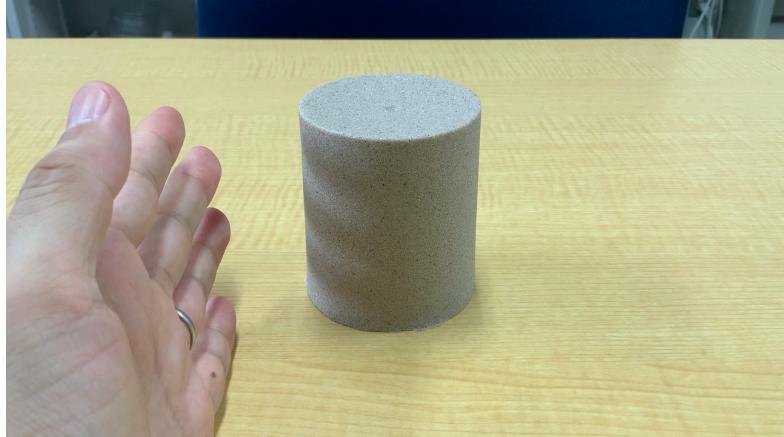


An exact expression of three-body system for the nonlinear response of frictional granular materials

Michio Otsuki (Osaka Univ.)
Hisao Hayakawa (Kyoto Univ.)

arXiv:2211.03294
Soft Matter (Advanced Article)

Rheology of granular materials

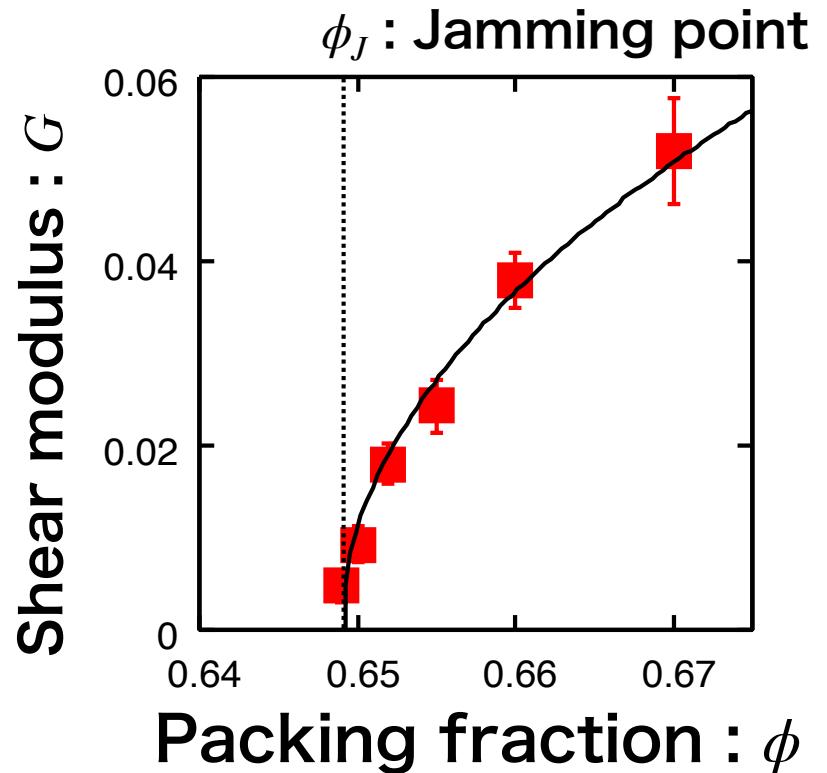


Dense granular materials have rigidity.

Shear modulus: G Packing fraction: ϕ

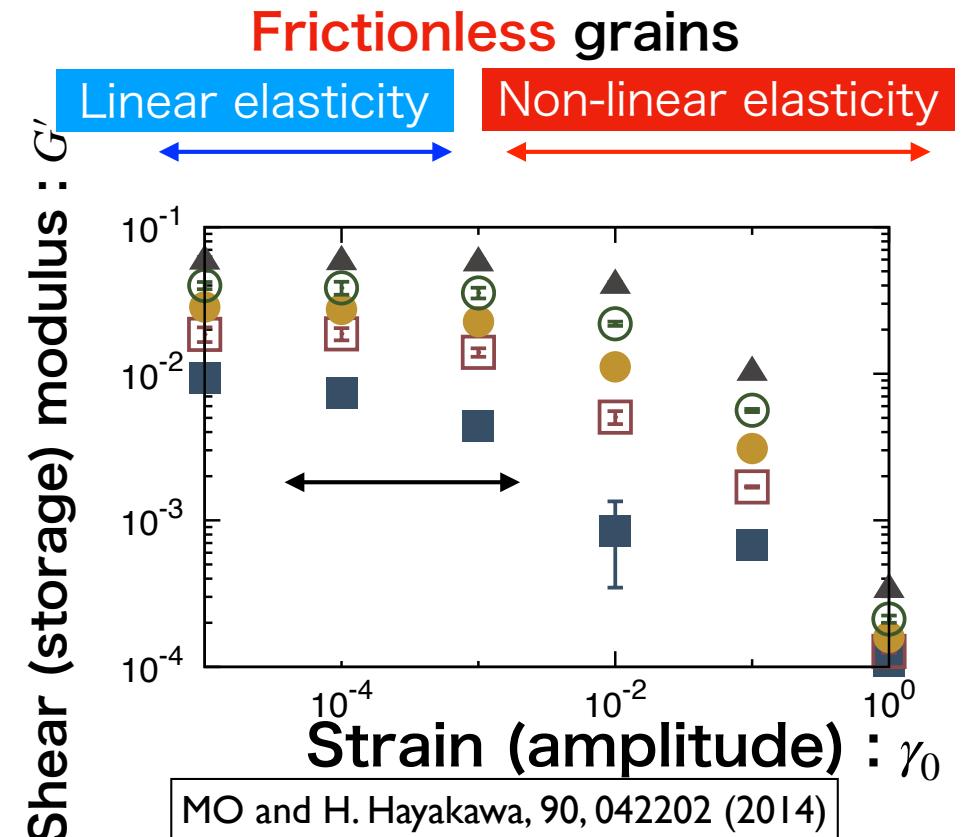
$$G > 0 \text{ for } \phi > \phi_J$$

Jamming point: ϕ_J



- Granular materials: Collection of solid particles (powder, sand, etc.).
- Above the jamming point ϕ_J , elastic response with $G > 0$ is observed.

Non-linear response?



Frictionless grains becomes “soft” as the strain increases.

Non-linear response of frictional granular materials?

Softening in frictionless grains:

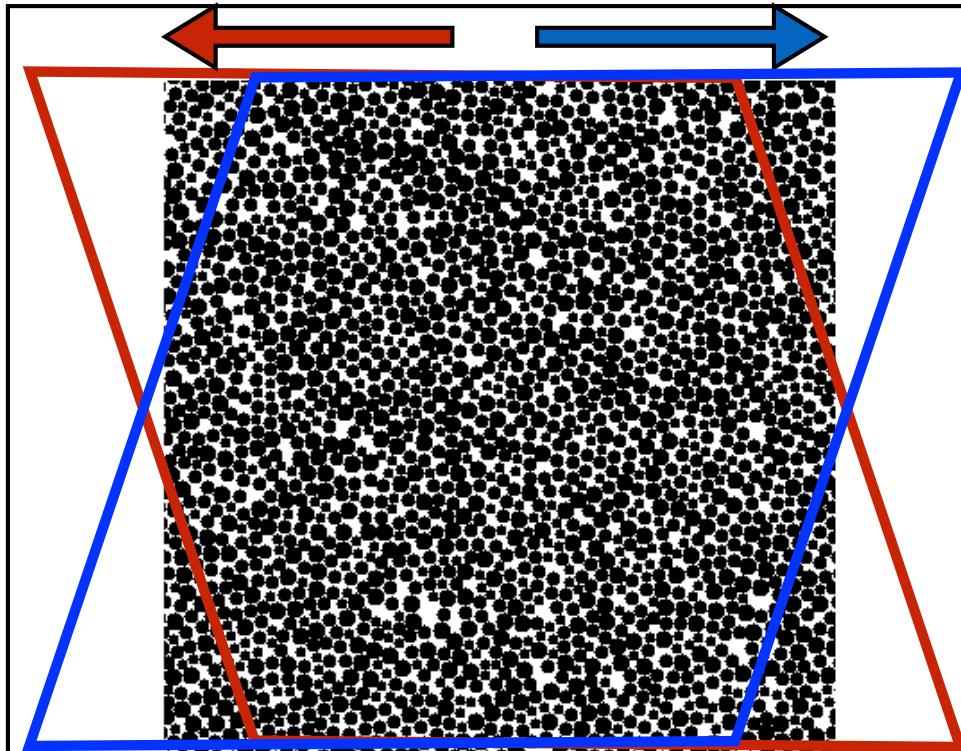
J. Boschan, et. al., (2016), S. Dagois-Bohy, et. al., (2017),

T. Kawasaki and K. Miyazaki, (2020),

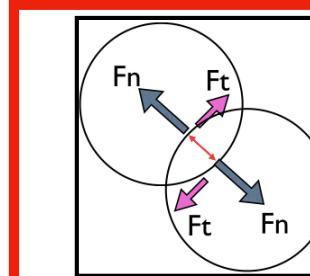
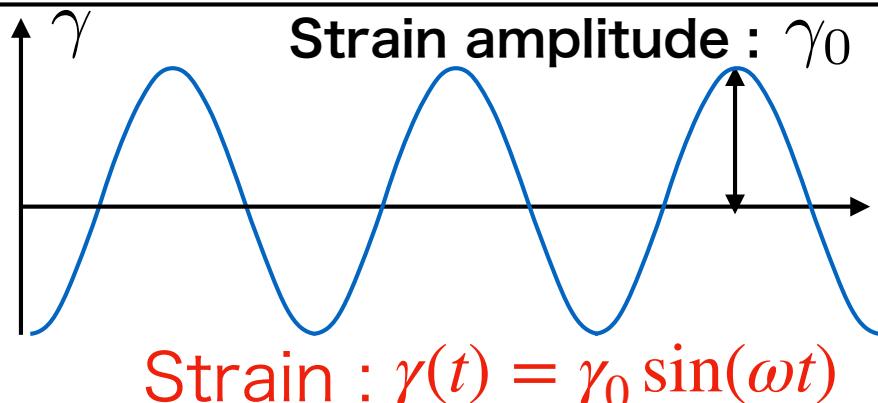
M. Otsuki and H. Hayakawa, Phys. Rev. Lett. 128, 208002 (2022).

Previous studies on G' and G'' : Model

MO and H. Hayakawa, Phys. Rev. E 95, 062902 (2017)



Oscillatory shear



F_n : Normal repulsive force

F_t : Tangential friction

Coulomb's law $F_t \leq \mu F_n$

μ : Friction coefficient

SLLOD eq. : Newton's second law with shear

$$\frac{d\mathbf{r}_i}{dt} = \frac{\mathbf{p}_i}{m_i} + \dot{\gamma}(t)r_{i,y}\hat{\mathbf{x}},$$

$$\frac{d\mathbf{p}_i}{dt} = \sum_{j \neq i} \mathbf{F}_{ij} - \dot{\gamma}(t)p_{i,y}\hat{\mathbf{x}},$$

Complex shear modulus:

Shear stress: $\sigma(t)$

Storage modulus: elasticity

$$G' = \frac{\omega}{\pi} \int_0^{2\pi/\omega} d\theta \frac{\sigma(t)\sin(\omega t)}{\gamma_0}$$

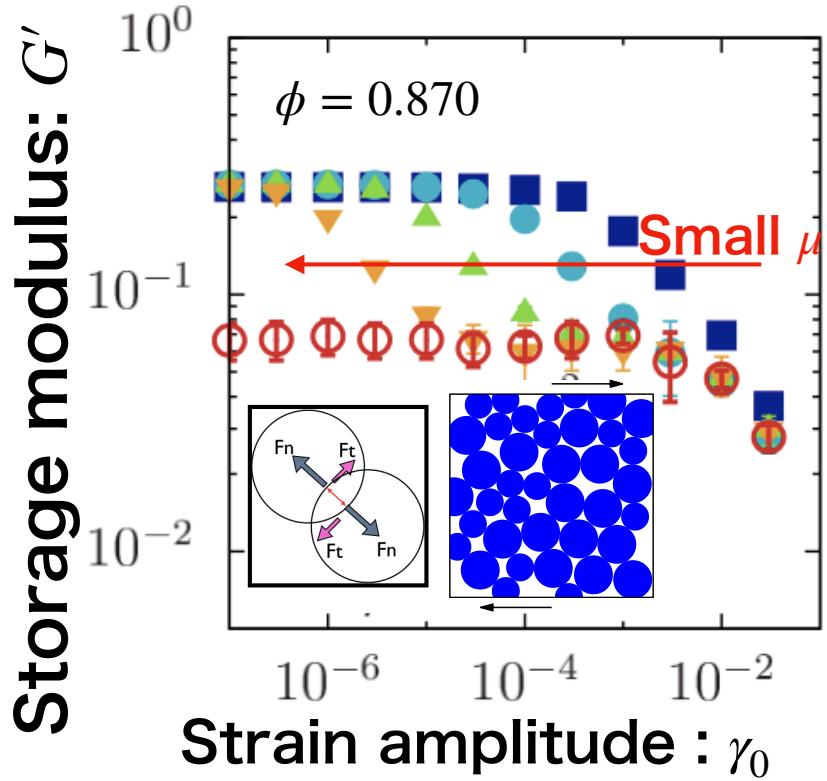
$G' = G$ for small γ_0

Loss modulus: dissipation

$$G'' = \frac{\omega}{\pi} \int_0^{2\pi/\omega} d\theta \frac{\sigma(t)\cos(\omega t)}{\gamma_0}$$

G' and G'' : Numerical results

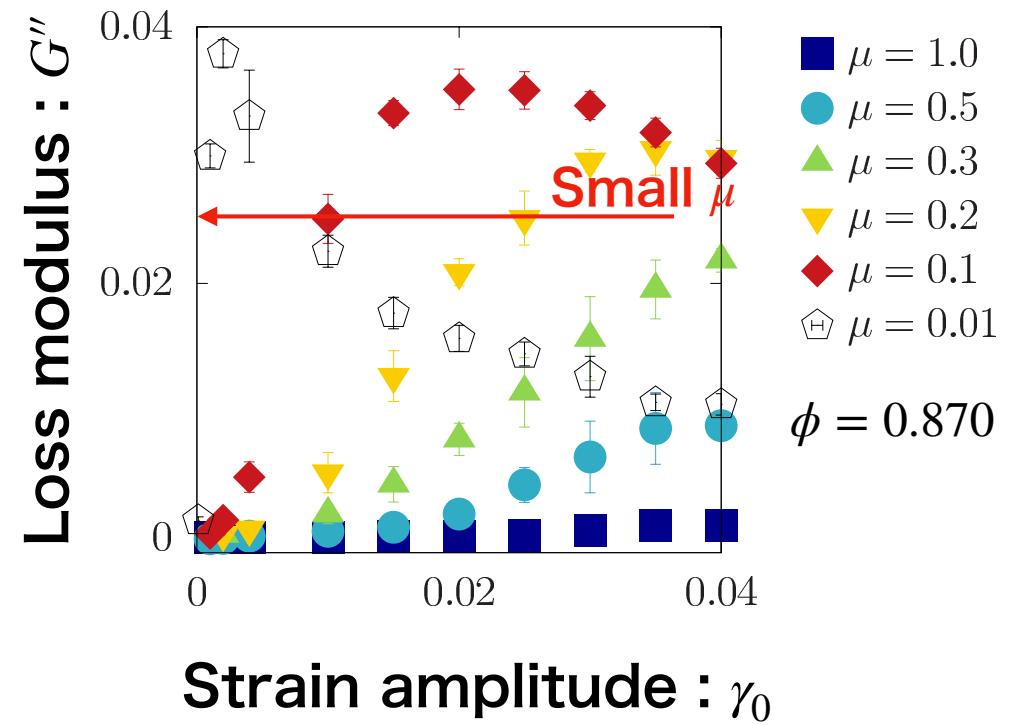
μ : Friction coefficient



MO and H. Hayakawa, Phys. Rev. E 95, 062902 (2017)

- G' decreases as γ_0 increases.
- The critical strain depends on μ .
- G' exhibits two step relaxation.

The number of grains : $N \geq 1000$



MO and H. Hayakawa, Eur. Phys. J. E 44, 106 (2021)

- G'' becomes finite as γ_0 increases.
- The critical strain depends on μ .
- G'' has a peak for small μ .

Theoretical analysis is difficult.

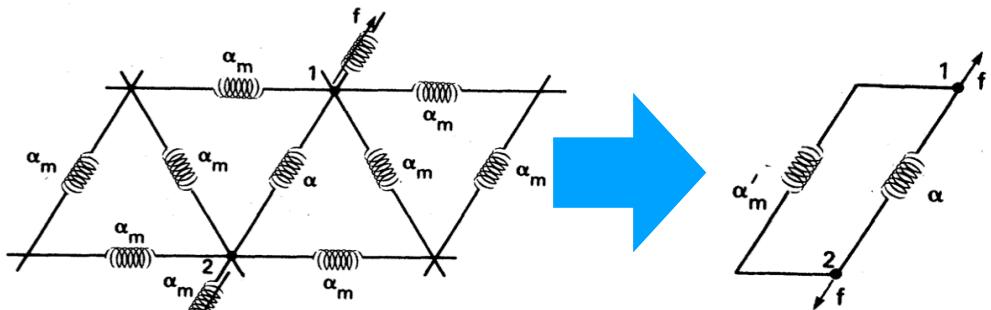
Many body systems, friction, etc.

Approach: simple effective model

Example: Mean field theory for Ising model

Elastic response of amorphous solid

Effective medium theory



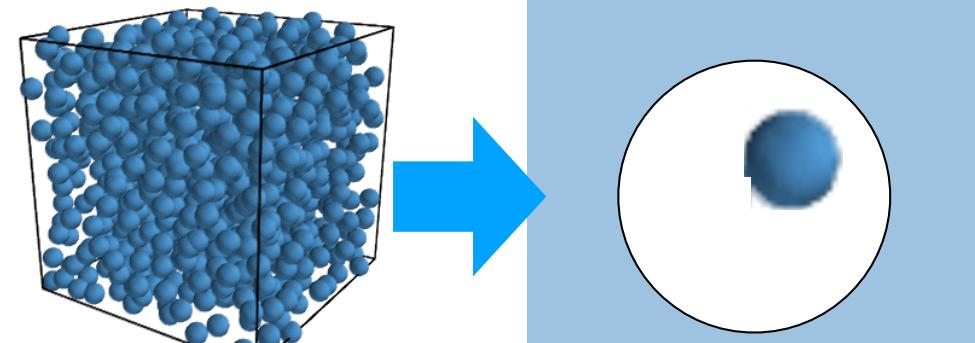
Random spring network

Two nodes with effective spring

Feng, Thorpe, and Garboczi, Phys. Rev. B 31, 276 (1995)

Gas-liquid transition

Cell model



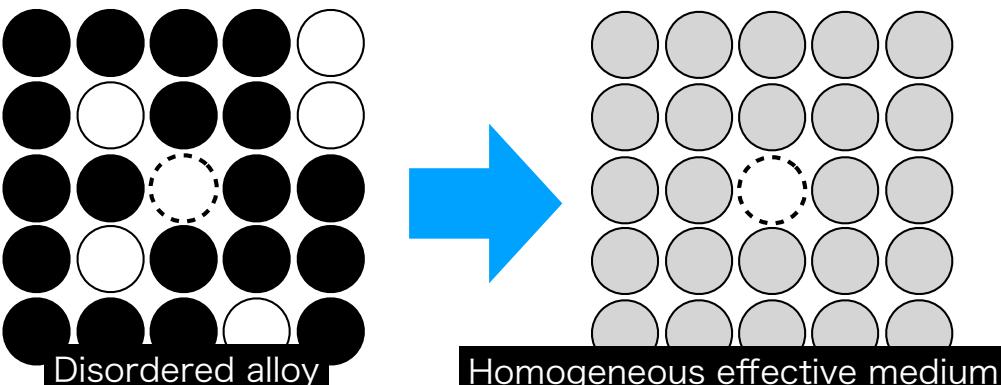
Many molecules

One molecule in an effective media

Lennard-Jones, Devonshire, Proc. R. Soc. Lond A 63, 53 (1937)

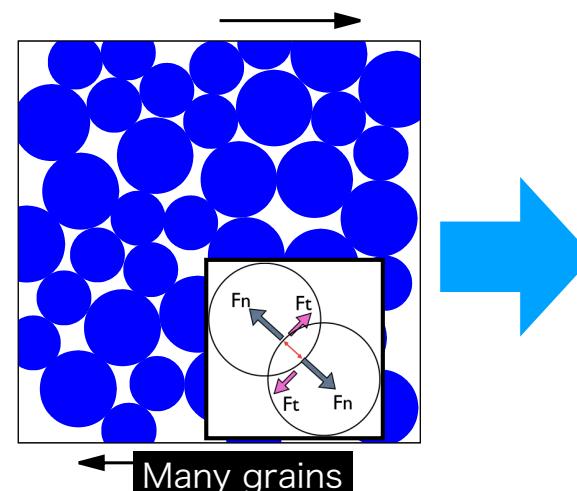
Electric band structure for disordered solid

Coherent potential approximation



Yonezawa, Morigaki, Prog. Theor. Supple. 53, I (1973)

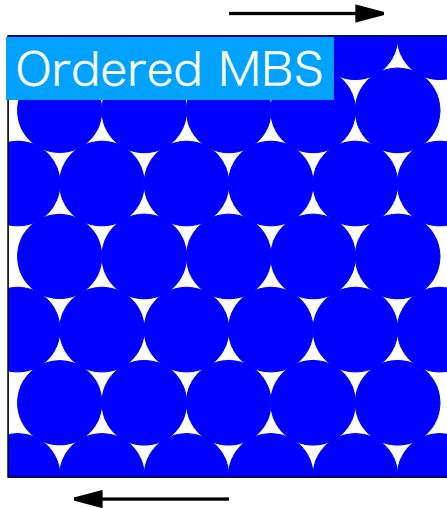
Response of frictional grains under shear



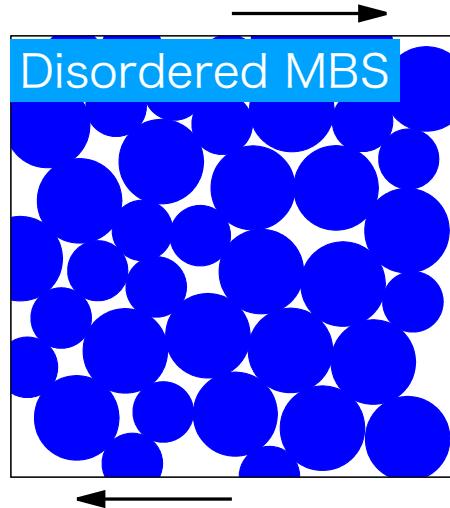
Many grains

Three body system: Dynamics

Many body system (MBS)



SLLOD eq. : Newton's second law with shear



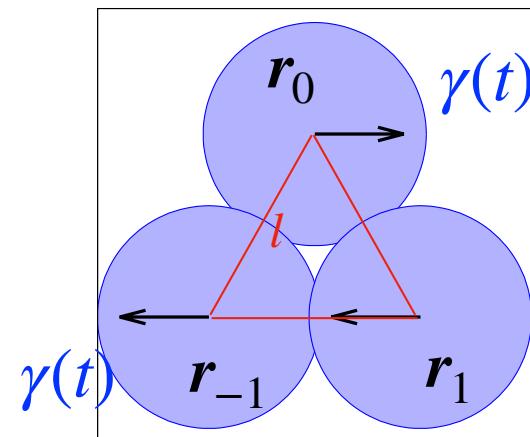
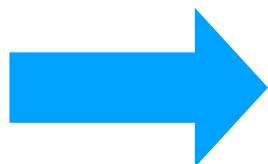
$$\frac{d\mathbf{r}_i}{dt} = \frac{\mathbf{p}_i}{m_i} + \dot{\gamma}(t)r_{i,y}\hat{\mathbf{x}},$$

$$\frac{d\mathbf{p}_i}{dt} = \sum_{j \neq i} \mathbf{F}_{ij} - \dot{\gamma}(t)p_{i,y}\hat{\mathbf{x}},$$

Shear strain: $\gamma(t) = \gamma_0 \sin(\omega t)$

γ_0 : Amplitude, ω : Frequency

Three body system (TBS):



$l = d(1 - \epsilon)$: Initial distance

d : Diameter of grains

ϵ : Compressive strain $\propto \phi - \phi_J$

$$\mathbf{r}_0 = (\sqrt{3}\gamma(t)l/4, \sqrt{3}l/4)$$

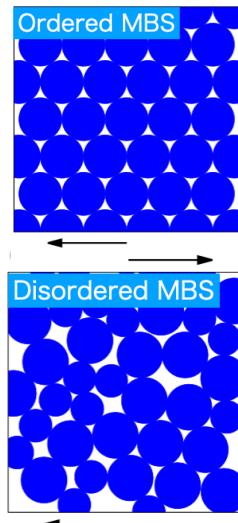
$$\mathbf{r}_{\pm 1} = (\pm l/2 - \sqrt{3}\gamma(t)l/4, -\sqrt{3}l/4)$$

Shear strain: $\gamma(t) = \gamma_0 \sin(\omega t)$

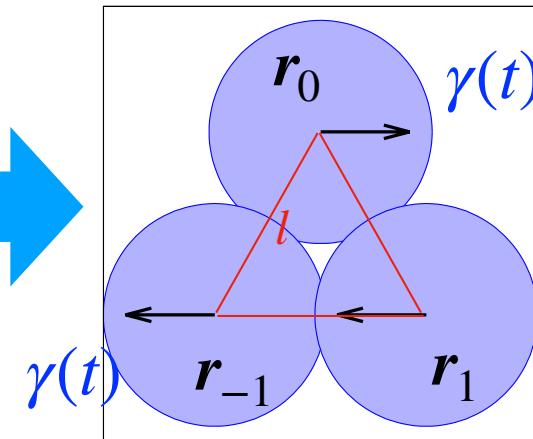
γ_0 : Amplitude, ω : Frequency

Three body system: Interaction force

Many body system



Three body system:



$l = d(1 - \epsilon)$: Initial distance

ϵ : Compressive strain $\propto \phi - \phi_J$

$$\mathbf{r}_0 = (\sqrt{3}\gamma(t)l/4, \sqrt{3}l/4)$$

$$\mathbf{r}_{\pm 1} = (\pm l/2 - \sqrt{3}\gamma(t)l/4, -\sqrt{3}l/4)$$

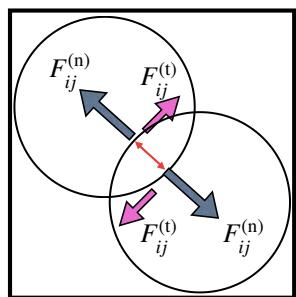
Shear strain: $\gamma(t) = \gamma_0 \sin(\omega t)$

γ_0 : Amplitude, ω : Frequency

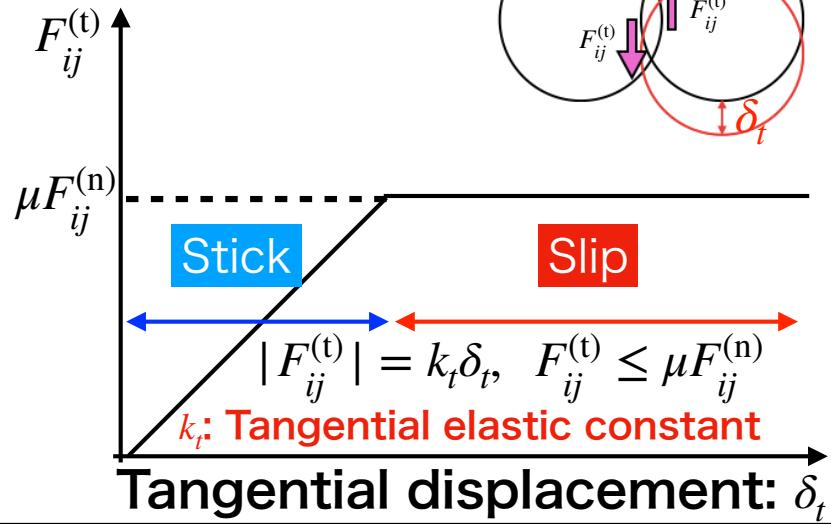
Particle interaction

The force is the same as that for usual simulations.

$F_{ij}^{(n)}$: Normal repulsive force



Tangential friction:

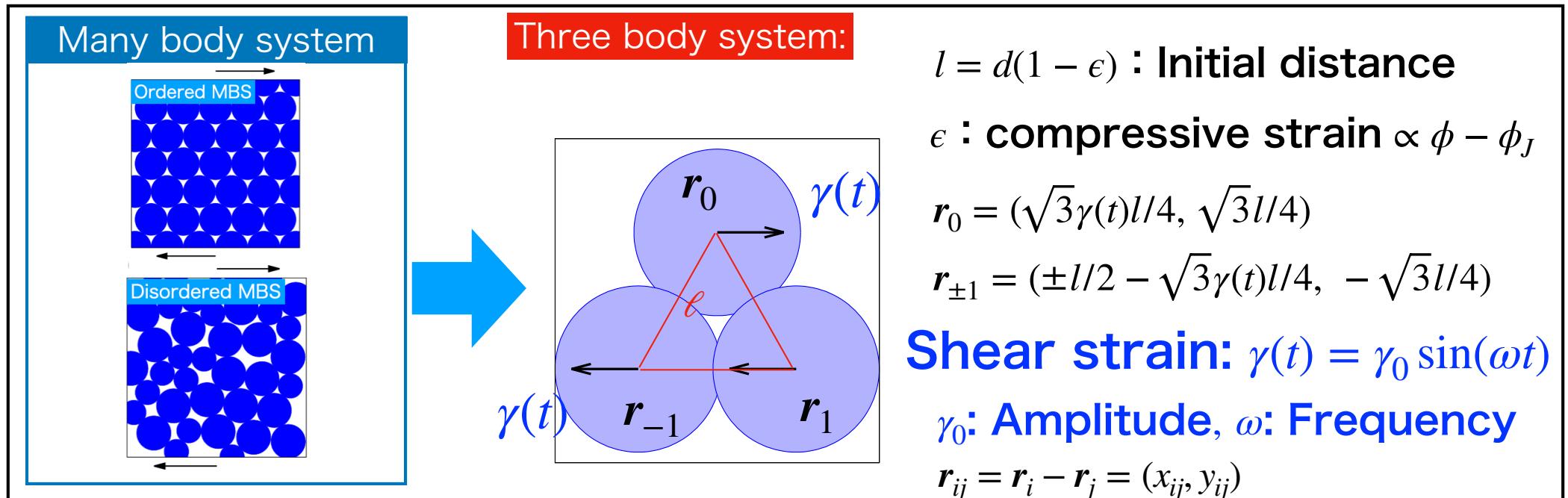


$F_{ij}^{(t)}$: Tangential friction

Coulomb's law $F_{ij}^{(t)} \leq \mu F_{ij}^{(n)}$

μ : Friction coefficient

Shear modulus in three body system



Interaction force: $F_{ij}^{(n)}, F_{ij}^{(t)}$

Stress: $\sigma = -\frac{1}{A} \sum_i \sum_j \left(\frac{x_{ij}y_{ij}}{r_{ij}} F_{ij}^{(n)} + \frac{x_{ij}^2 - y_{ij}^2}{r_{ij}} F_{ij}^{(t)} \right)$

Pressure: $P = \frac{1}{A} \sum_i \sum_j r_{ij} F_{ij}^{(n)}$

A : the area of the system

Storage modulus (Elasticity)

$$G' = \frac{\omega}{\pi} \int_0^{2\pi/\omega} d\theta \frac{\sigma(t)\sin(\omega t)}{\gamma_0}$$

Loss modulus (Dissipation)

$$G'' = \frac{\omega}{\pi} \int_0^{2\pi/\omega} d\theta \frac{\sigma(t)\cos(\omega t)}{\gamma_0}$$

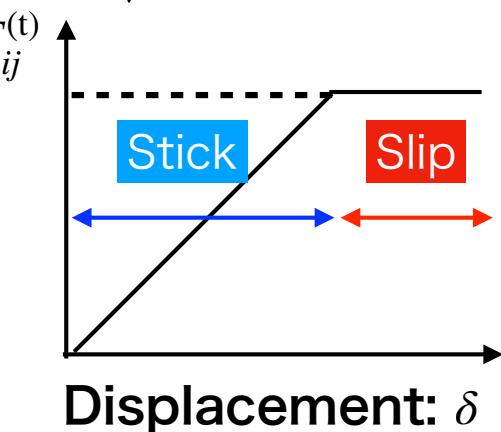
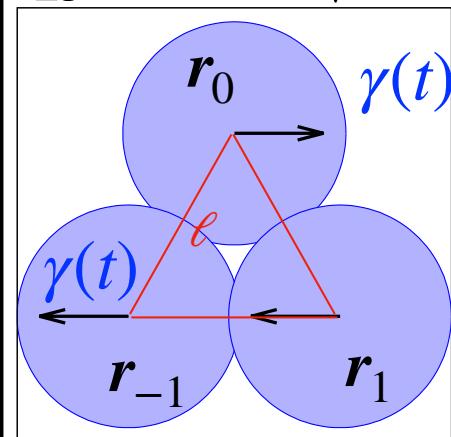
Analytical solution: stress σ

Assumption: $\gamma_0 \ll 1$

Three body system:

$$\mathbf{r}_0 = (\sqrt{3}\gamma(t)l/4, \sqrt{3}l/4)$$

$$\mathbf{r}_{\pm 1} = (\pm l/2 - \sqrt{3}\gamma(t)l/4, -\sqrt{3}l/4)$$



Transition to slip state when γ becomes large

Displacement for slip : $\delta_c = \mu F_{ij}^{(n)}/k_t$

Pressure : $P \sim F_{ij}^{(n)}/d$

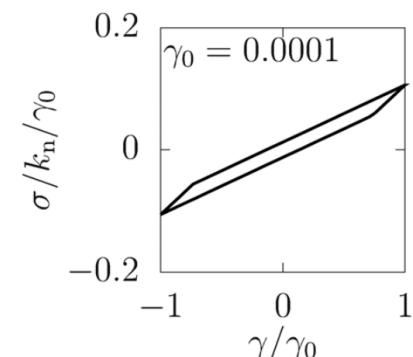
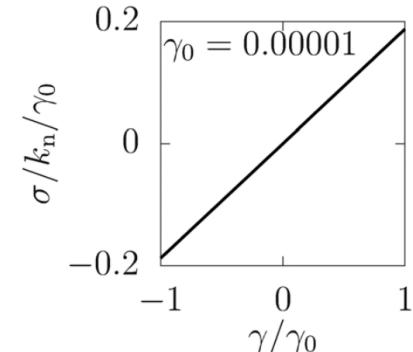
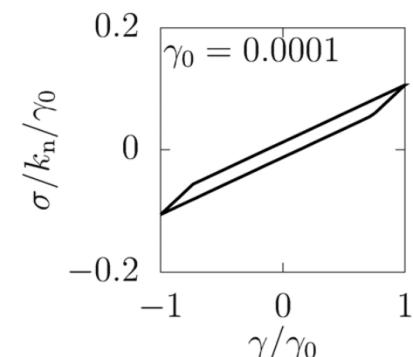
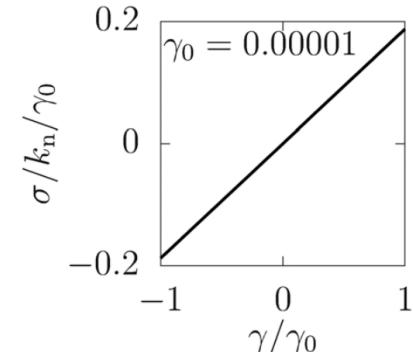
Normal force: $F_{ij}^{(n)}$, Diameter of grain: d

Critical strain : $\gamma_c \sim \delta_c/d \sim \mu P/k_t$

Analytical solution for γ_c

$$\gamma_c(\mu) = \frac{4}{3\sqrt{3}} \frac{\mu P}{k_t}$$

Analytical solution for σ

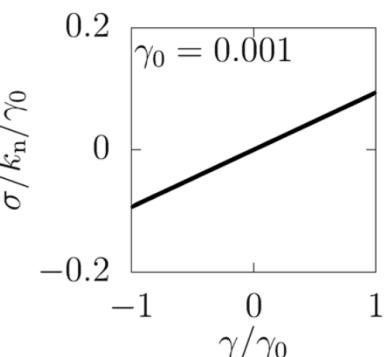
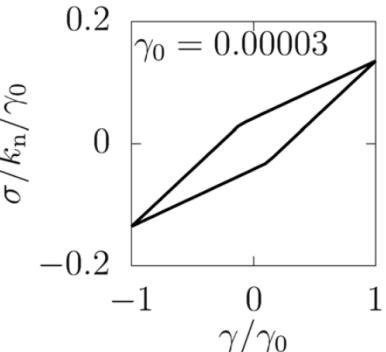


$\gamma_0 \leq \gamma_c(\mu)$: No slip

Linear elasticity

$$\sigma(t) = \frac{\sqrt{3}(k_n + k_t)}{4} \gamma(t)$$

k_n : elastic constant for $F_{ij}^{(n)}$



$\gamma_0 > \gamma_c(\mu)$: Slip

Elasto-plasticity

Plastic region:

$$\sigma(t) \simeq \sqrt{3}k_n \gamma(t)/4$$

→ Calculation of G' and G''

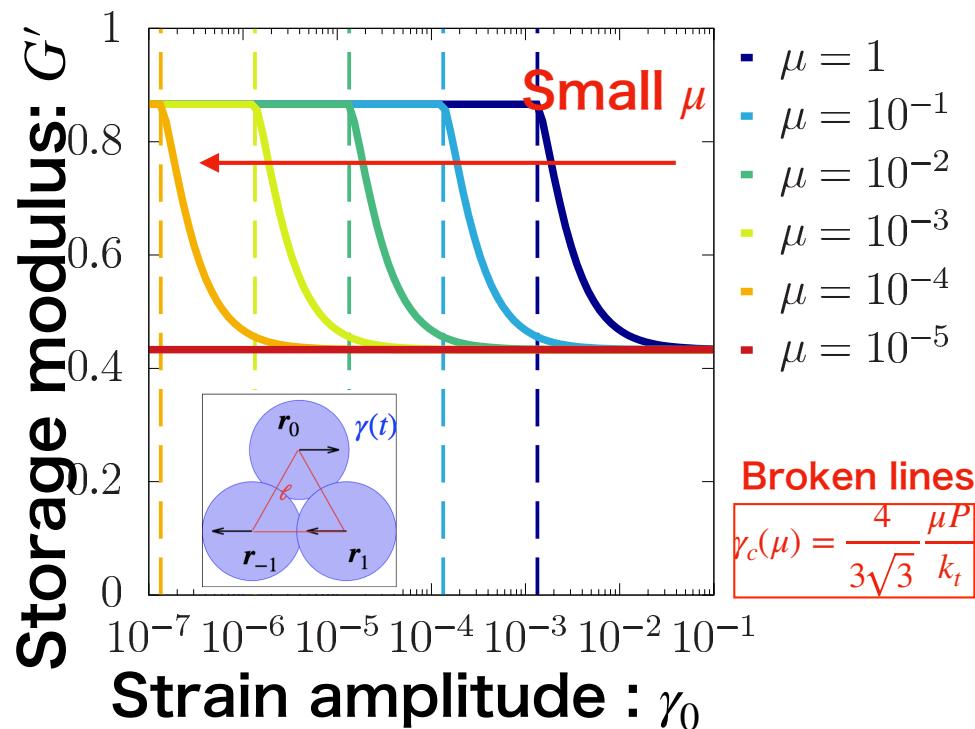
Analytical solution: G' and G''

Storage modulus

$$\epsilon = 0.001$$

$$G' = \begin{cases} \frac{\sqrt{3}(k_n + k_t)}{4}, & \gamma_0 < \gamma_c(\mu) \\ \frac{\sqrt{3}}{4} \left\{ k_n + \frac{k_t}{\pi} (\Theta(\gamma_0) - \sin \Theta(\gamma_0) \cos \Theta(\gamma_0)) \right\}, & \gamma_c(\mu) \leq \gamma_0 \end{cases}$$

$$\Theta(\gamma_0) = \cos^{-1} (1 - 2\gamma_c(\mu)/\gamma_0)$$

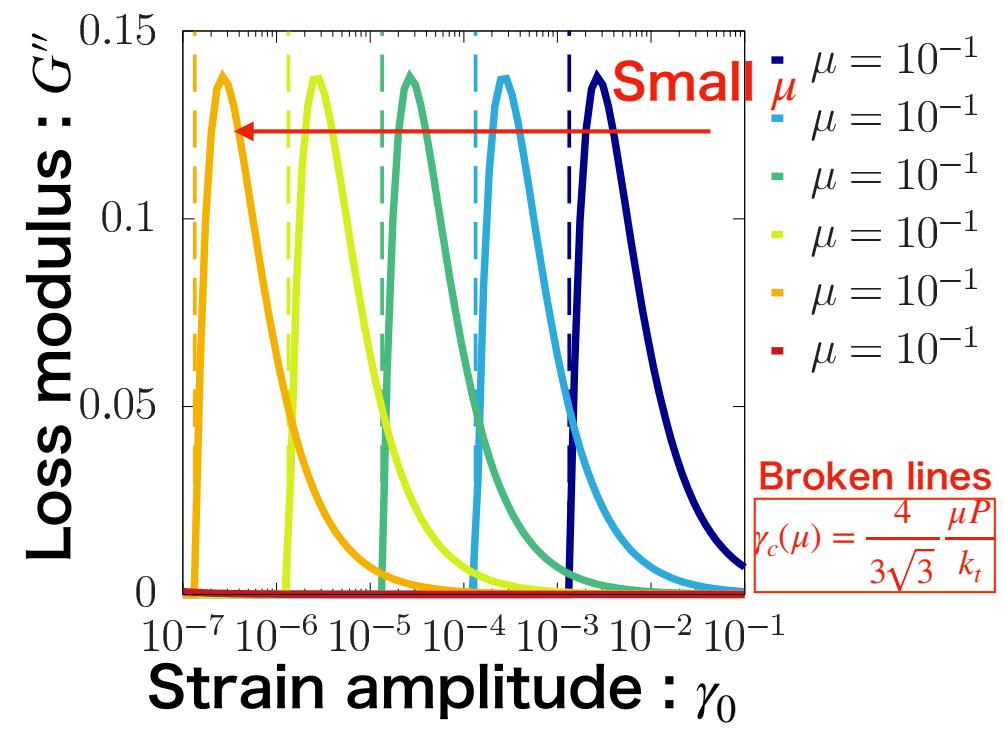


Loss modulus

$$\epsilon = 0.001$$

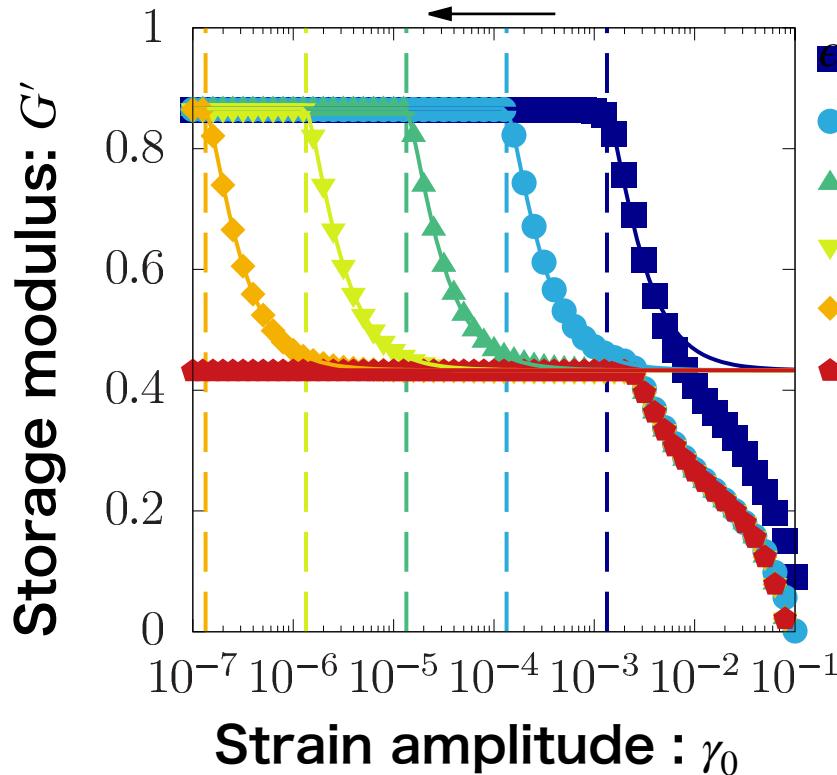
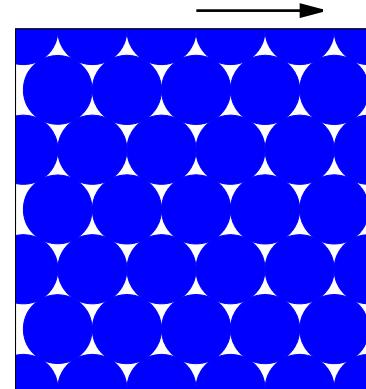
$$G'' = \begin{cases} 0, & \gamma_0 < \gamma_c(\mu) \\ \frac{\sqrt{3}k_t}{4\pi} \{1 - \cos^2 \Theta(\gamma_0)\}, & \gamma_c(\mu) \leq \gamma_0 \end{cases}$$

$$\Theta(\gamma_0) = \cos^{-1} (1 - 2\gamma_c(\mu)/\gamma_0)$$

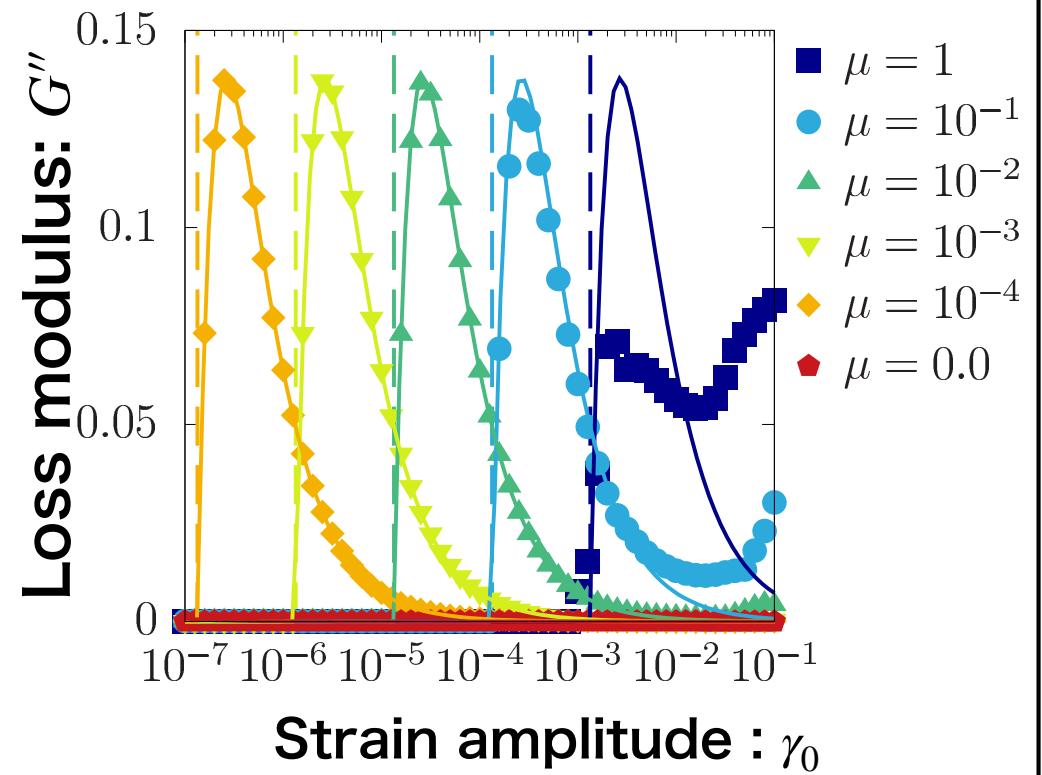


Comparison with ordered MBS: G' and G''

Ordered MBS



Points: Ordered many body system
Lines: Analytical results



- The analytical results perfectly agree with ordered MBS except for large γ_0 .

Comparison with ordered MBS: Scaling

Analytical solution of TBS:

Assumption: $\gamma_0 \ll 1$

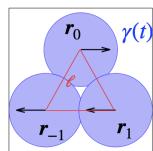
$$G'' = \begin{cases} 0, & \gamma_0 < \gamma_c(\mu) \\ \frac{\sqrt{3}k_t}{4\pi} \left\{ 1 - \cos^2 \Theta(\gamma_0) \right\}, & \gamma_c(\mu) \leq \gamma_0 \end{cases}$$

$$\Theta(\gamma_0) = \cos^{-1} \left(1 - 2\gamma_c(\mu)/\gamma_0 \right) \quad \gamma_c(\mu) = \frac{4}{3\sqrt{3}} \frac{\mu P}{k_t}$$

Prediction: Scaling laws for G' and G''

$$G'(\gamma_0, \mu) = G'_M(\mu) \mathcal{F}_1 \left(\frac{k_t \gamma_0}{\mu P(\mu)} \right) \sim \gamma_0 / \gamma_c$$

$$G''(\gamma_0, \mu) = G''_M(\mu) \mathcal{F}_2 \left(\frac{k_t \gamma_0}{\mu P(\mu)} \right)$$



μ : Friction coefficient P : Pressure
 k_t : Tangential elastic constant

$G'_M(\mu)$: Maximum value of G'

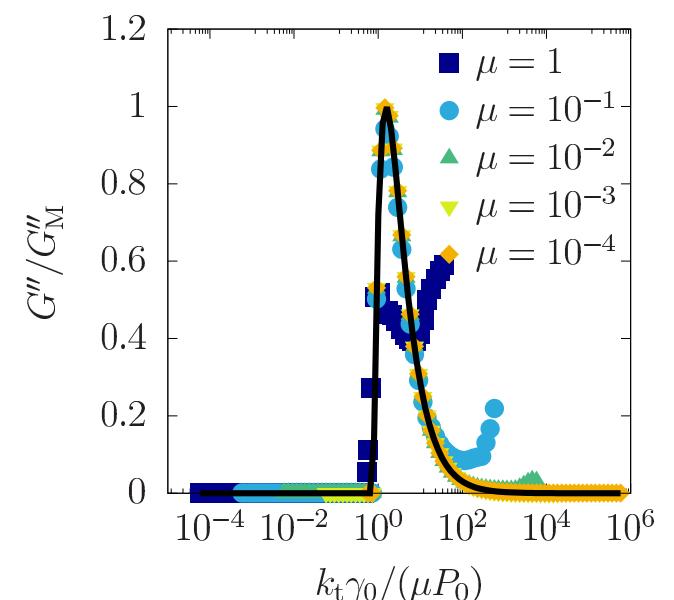
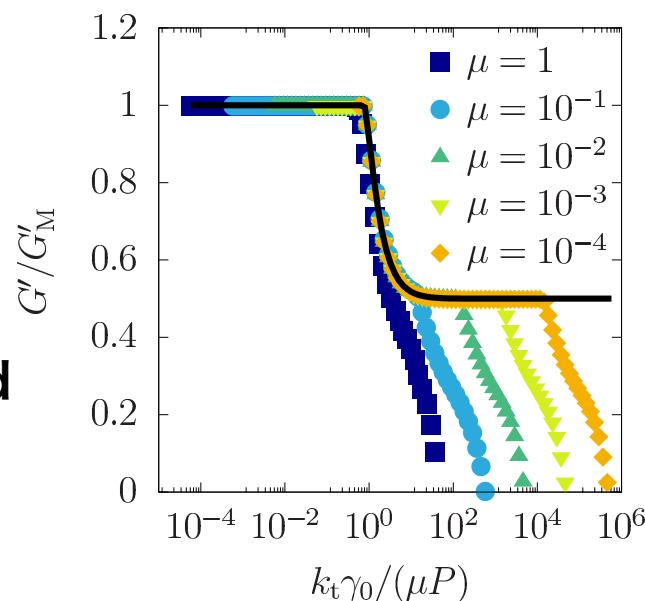
$G''_M(\mu)$: Maximum value of G''

$\mathcal{F}_1(x), \mathcal{F}_2(x)$: Scaling functions

Ordered MBS

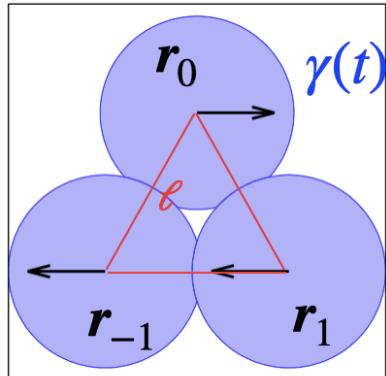
Solid line: Analytical results

Scaling laws are satisfied except for large γ_0 .

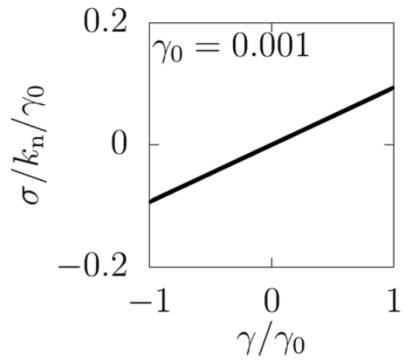
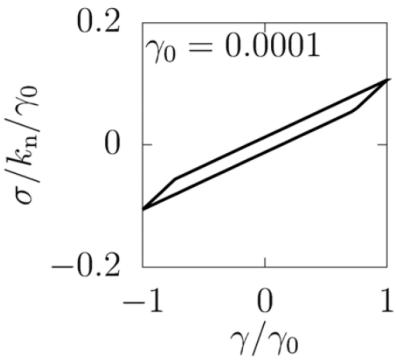
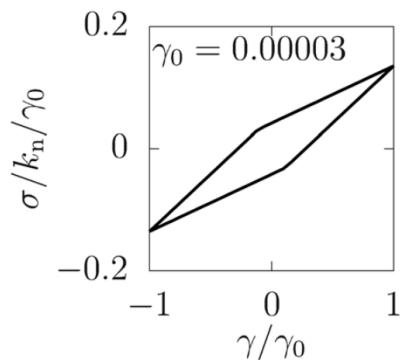
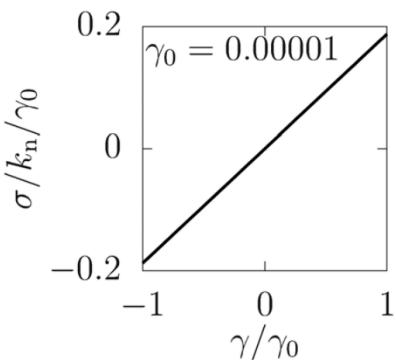


Comparison with disordered MBS: σ - γ

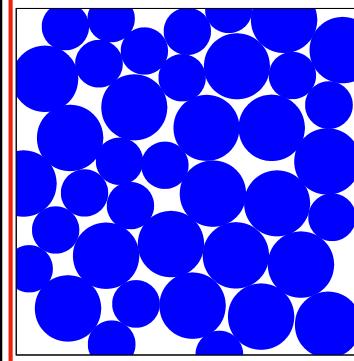
Three body system



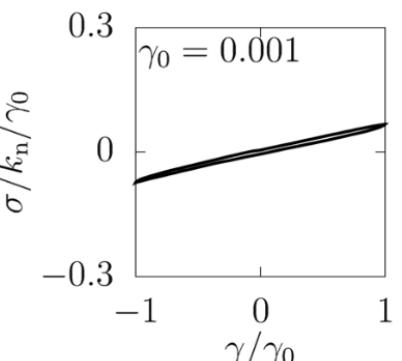
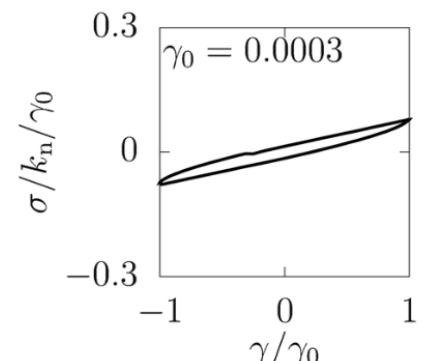
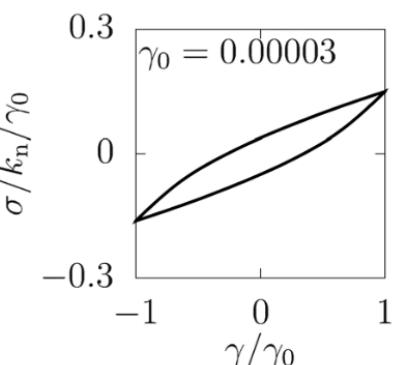
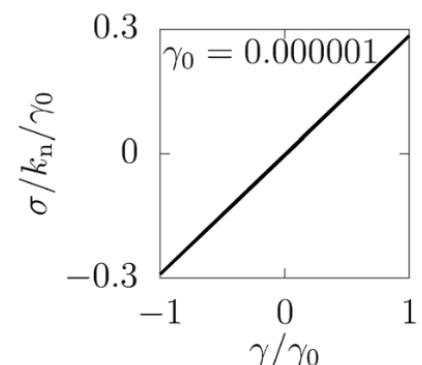
Loops appear for $\gamma_0 > \gamma_c$.
Loops are parallelograms.
(Elastoplasticity)



Disordered MBS

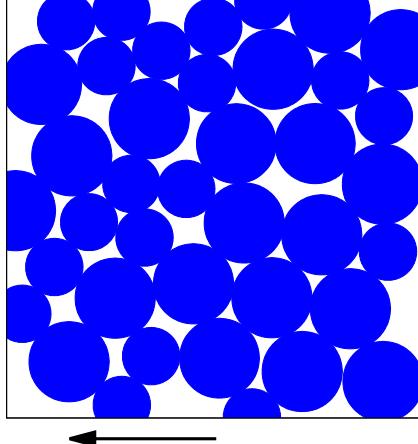


Loops appear for $\gamma_0 > \gamma_c$.
Loops are not parallelograms
due to the disorder.
(visco-elastoplasticity)

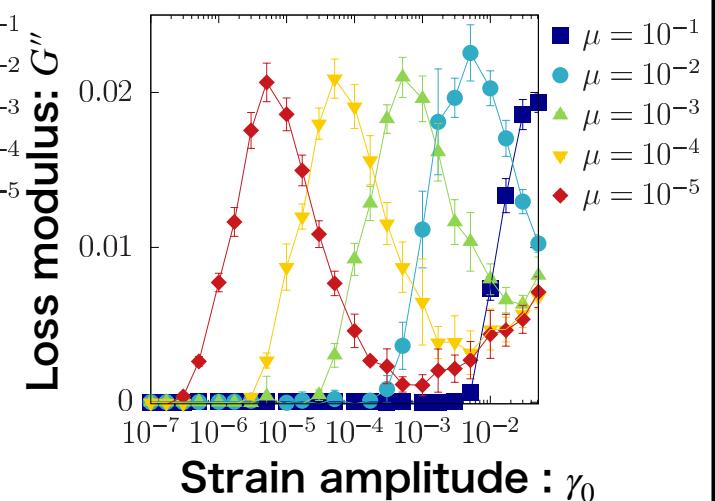
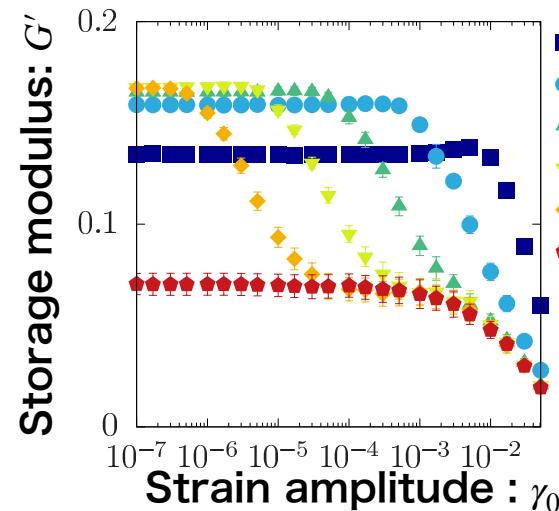


Comparison with disordered MBS: G' and G''

G' and G'' for disordered MBS



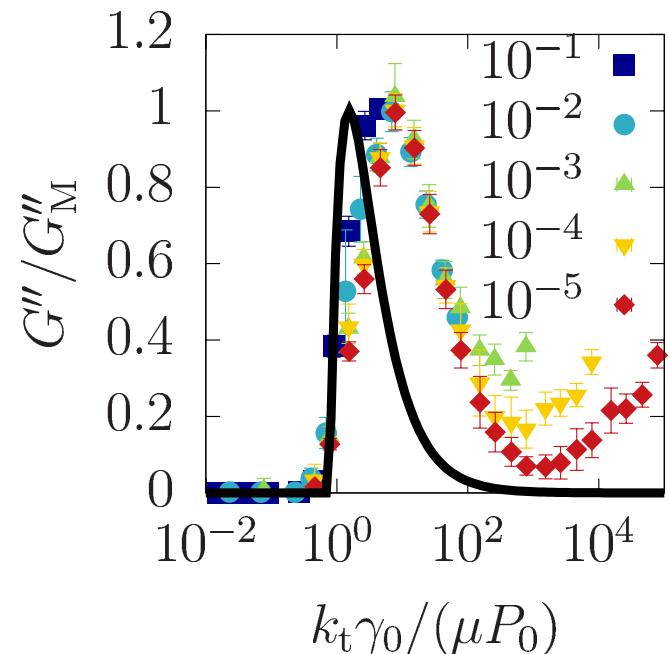
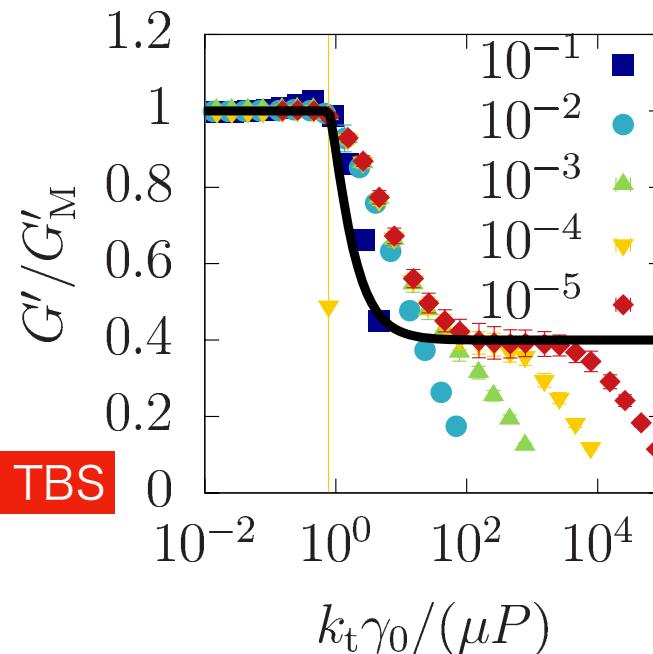
$\phi = 0.87$



Critical scaling

$$G'(\gamma_0, \mu) = G'_M(\mu) \mathcal{F}_1\left(\frac{k_t \gamma_0}{\mu P(\mu)}\right)$$

$$G''(\gamma_0, \mu) = G''_M(\mu) \mathcal{F}_2\left(\frac{k_t \gamma_0}{\mu P(\mu)}\right)$$



Solid line: Analytical results of TBS

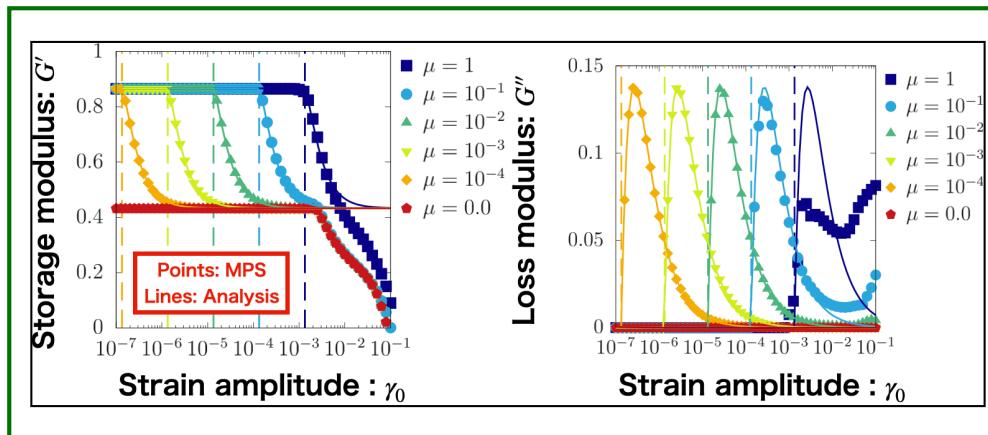
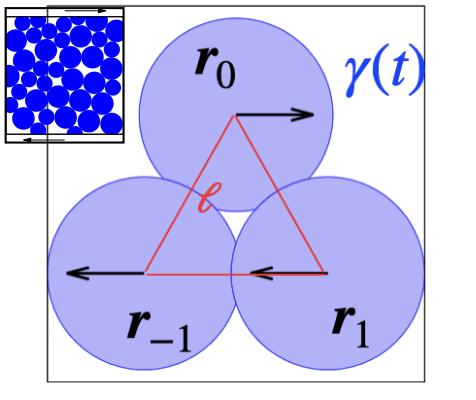
k_t is used as a fitting parameter.

- Analytical results of TBS qualitatively reproduced disordered MBS except for large γ_0 .

Summary

arXiv:2211.03294
Soft Matter

- We have proposed a model comprising three particles.
- Three body system perfectly reproduces G' and G'' in ordered MBS for small γ_0 .
- Three body system qualitatively reproduces G' and G'' in disordered MBS for small γ_0 .
- We derive scaling laws, which are satisfied even in disordered MBS.
- Disorder effect: Visco-plastoelasticity, ϕ -dependence.
- Problem: we need a fitting parameter for disordered MBS.
- Future work: Self-consistent determination of a fitting parameter.



Scaling laws:

$$G' = G'_M F_1 \left(\frac{k_t \gamma_0}{\mu P(\mu)} \right)$$

$$G'' = G''_M \mathcal{F}_2 \left(\frac{k_t \gamma_0}{\mu P(\mu)} \right)$$