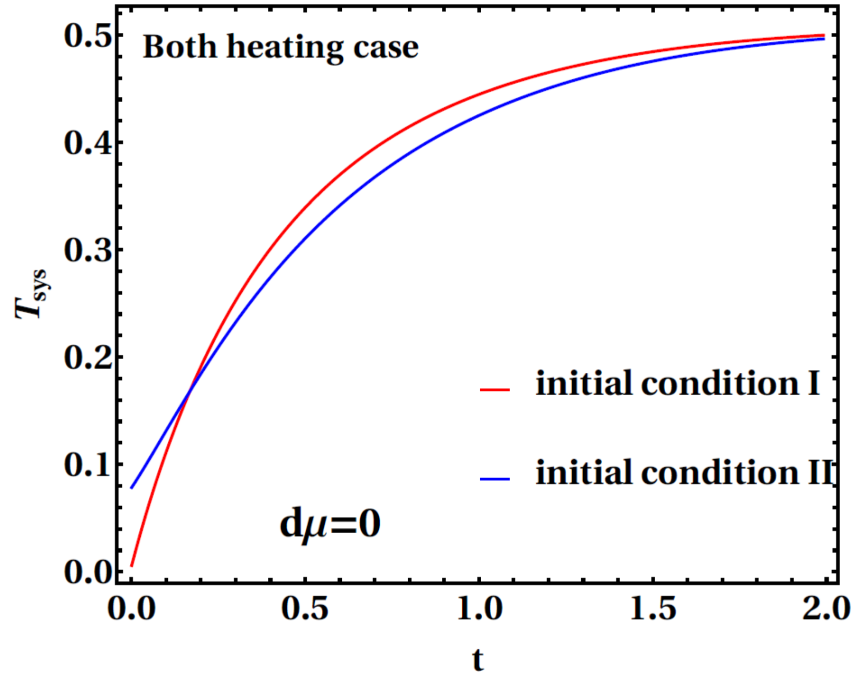




Temperature

- The temperature of the system is defined by a thermodynamic relation.
- QMPE can be observed for both heating cases (inverse MPE).

$$1/T_{sys} := \partial S_{vN}/\partial E_{sys}$$



Summary of quantum Mpemba effect

- We have demonstrated the existence of Mpemba-like phenomena after a sudden change of system.
- Such effects can be observed in the density matrix elements, von Neumann , energy and temperature.
- Inverse Mpemba (heating) effect is easier than the normal Mpemba effect after the quench.
- Mpemba effect may be useful to speed-up to get a desired state.

Contents

Mpemba effect in a double well potential

- Experiment by Kumar & Bechhoeffer
- Successive papers by Chetrite, Kumar & Bechhoeffer (Frontiers in Physics) and Walker & Vucelja (J. Stat. Mech.) only emphasized the strong Mpemba condition of a_2=0.
  - This is contrast to the quantum Mpemba case.
- The explicit calculation by us seems to be different from the previous scenario.

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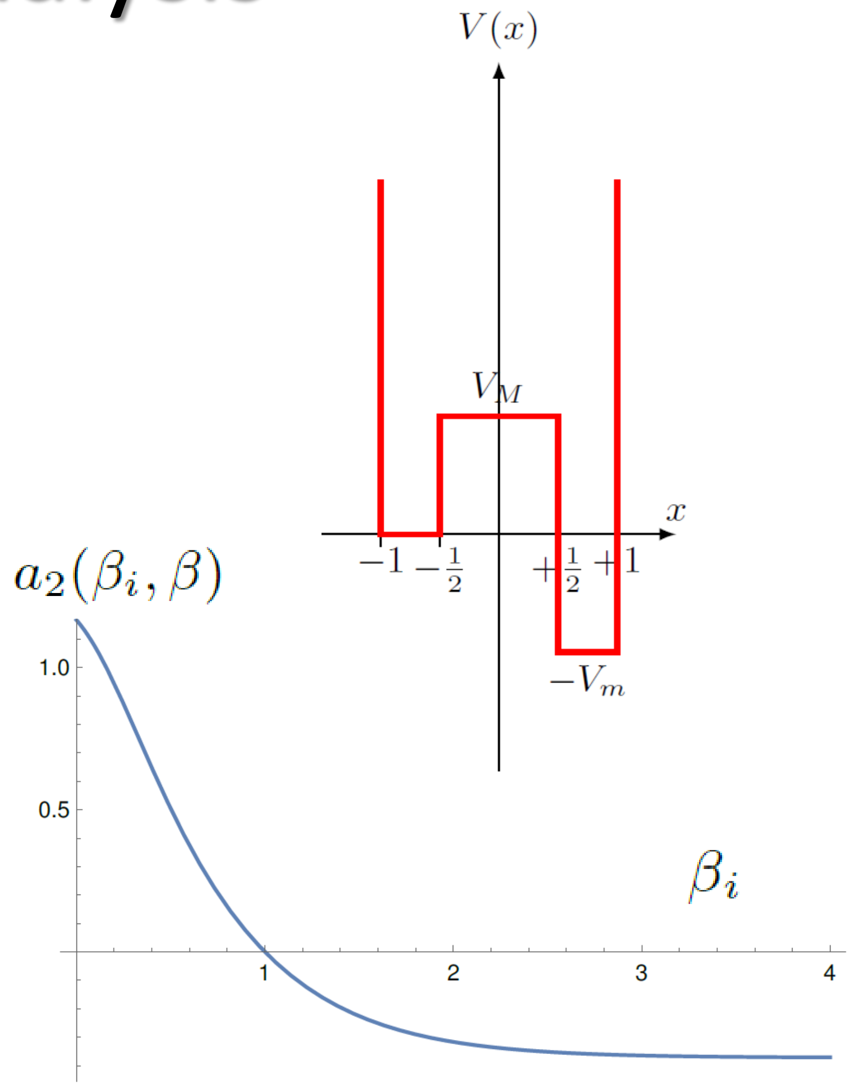
Our analysis

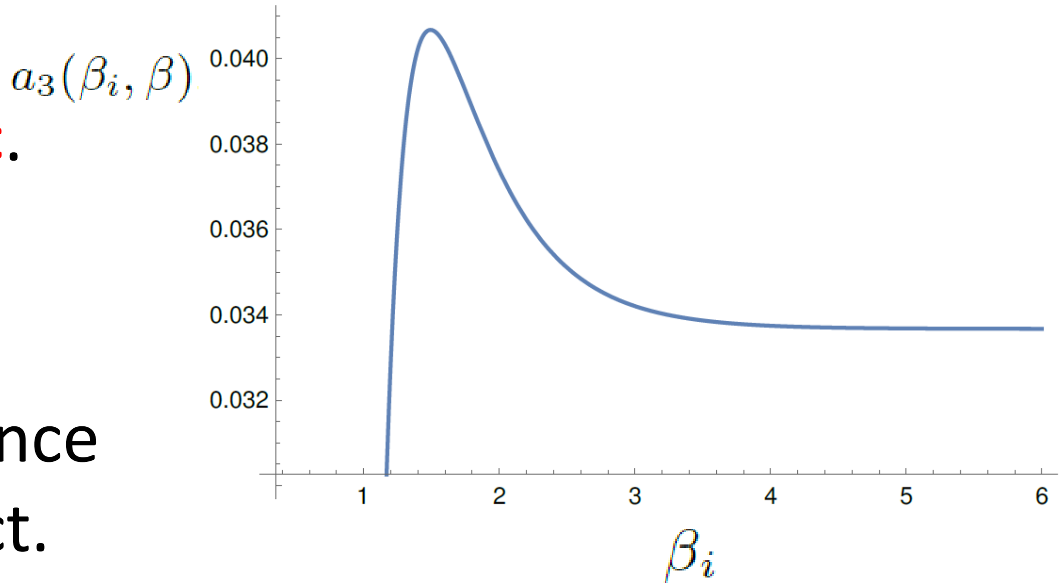
- We solve the Fokker-Planck equation describing a particle confined in a square-box double well potential.
- We found a_2 is a monotonic function of the initial temperature.



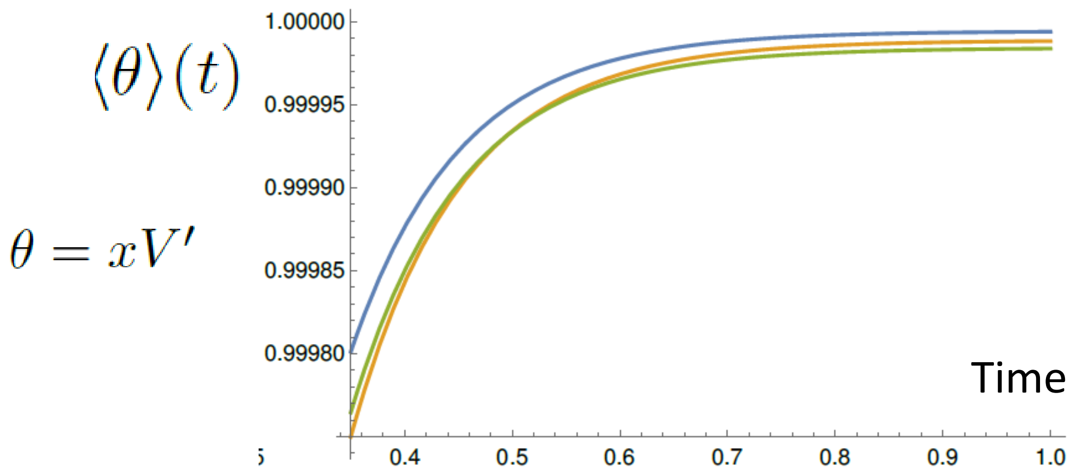
β is the inverse temperature for t>0.

Typical results

But a_3 is non-monotonic.



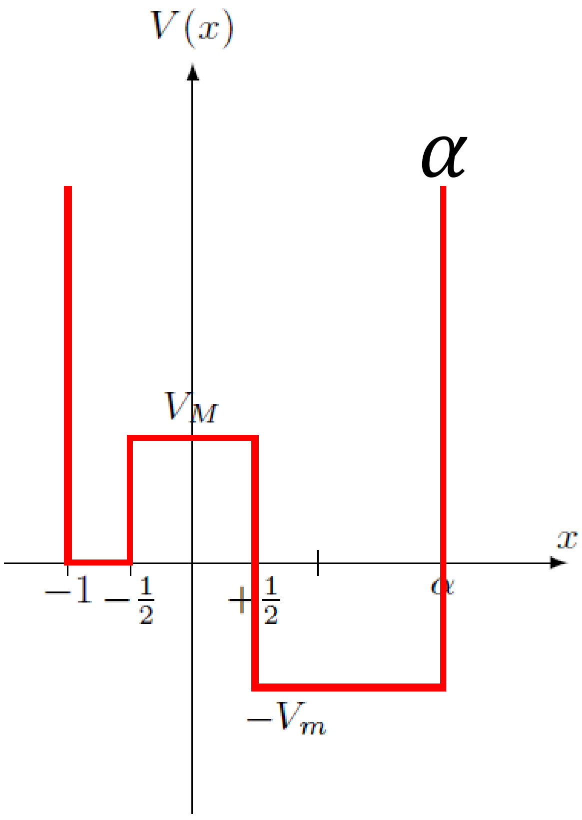
We also found the existence of inverse-Mpemba effect.



β is the inverse temperature for t>0.

What happens when we control α?

- We can modify the potential shape as the right figure.
- α is a new parameter.
- If $\alpha < 1$, we observe a non-monotonous a_2.
- Now the analysis is in progress.



Potential for $\alpha > 1$

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Summary

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- From couple of examples the condition a_2=0 is not always important.
- In general, we cannot understand Mpemba effect only based on the slowest mode.
  – The second & third slowest modes play major roles.
- To control the viscous heating is a heuristic example to observe Mpemba effect.

See you soon.

This is one of satellite meeting of Statphys 28.
There will be a long-term workshop in July of 2024.