

Exact Solution for the Darcy law of yield stress fluids on the Bethe lattice

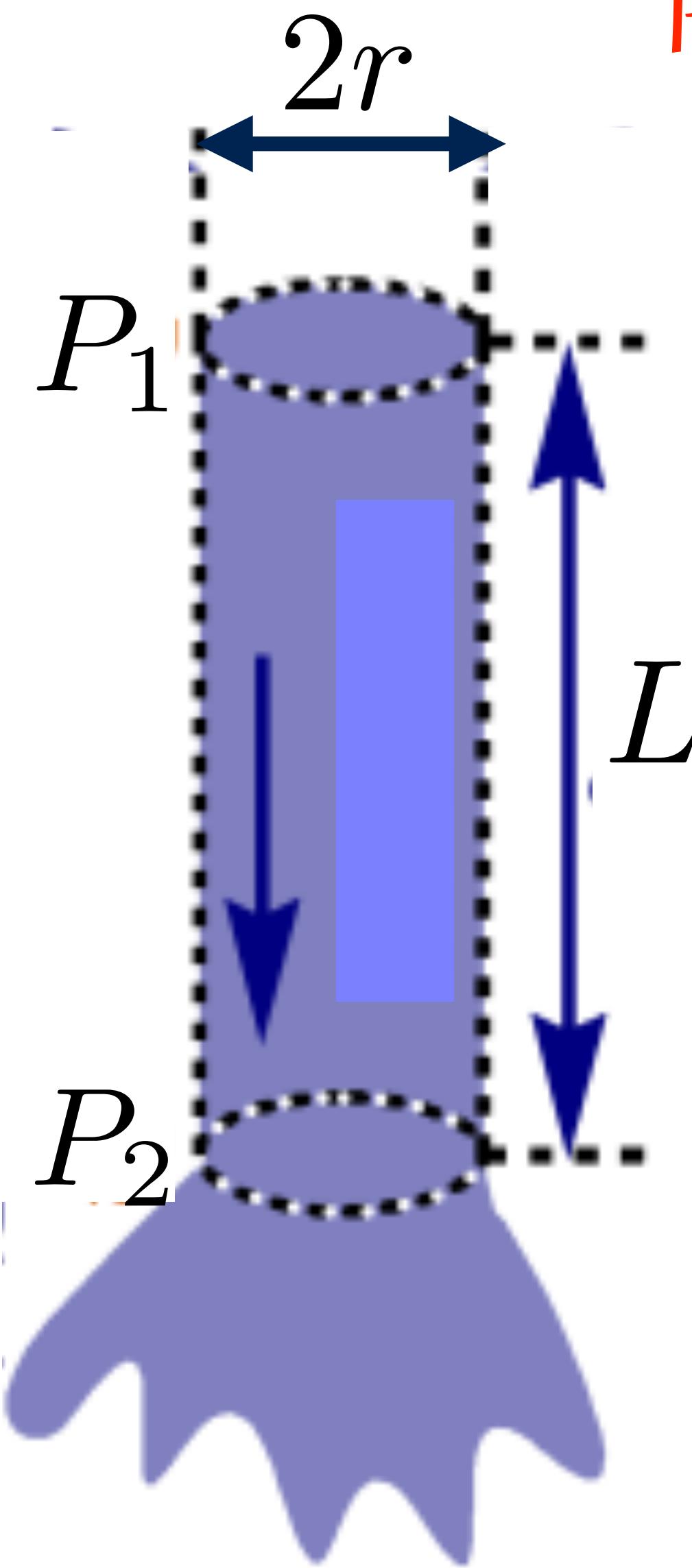
Alberto Rosso

LPTMS (Univ. Paris-Saclay, CNRS)

- ◆ Collaborations: V. Schimmenti, A. De Luca, S. Franz, F. Lanza, A. Hansen, L. Talon. *arXiv:2208.06048*
- ◆ Special Thanks: Chen Liu (Columbia Univ., NY)



Poiseuille law: yield stress vs Newtonian fluid



Newtonian fluids:

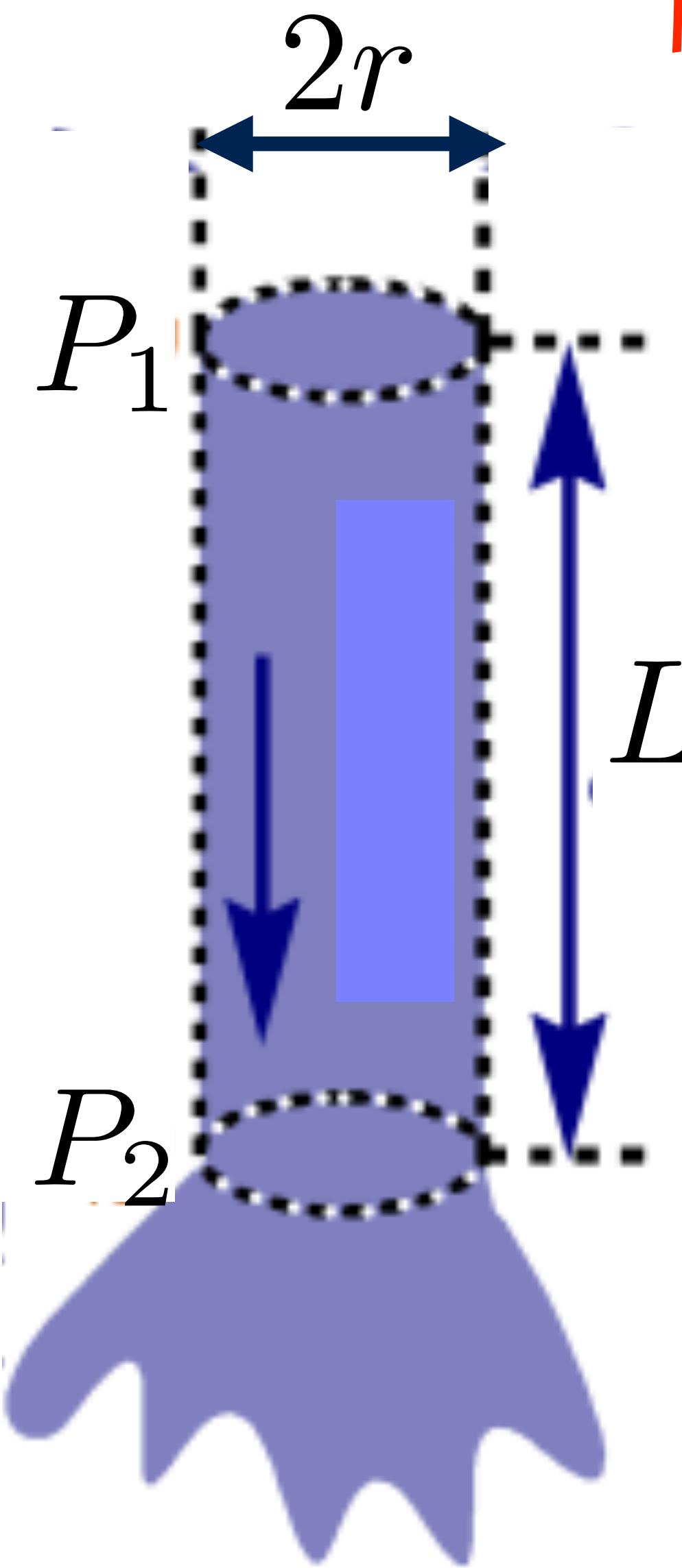
$$Q_{\text{Poise}} = \frac{\pi r^4}{8\eta} \frac{P}{L}$$

Yield stress fluids:

$$Q_{\text{Poise}} = \frac{\pi r^4}{8\eta} \left(\frac{P}{L} - \frac{\tau_y}{r} \right)$$

$$P = P_1 - P_2$$

Poiseuille law: yield stress vs Newtonian fluid



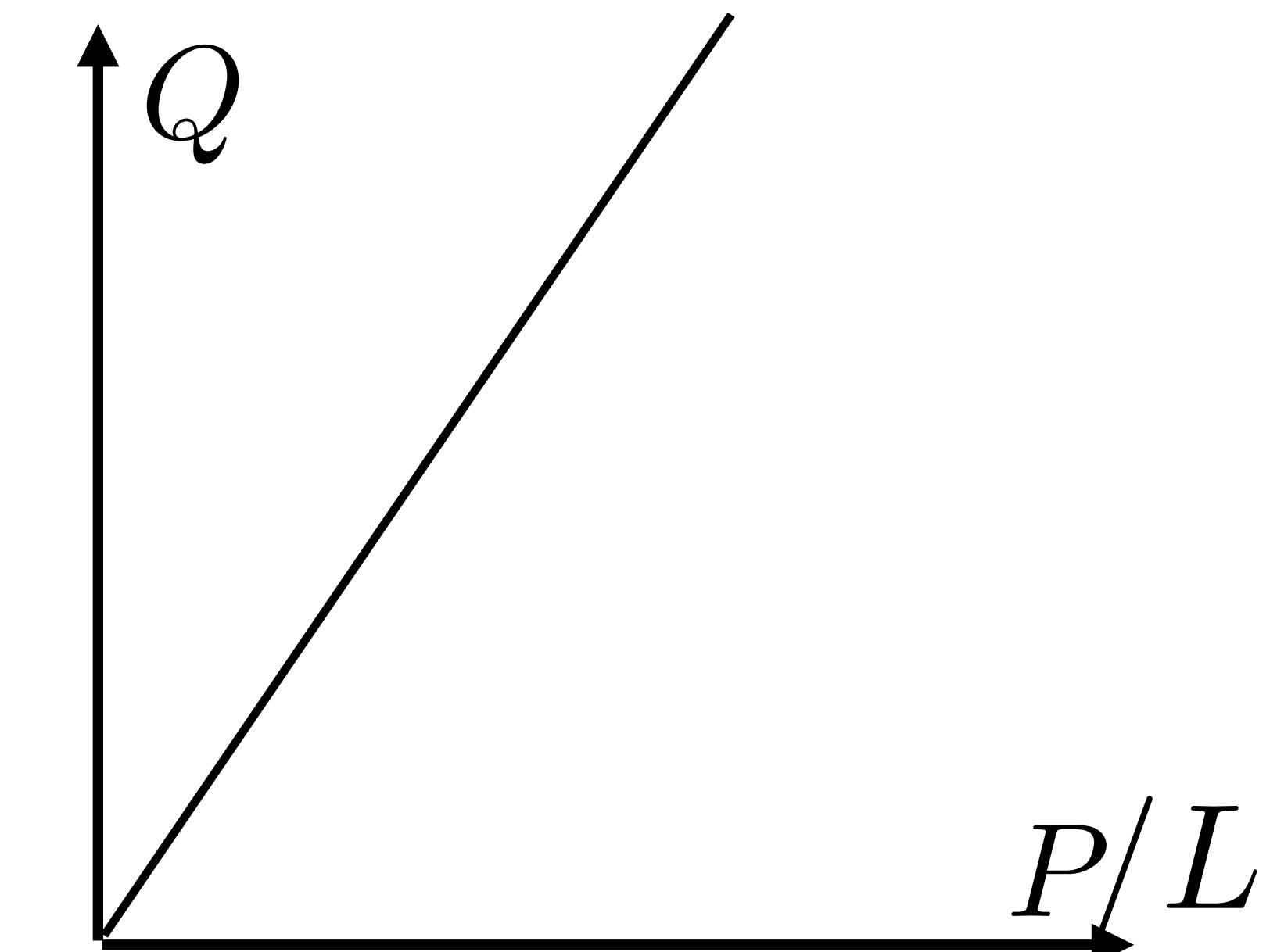
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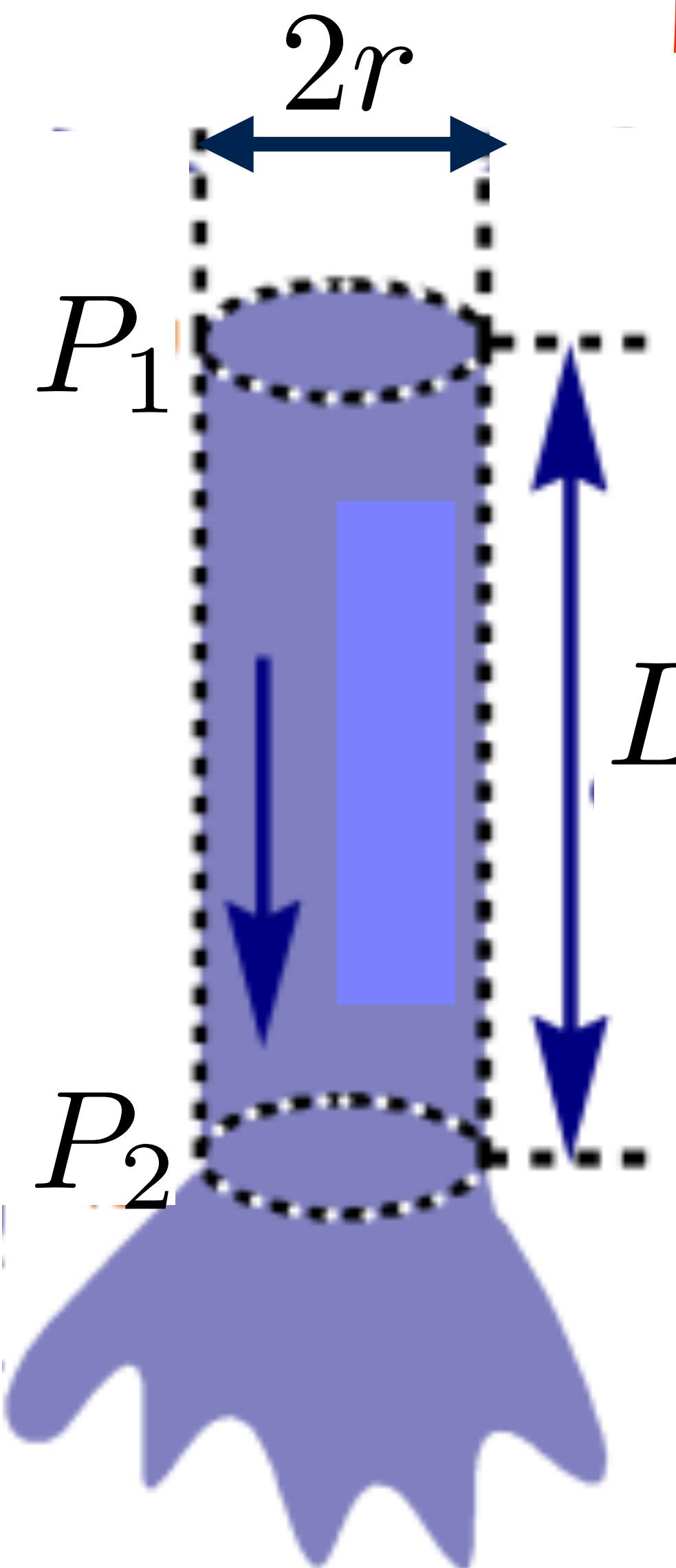
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Poiseuille law: yield stress vs Newtonian fluid



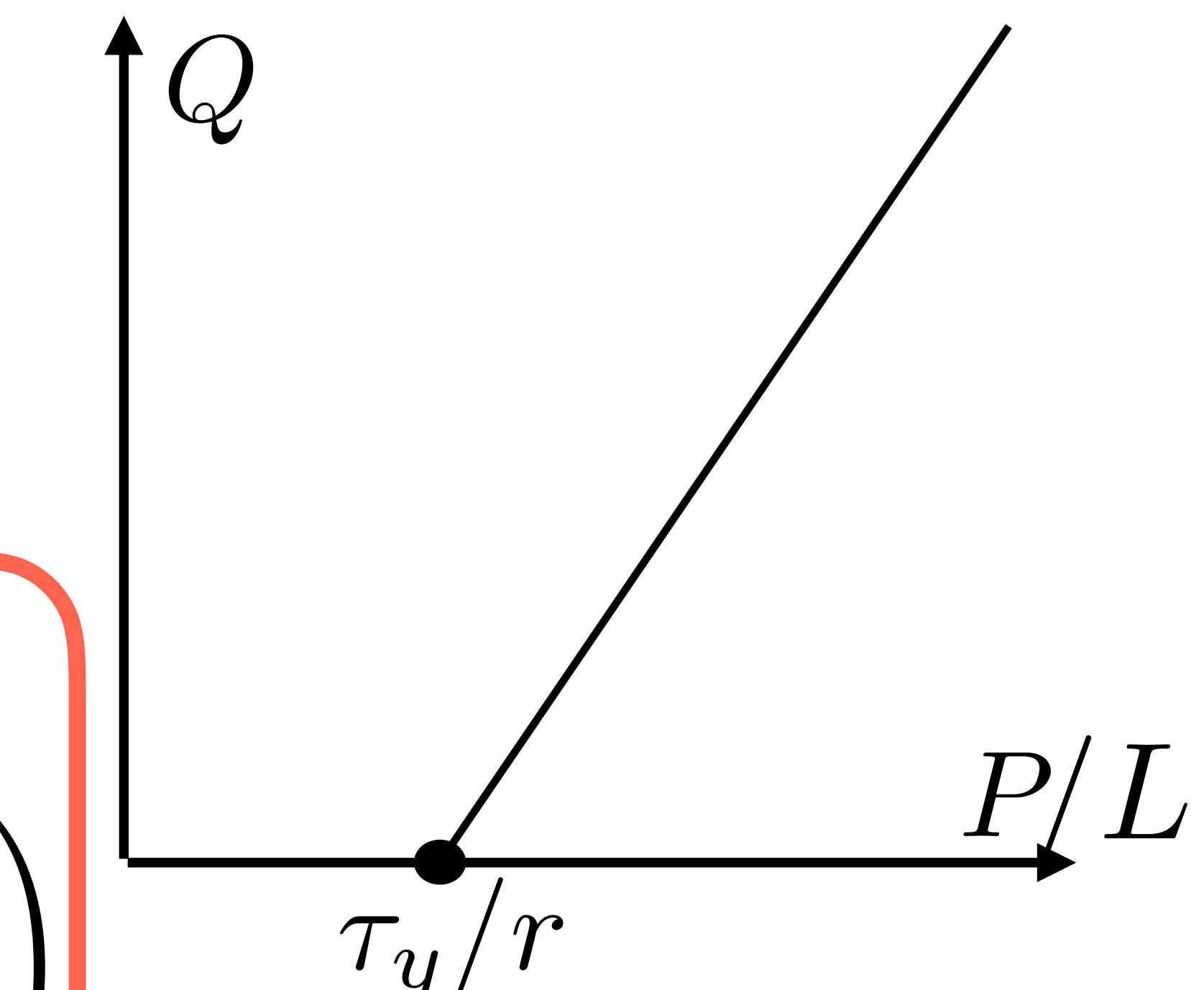
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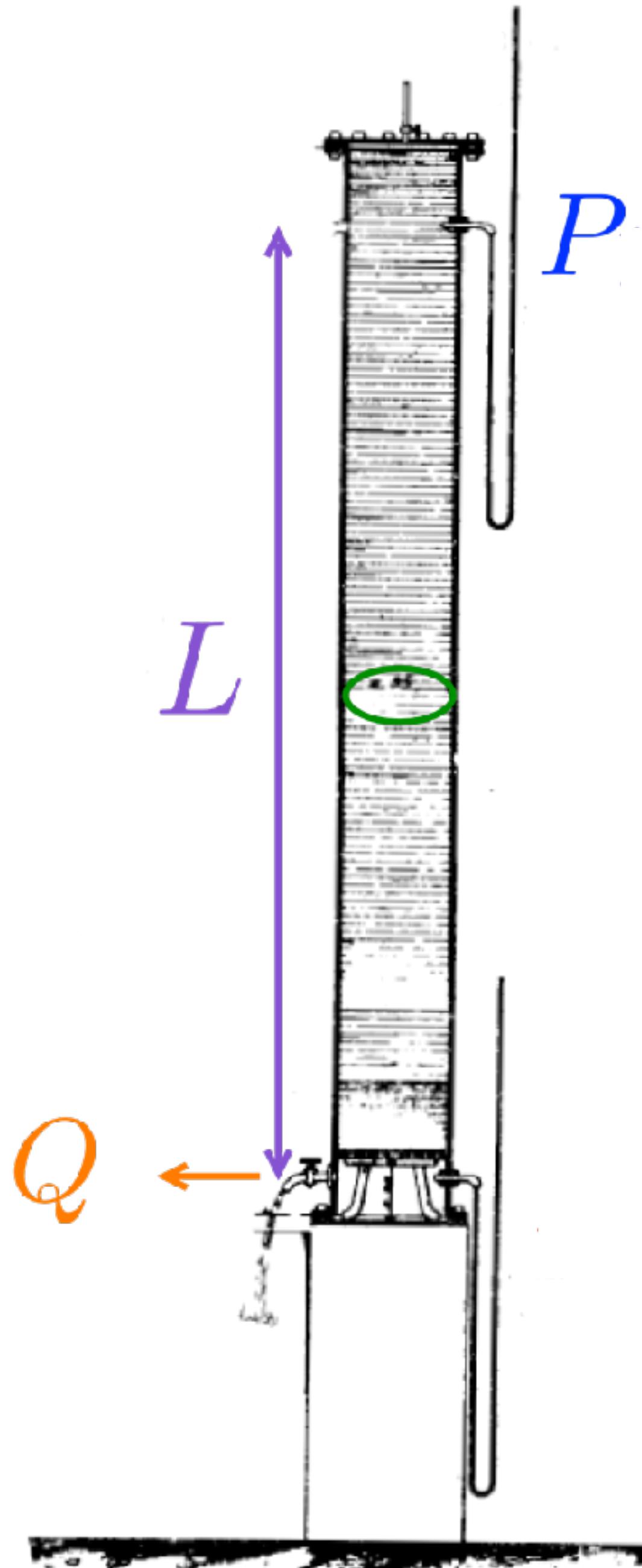
$$P = P_1 - P_2$$



Water reservoir in Dijon (H. Darcy 1840)

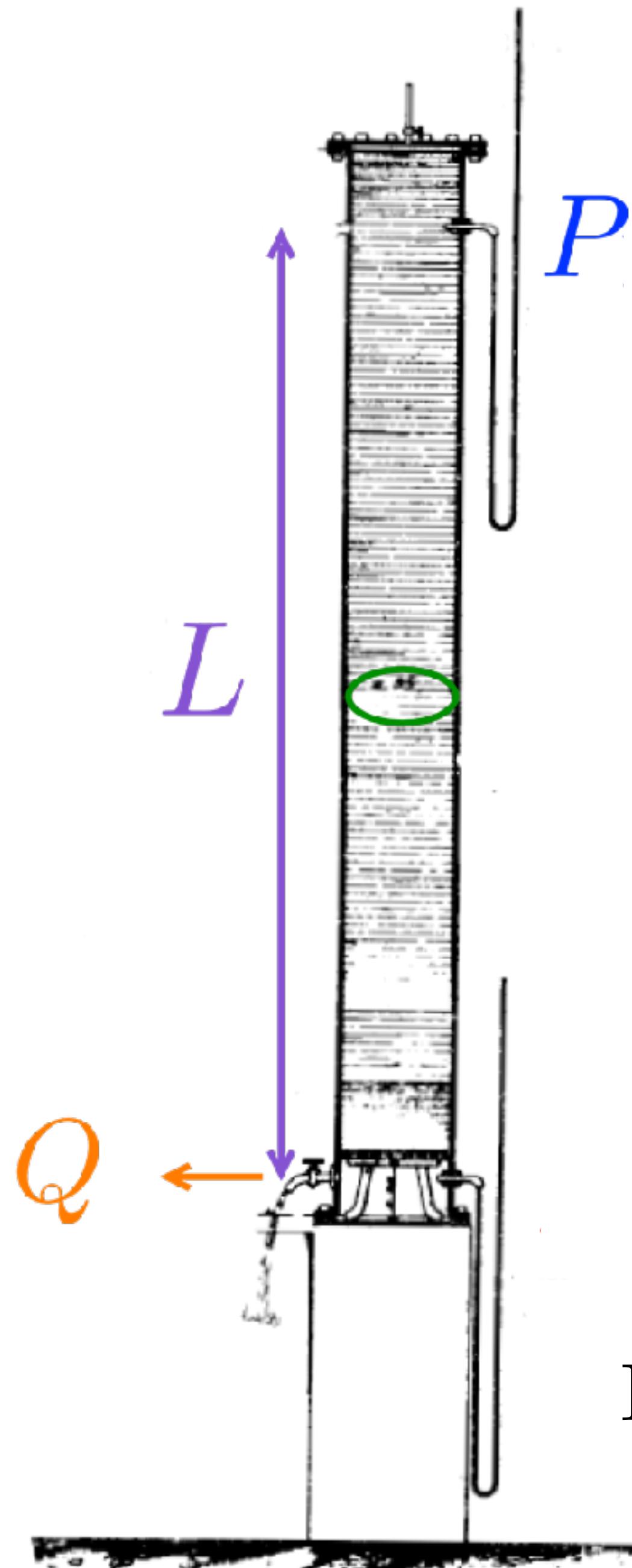


Les fontaines publiques de la ville de Dijon (H. Darcy 1856)



$$Q_{\text{Darcy}} = \kappa \frac{\pi r^2}{\eta} \frac{P}{L}$$

Les fontaines publiques de la ville de Dijon (H. Darcy 1856)



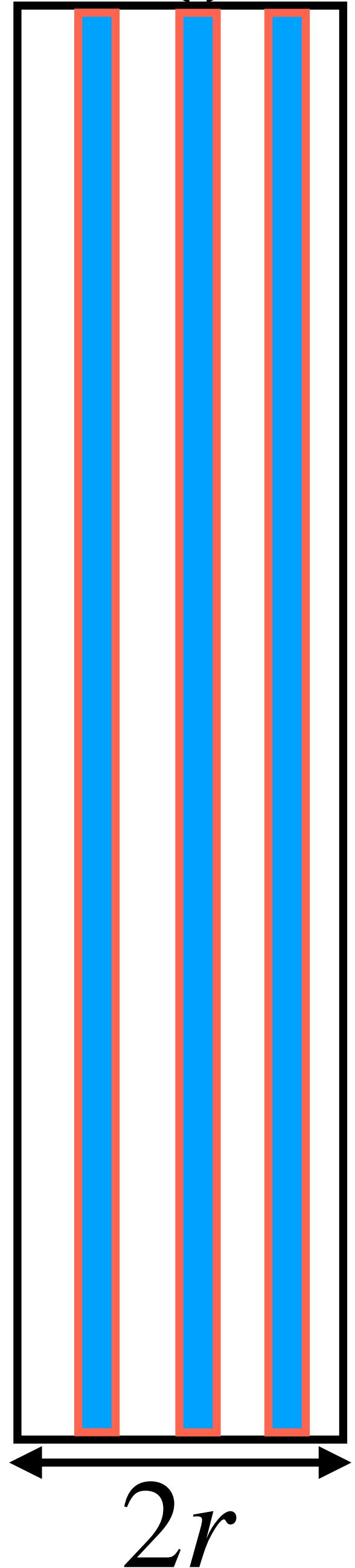
$$Q_{\text{Darcy}} = \kappa \frac{\pi r^2}{\eta} \frac{P}{L}$$

$$Q_{\text{Poise}} = \frac{r^2}{8} \frac{\pi r^2}{\eta} \frac{P}{L}$$

$$\frac{r^2}{8} \rightarrow \kappa \text{ (permeability)}$$

Permeability is strongly material dependent

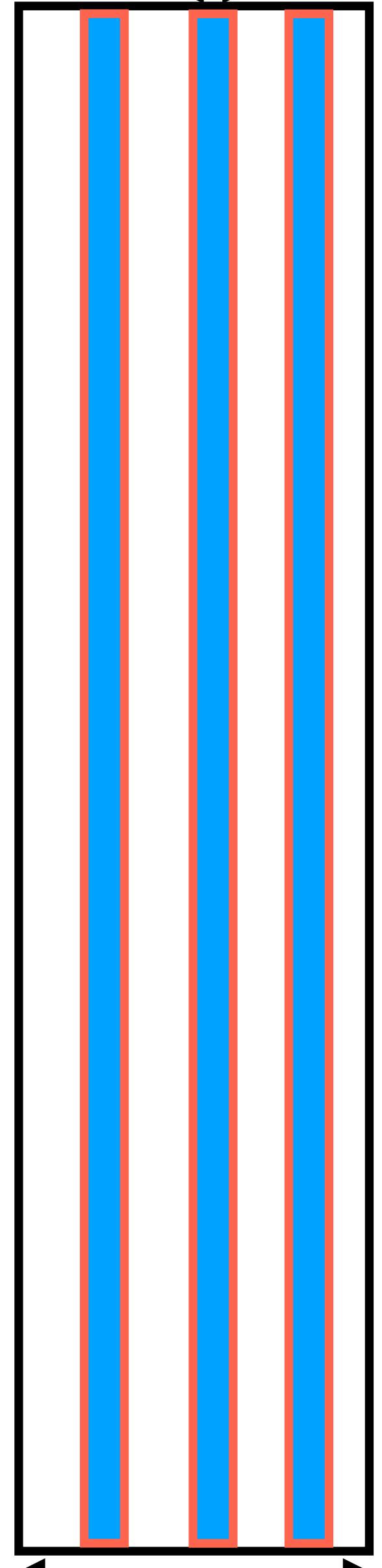
$$2r_c$$



Les fontaines publiques de la ville de Dijon (H. Darcy 1856)

$$Q_{\text{Darcy}} = \pi R^2 n^{\text{ch}} \frac{\pi R_c^4 P}{8\eta L}$$

$2r_c$

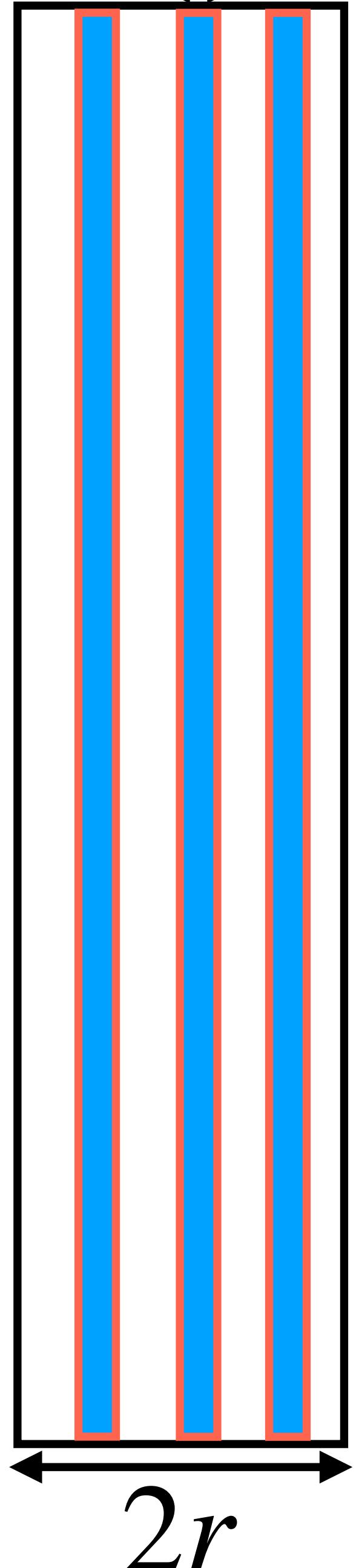


Les fontaines publiques de la ville de Dijon (H. Darcy 1856)

$$Q_{\text{Darcy}} = \pi R^2 n^{\text{ch}} \frac{\pi R_c^4 P}{8\eta L}$$

$$Q_{\text{Darcy}} = \frac{\pi R_c^4 n^{\text{ch}}}{8} \frac{\pi R^2 P}{8\eta L}$$

$$2r_c$$



Les fontaines publiques de la ville de Dijon (H. Darcy 1856)

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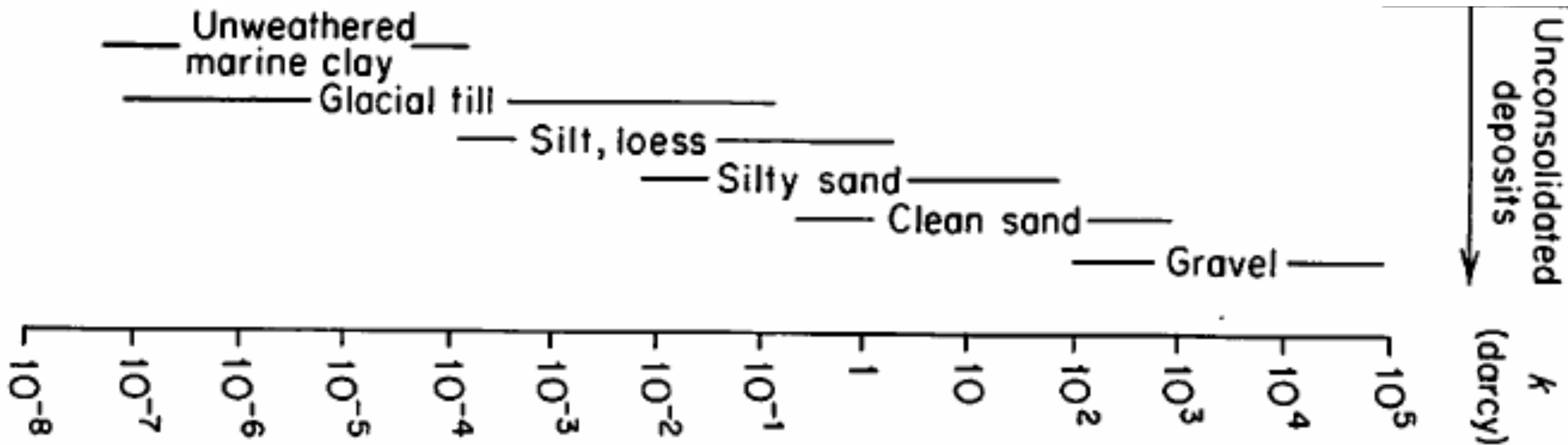
$$\kappa$$

Permeability & material dependence

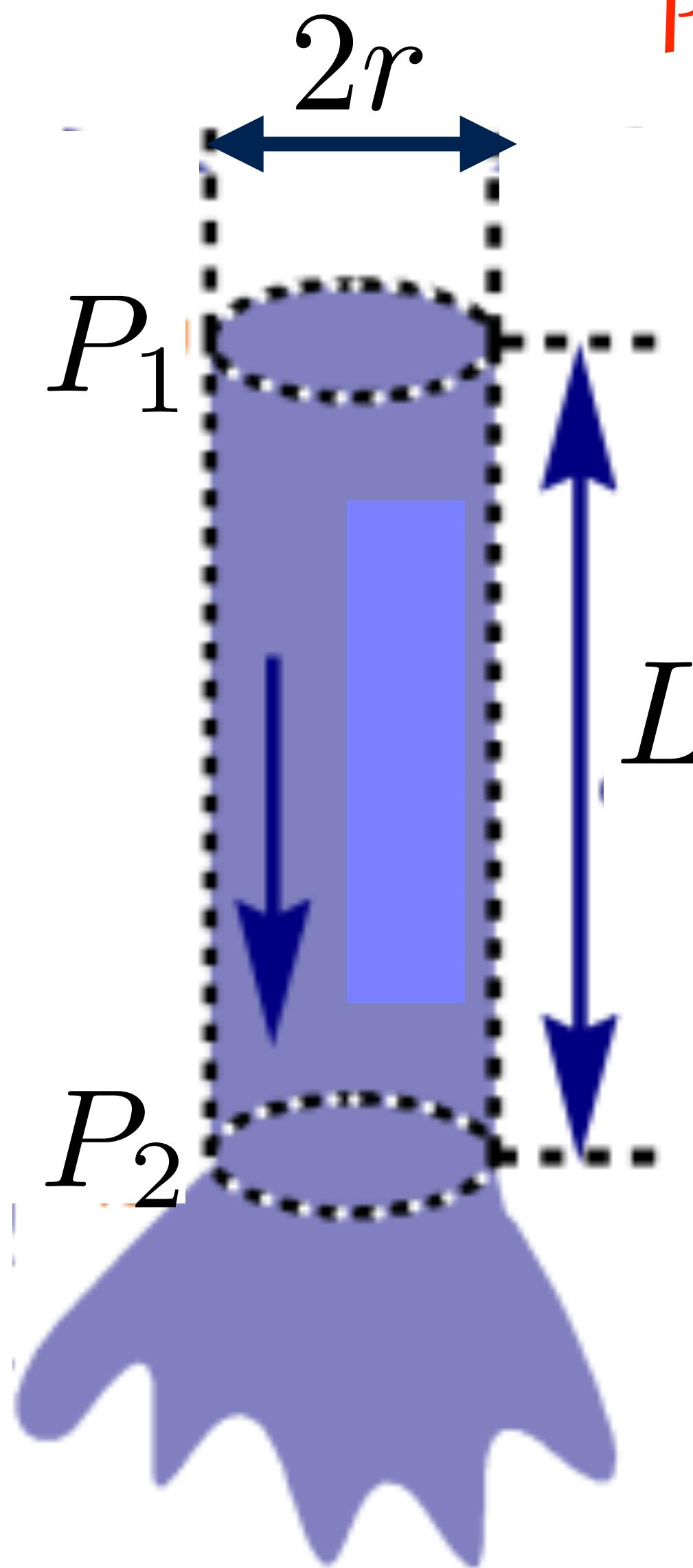
$$\kappa \sim r_c^2$$

$$1 \text{ darcy} = 10^{-12} \text{ m}^2$$

$$r_c \approx 1 \mu\text{m}$$



Poiseuille law: yield stress vs Newtonian fluid



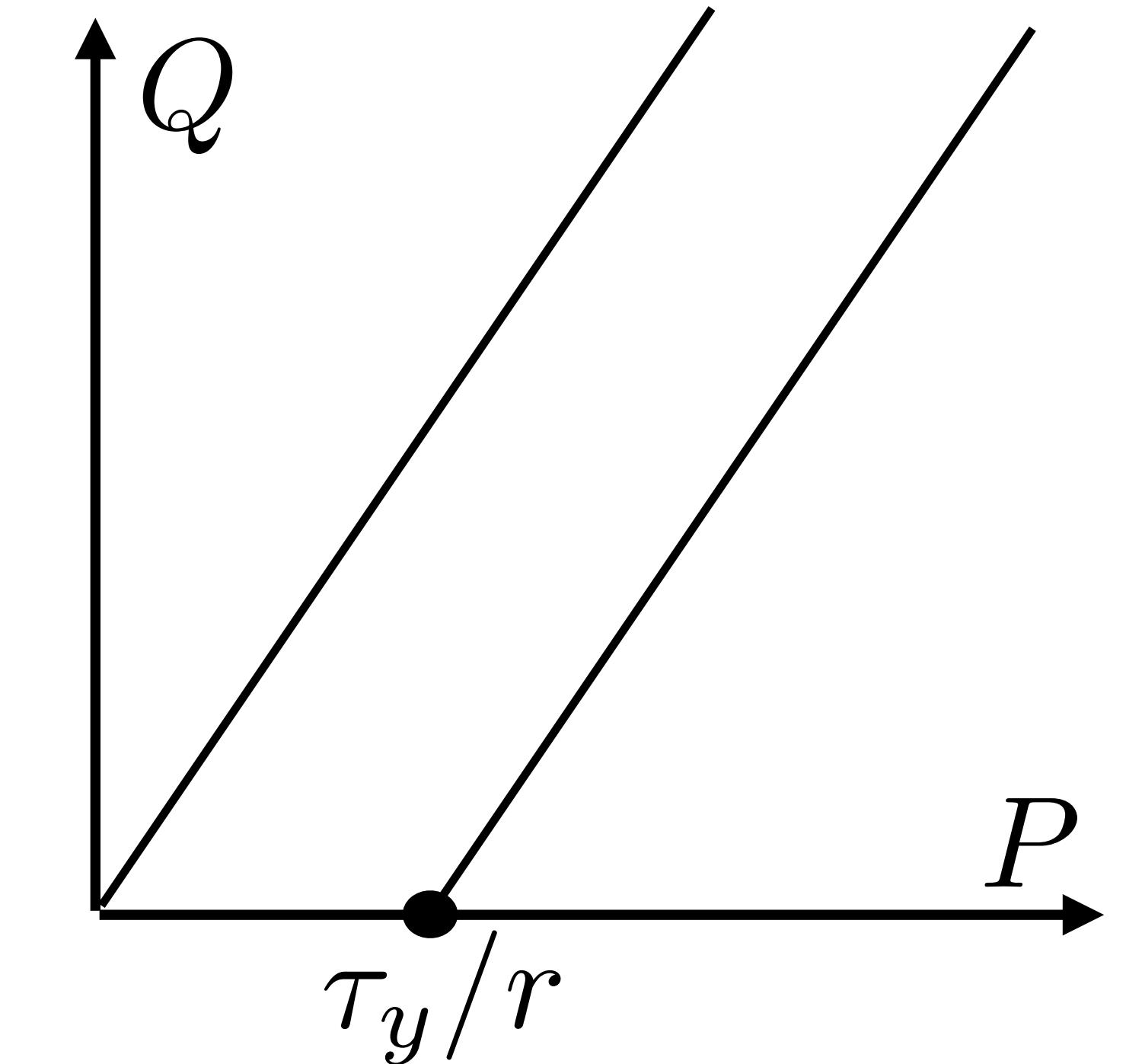
Newtonian fluids:

$$Q_{\text{Poise}} = \frac{\pi r^4}{8\eta} \frac{P}{L}$$

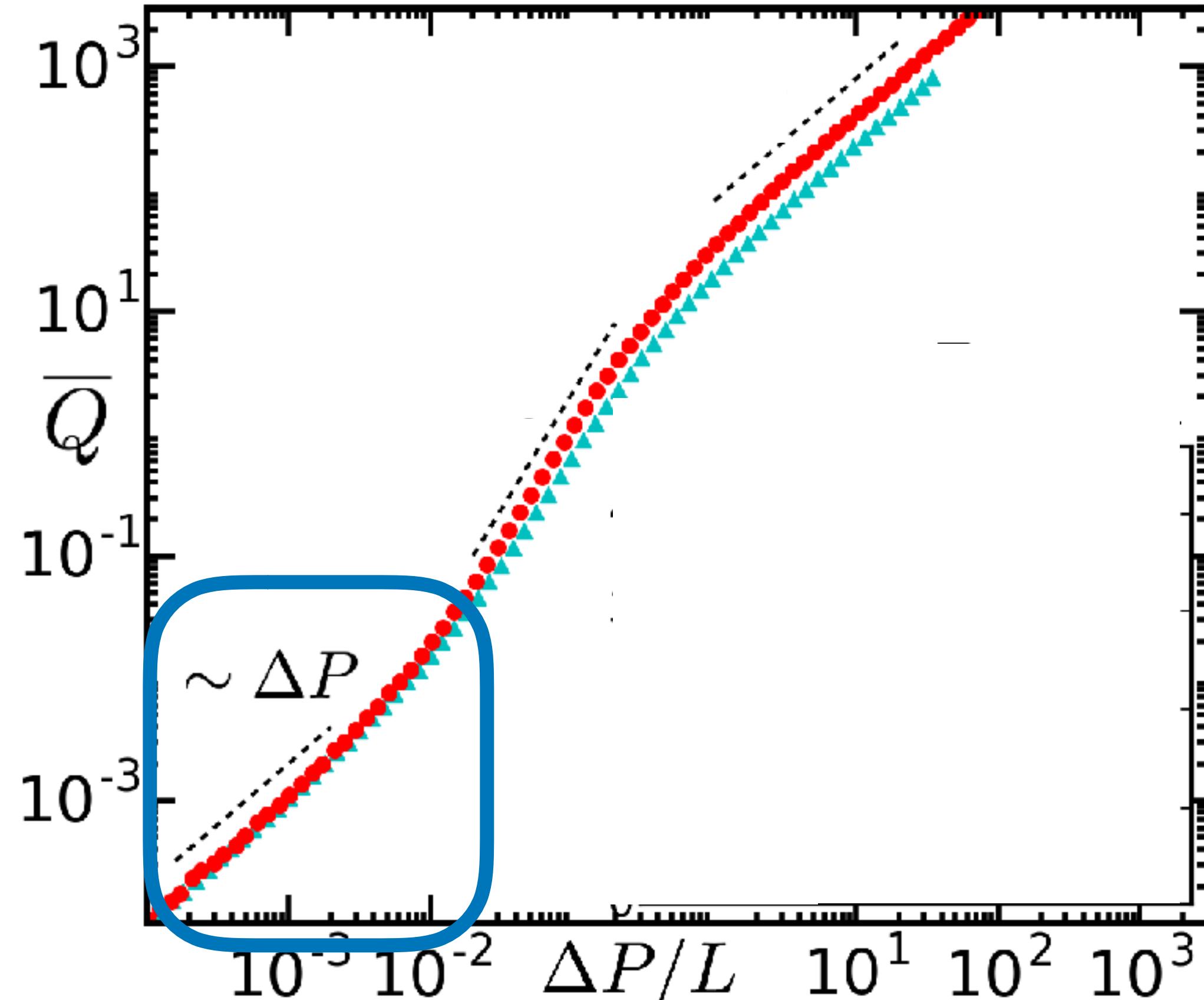
Yield stress fluids:

$$Q_{\text{Poise}} = \frac{\pi r^4}{8\eta} \left(\frac{P}{L} - \frac{\tau_y}{r} \right)$$

$$P = P_1 - P_2$$



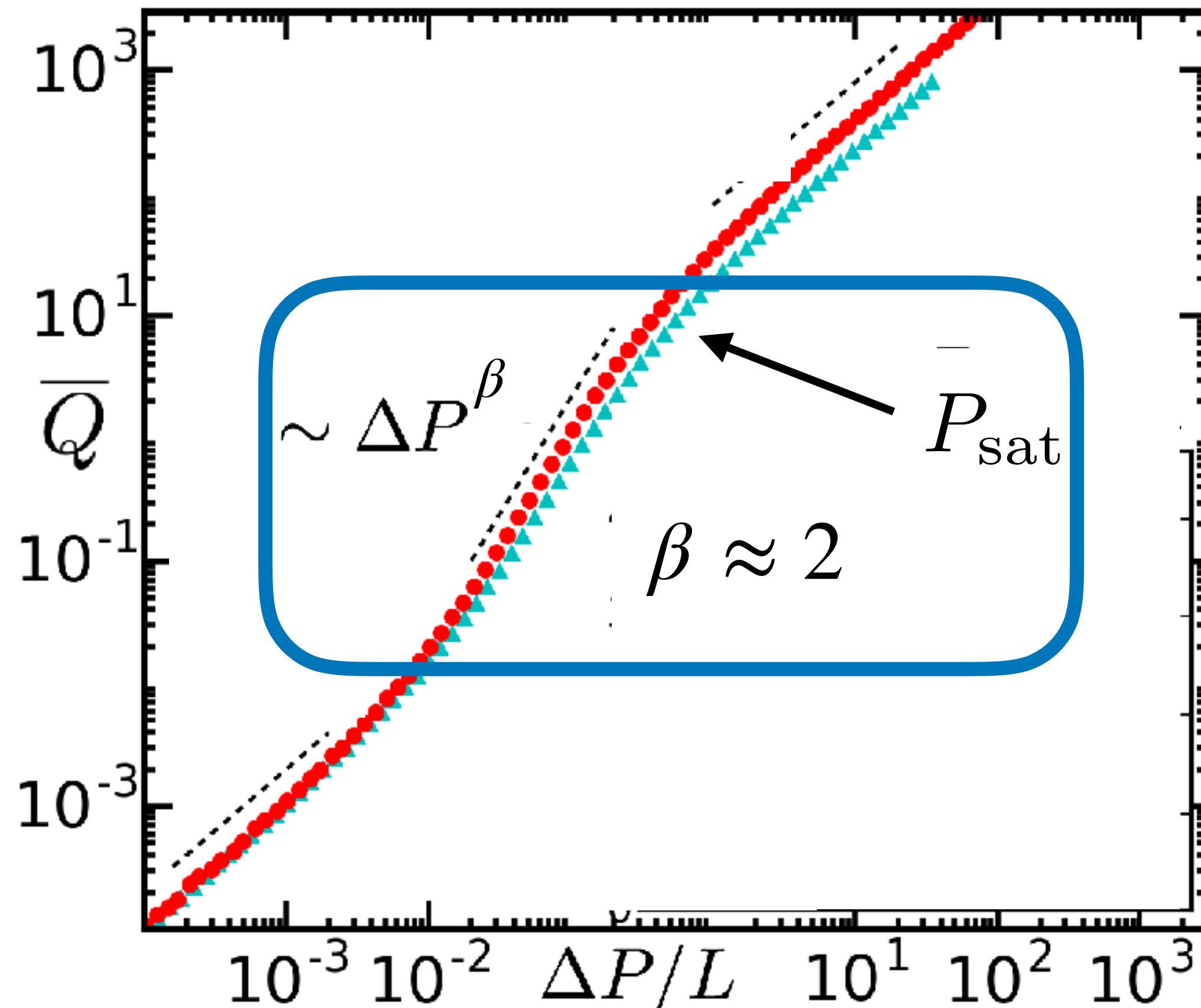
Darcy's law for yield stress fluid (numerics)



$$Q(P) = \begin{cases} \sim \frac{P - P_0}{L} & P \rightarrow P_0^+ \end{cases}$$

$$\Delta P \approx P - P_0$$

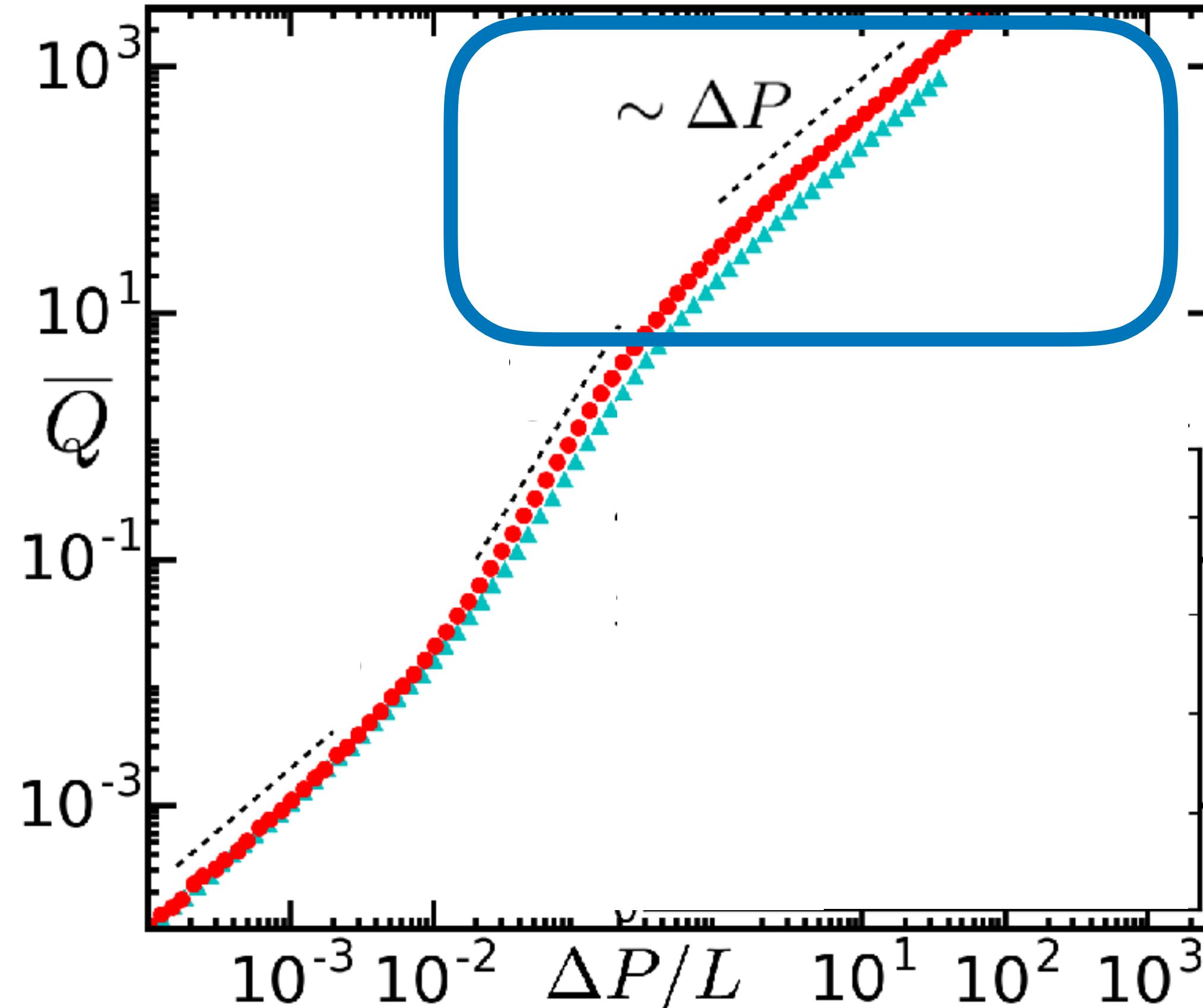
Darcy's law for yield stress fluid (numerics)



$$Q(P) = \begin{cases} \sim \frac{P - P_0}{L} & P \rightarrow P_0^+ \\ \sim \left(\frac{P - P_0}{L} \right)^\beta & P_0 < P < P_{\text{sat}} \end{cases}$$

$$\Delta P \approx P - P_0$$

Darcy's law for yield stress fluid (numerics)



$$\Delta P \approx P - P_0$$

$$Q(P) = \begin{cases} \sim \frac{P - P_0}{L} & P \rightarrow P_0^+ \\ \sim \left(\frac{P - P_0}{L} \right)^\beta & P_0 < P < P_{\text{sat}} \\ = \kappa \frac{\pi r^2}{\eta} \frac{P - P^*}{L} & P \gg P_{\text{sat}} \end{cases}$$

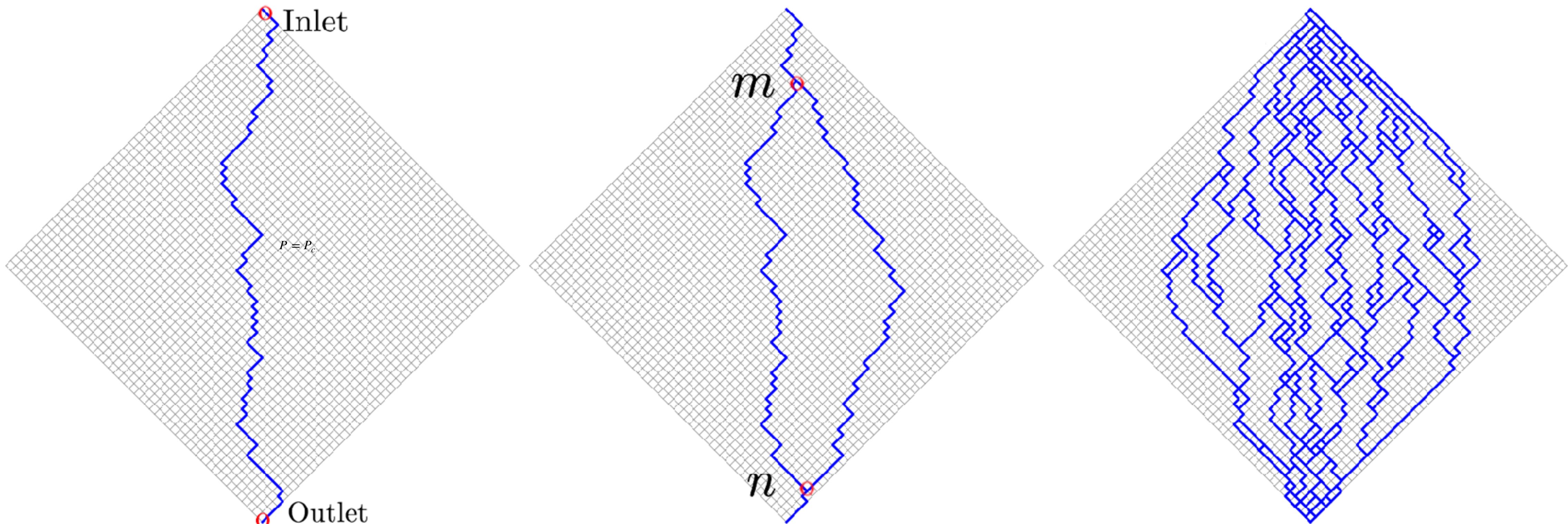
$$P_0 \ll P_{\text{sat}} \ll P^*$$

Darcy's law for yield stress fluid (numerics)

$$P \rightarrow P_0^+$$

$$P_0 < P < P_{\text{sat}}$$

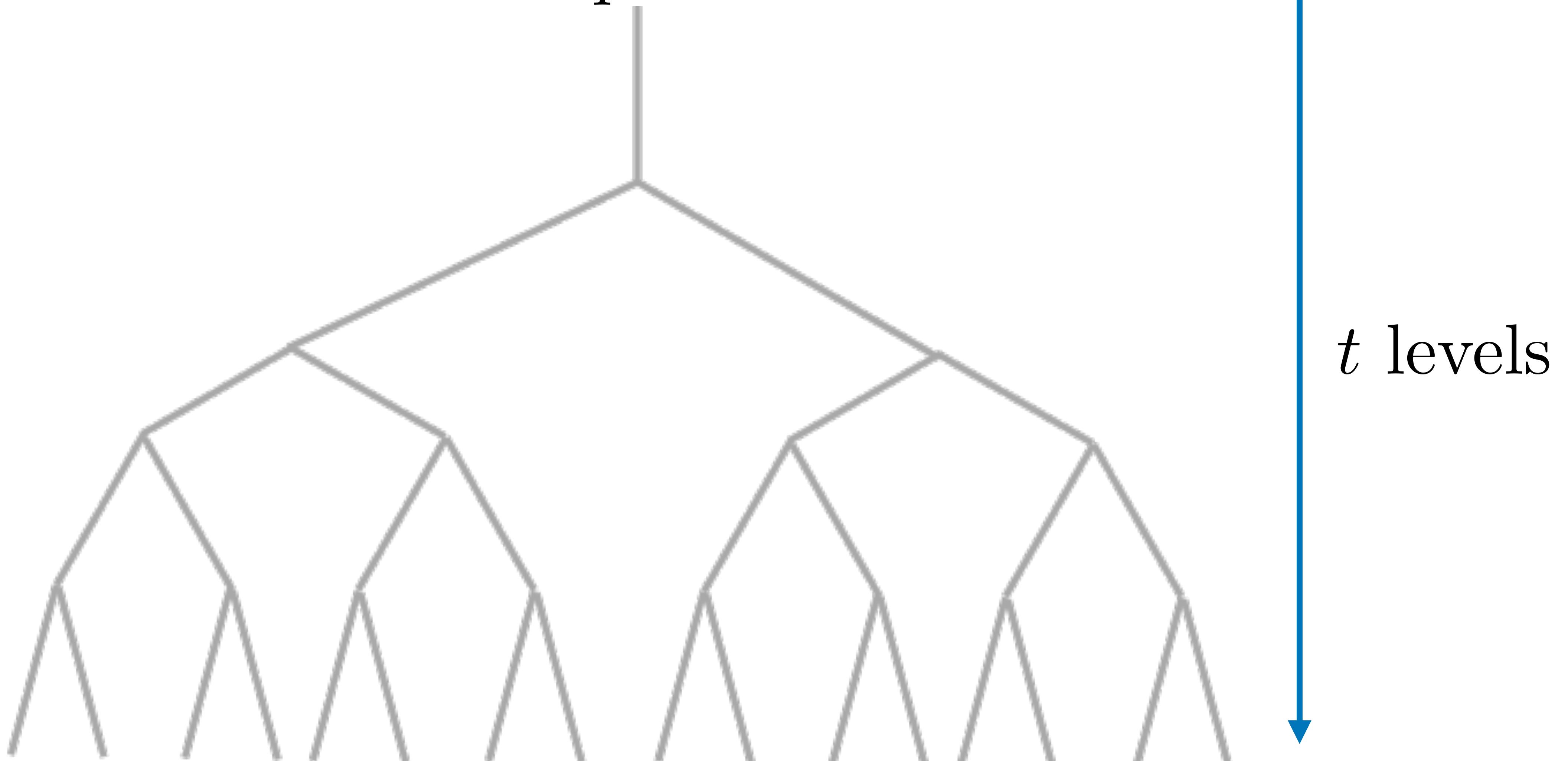
$$P \gg P_{\text{sat}}$$



$$Q_{\text{Darcy}} = \pi R_c^2 n^{\text{ch}} \frac{\pi R_c^4 P}{8\eta L}$$

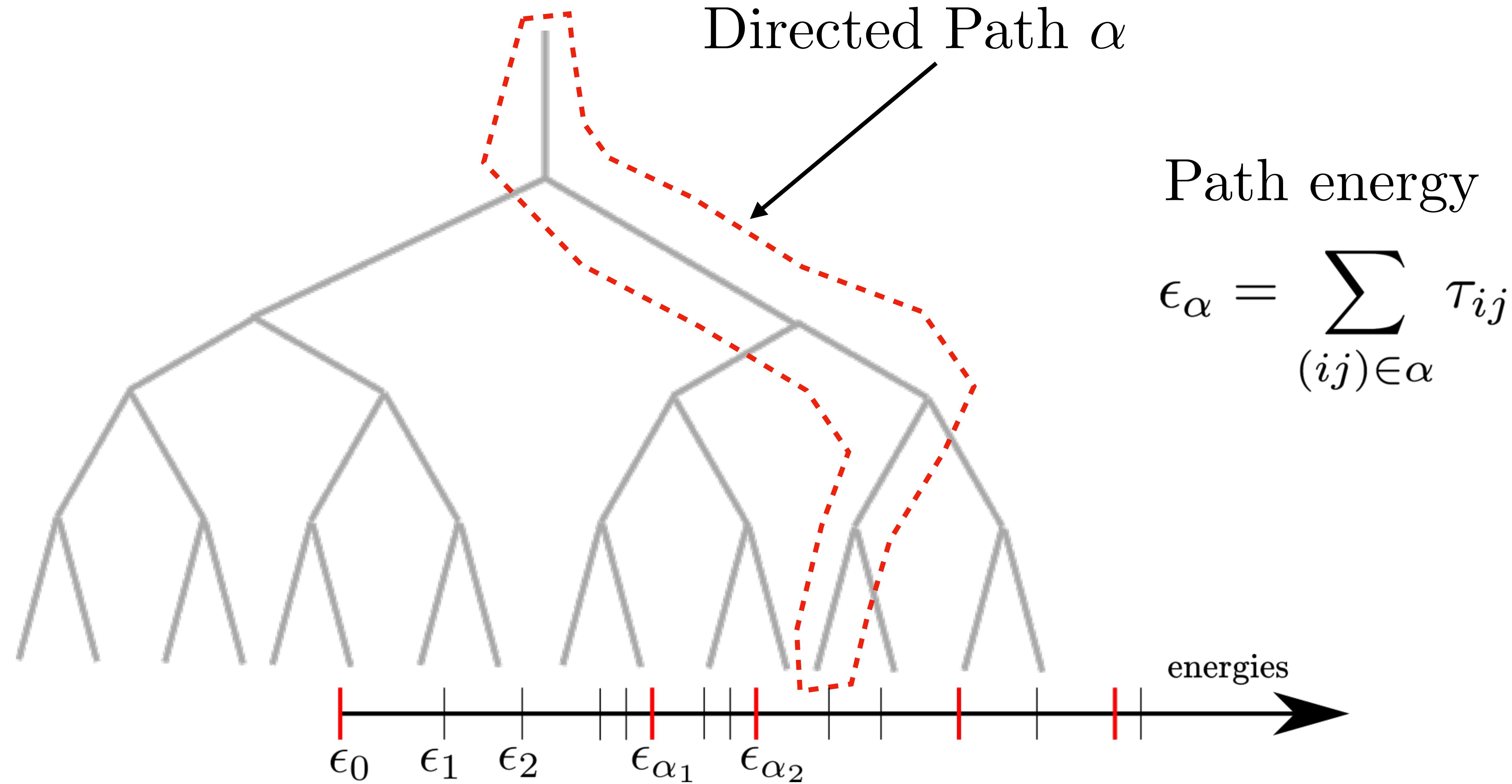
Darcy's law for the Bethe lattice

Inlet at pressure P

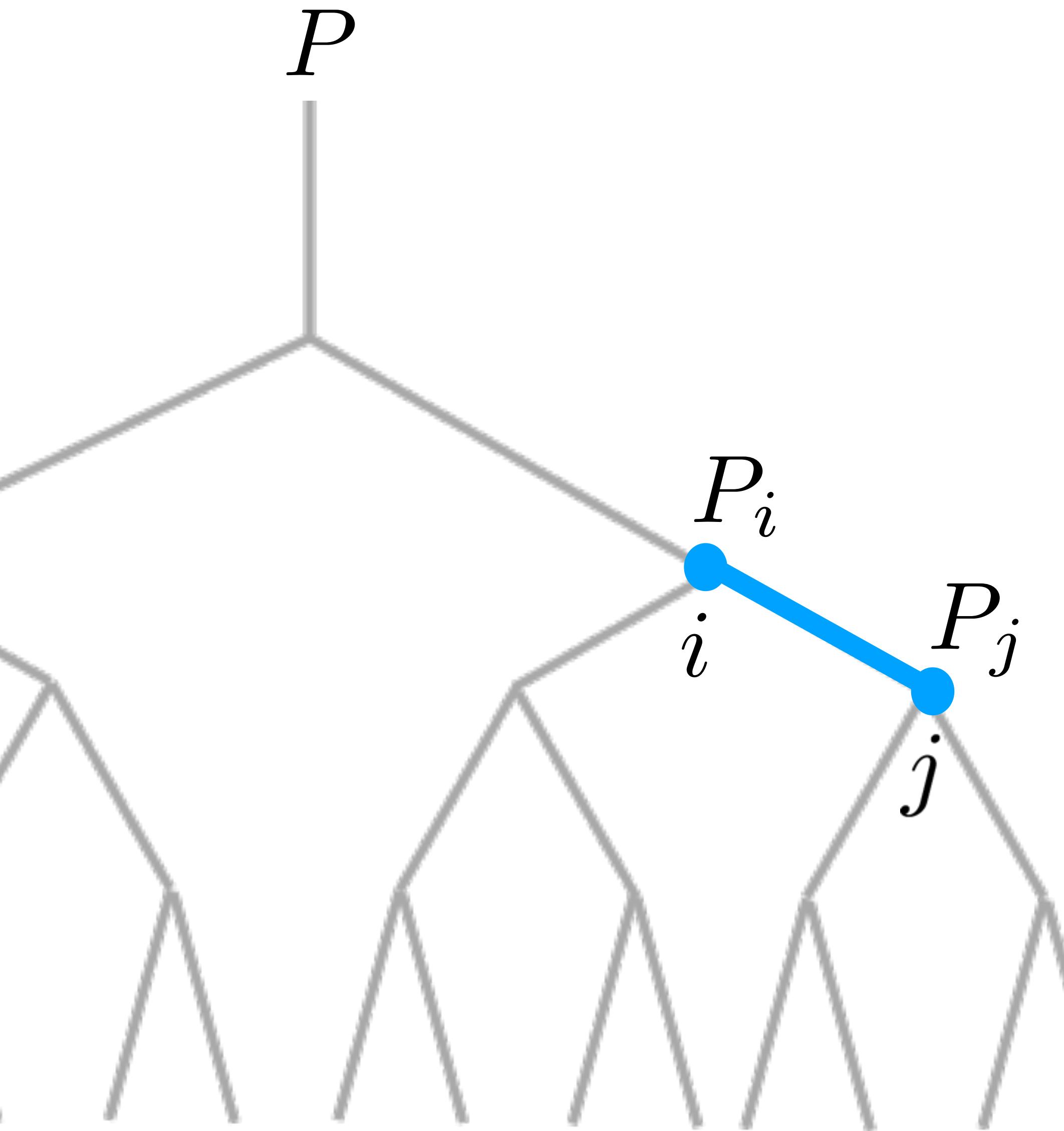


2^{t-1} outlets at zero pressure

Mapping to the directed polymer problem

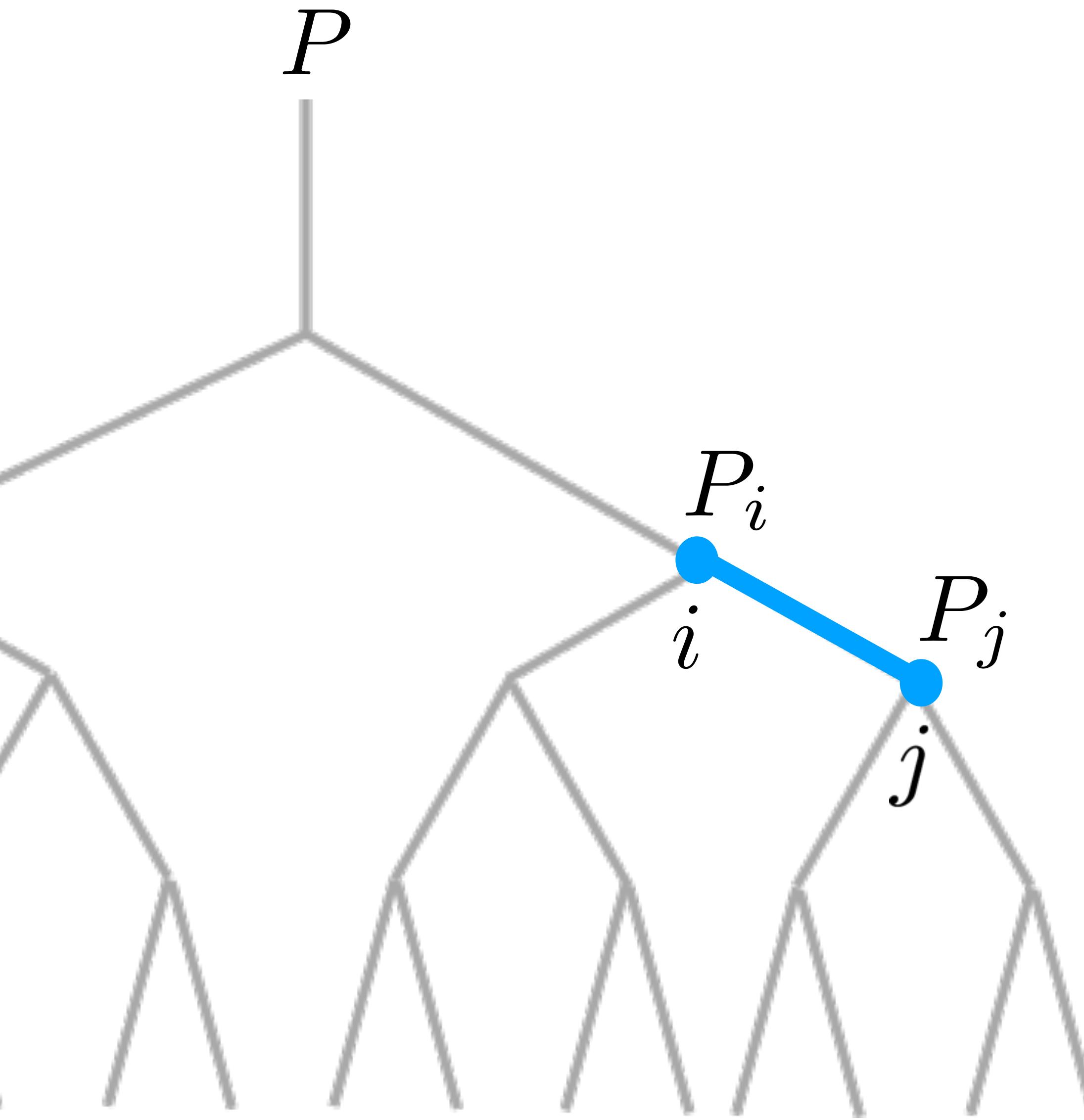


Darcy's law for the Bethe lattice

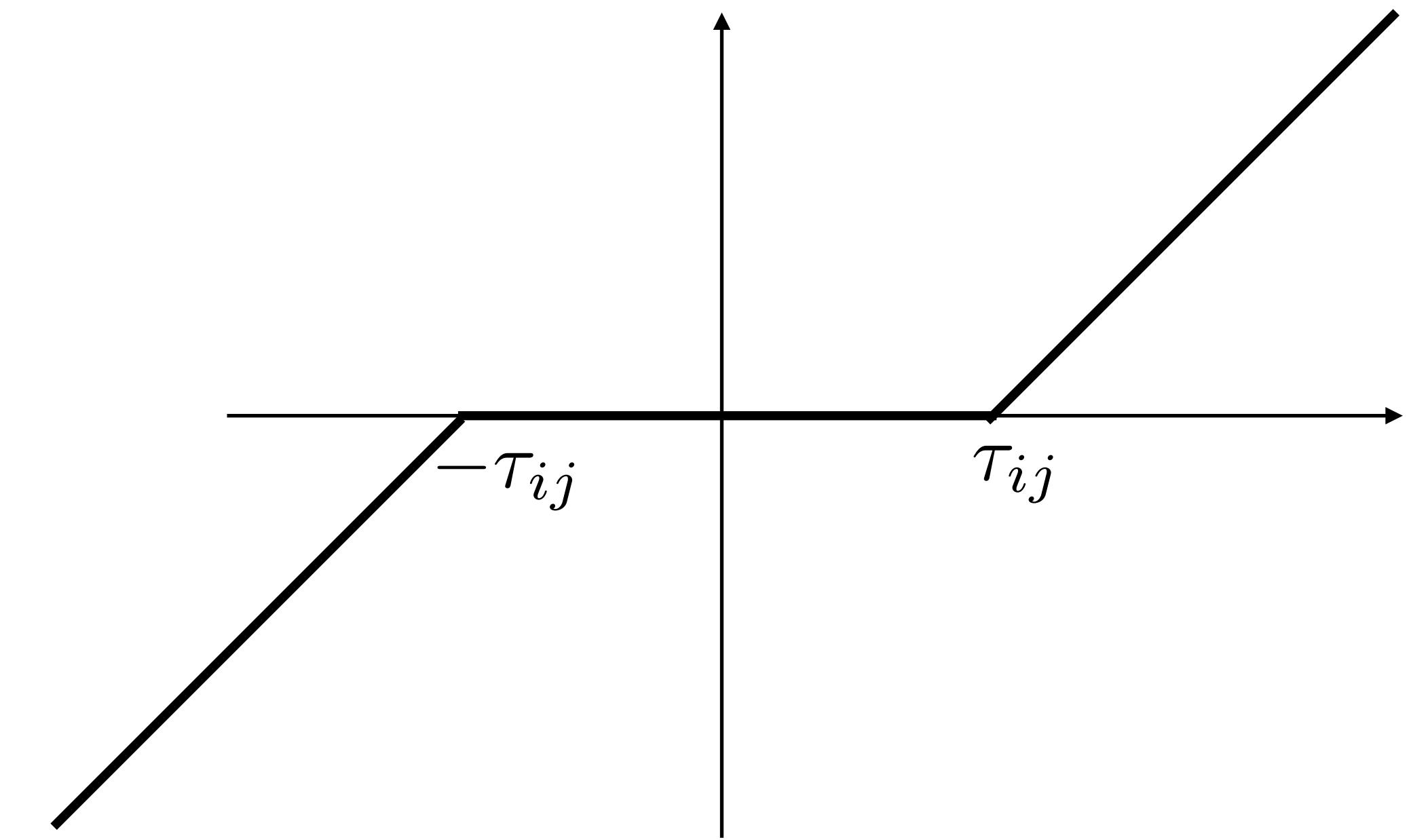


$$Q_{ij}(P) = \frac{\pi r_{ij}^4}{8\eta\ell} \left(P_i - P_j - \frac{l\tau_y}{r_{ij}} \right)_+$$

Darcy's law for the Bethe lattice

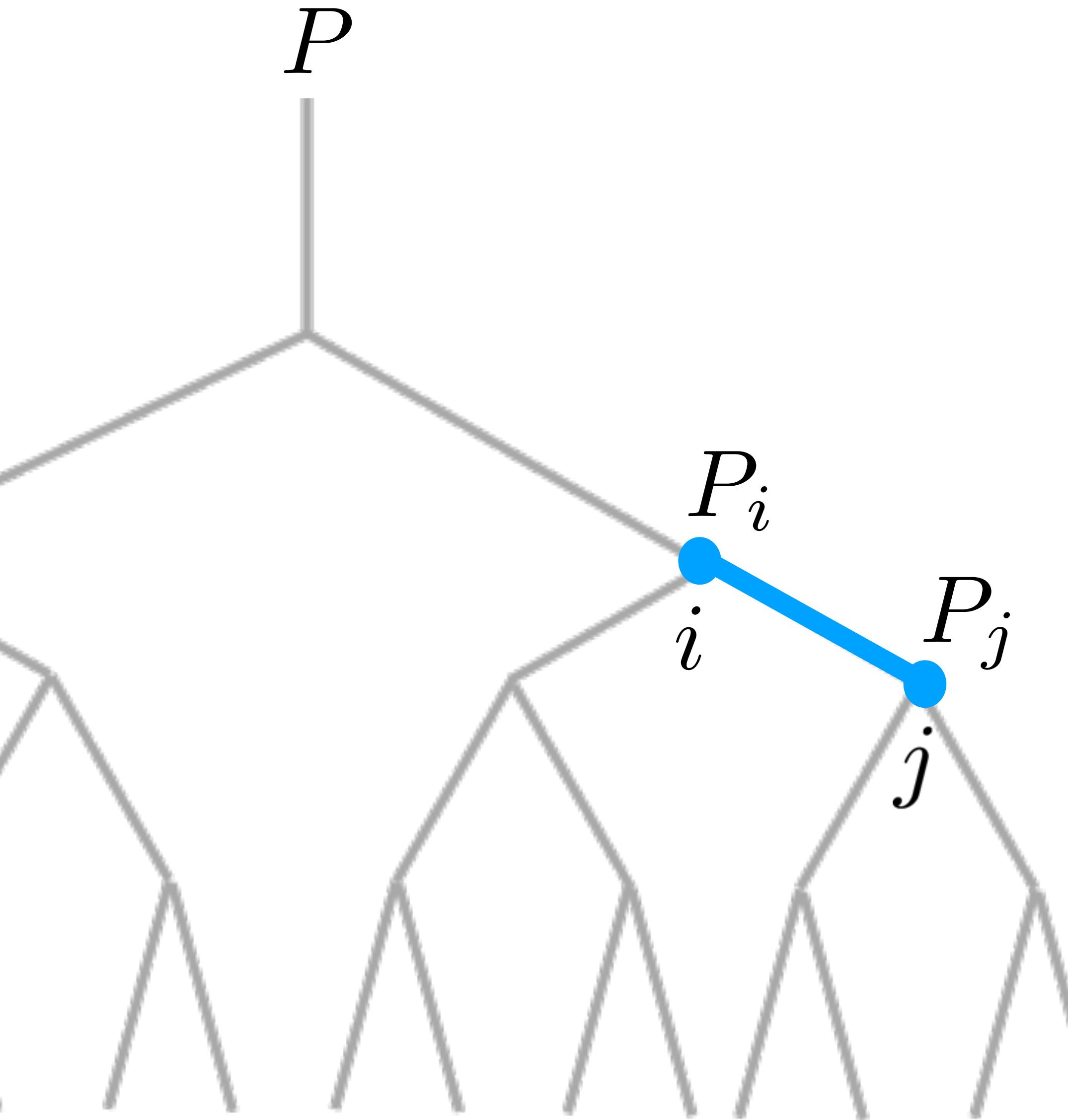


$$Q_{ij}(P) = (P_i - P_j - \tau_{ij})_+$$



τ_{ij} drawn from $p(\tau)$

Darcy's law for the Bethe lattice

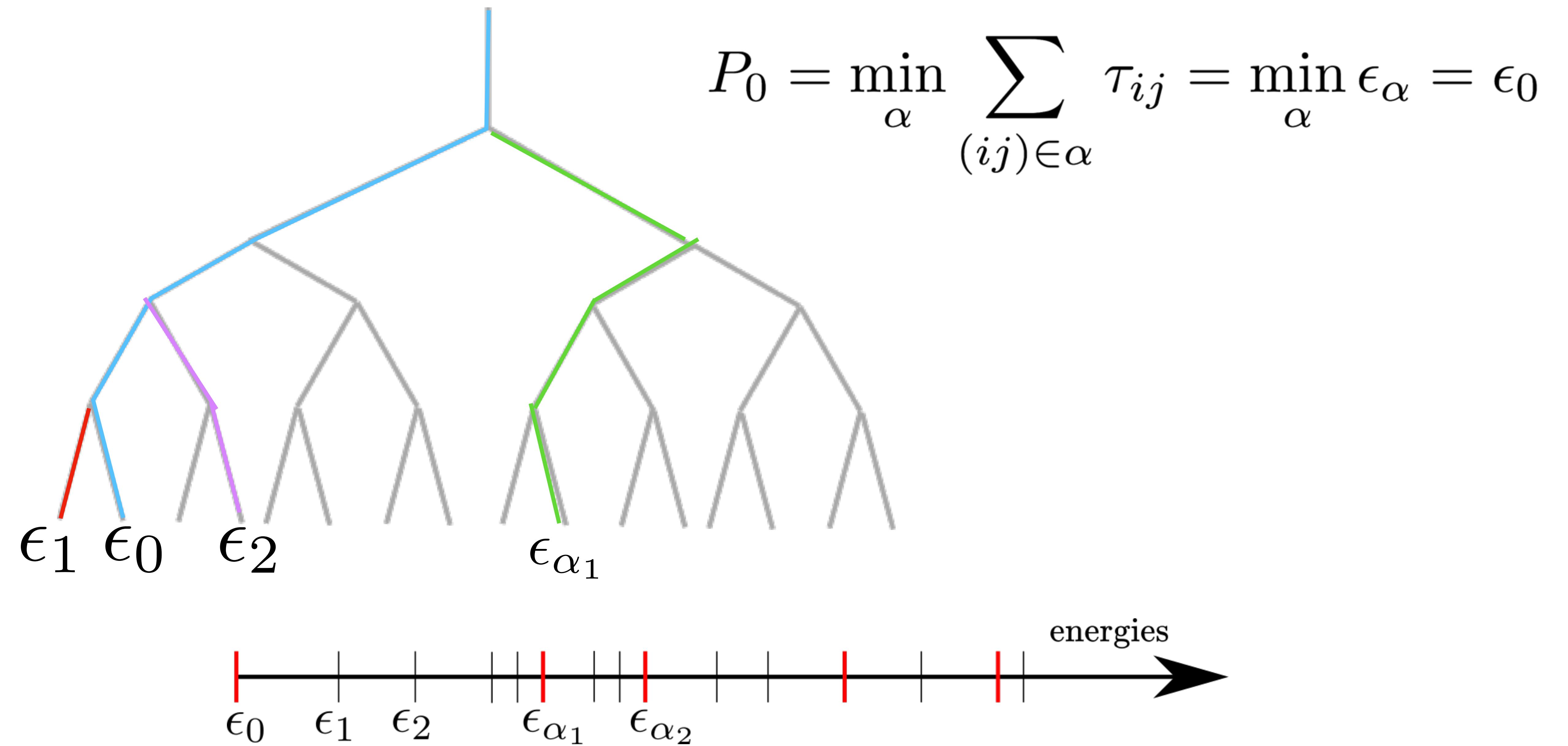


$$Q_{ij}(P) = (P_i - P_j - \tau_{ij})_+$$

$$\sum_{j \in n(i)} Q_{ij}(P) = 0$$

Kirchhoff non-linear problem

Mapping to the directed polymer problem



Exact results 1: the ground state energy (KPP approach)

$$P_0 = \epsilon_0 = -c(\beta_c)t + \frac{3}{2\beta_c} \log t + \chi$$

fluctuations

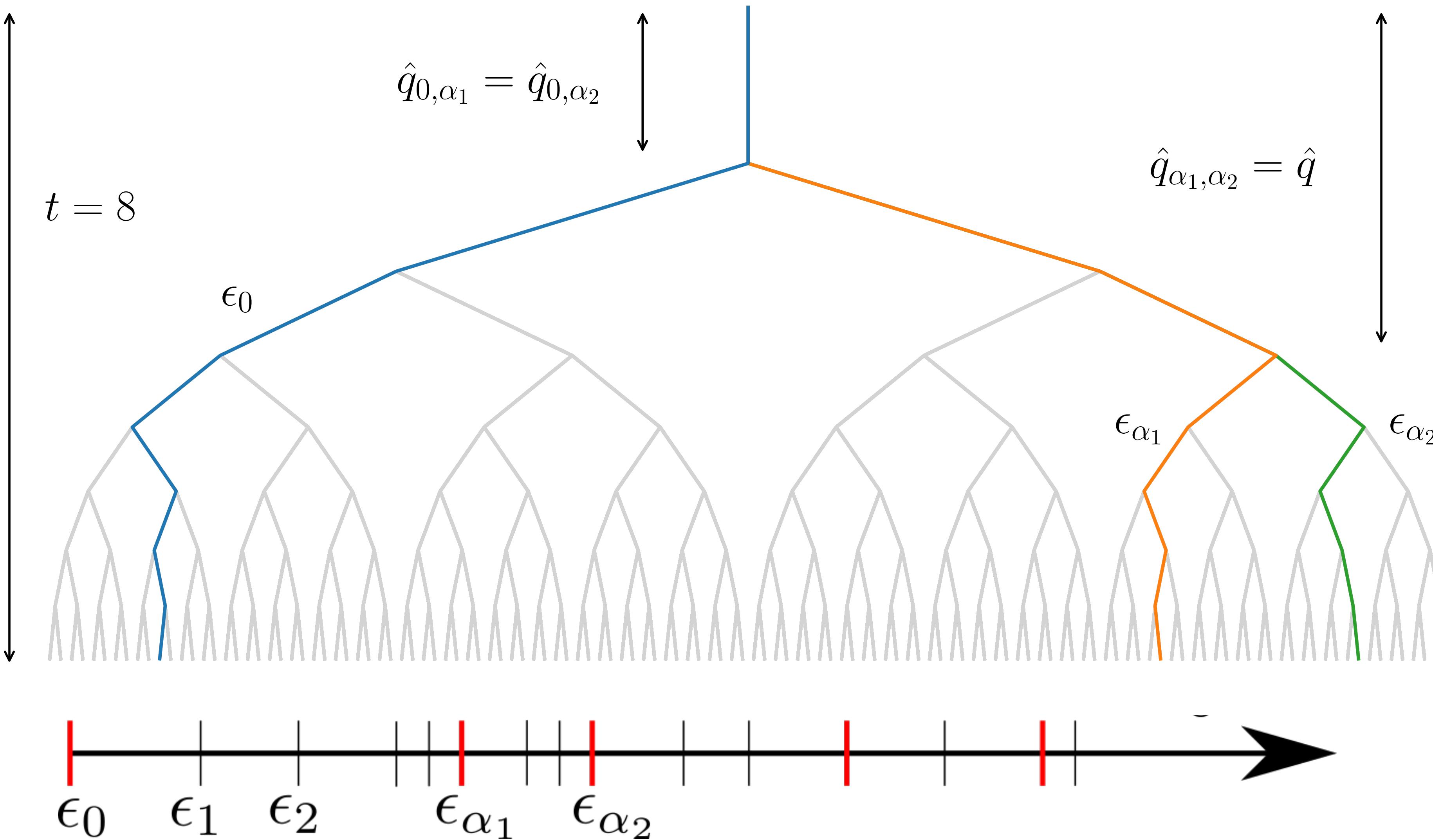
The diagram shows the expression for the ground state energy $P_0 = \epsilon_0$. It is decomposed into three parts: a linear term $-c(\beta_c)t$, a logarithmic term $\frac{3}{2\beta_c} \log t$, and a constant term χ . The logarithmic term and the constant term are grouped together under a large black bracket at the bottom, labeled "Extensive and sub-extensive parts". A red arrow points from the word "fluctuations" above the expression to the constant term χ , which is enclosed in a red circle.

$c(\beta_c), \beta_c$ depends on $p(\tau)$, the threshold distribution

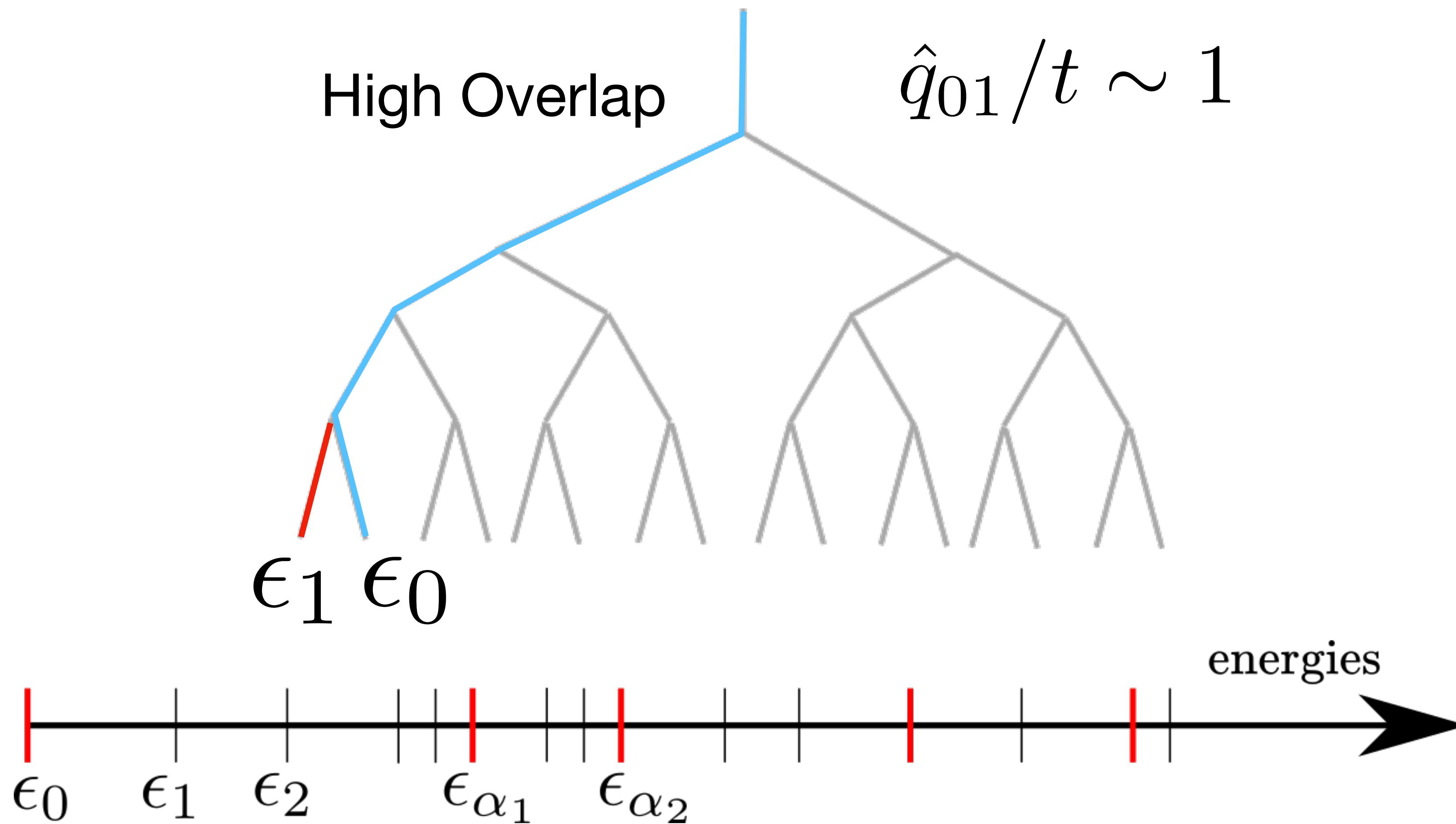
Refs: Derrida, Spohn, Brunet, Majumdar and Mottishaw

Sequence of subsequent channels: energy and overlaps

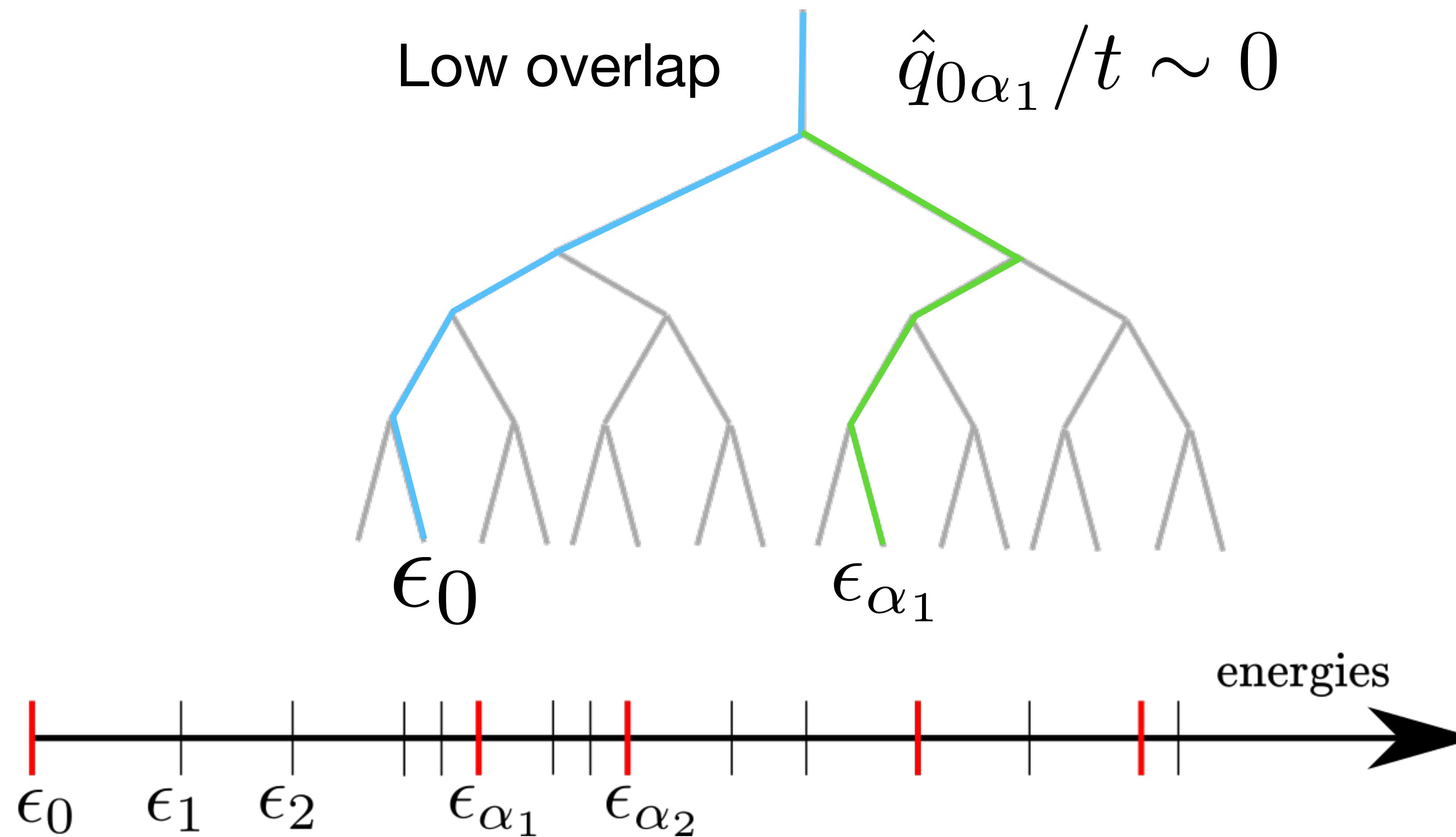
$$P_1 = \epsilon_0 + \min_{\alpha \neq 0} \frac{\epsilon_\alpha - \epsilon_0}{1 - \hat{q}_{0\alpha}/t}$$



$$P_1 = \epsilon_0 + \min_{\alpha \neq 0} \frac{\epsilon_\alpha - \epsilon_0}{1 - \hat{q}_{0\alpha}/t} \sim \epsilon_0 + t(\epsilon_1 - \epsilon_0)$$

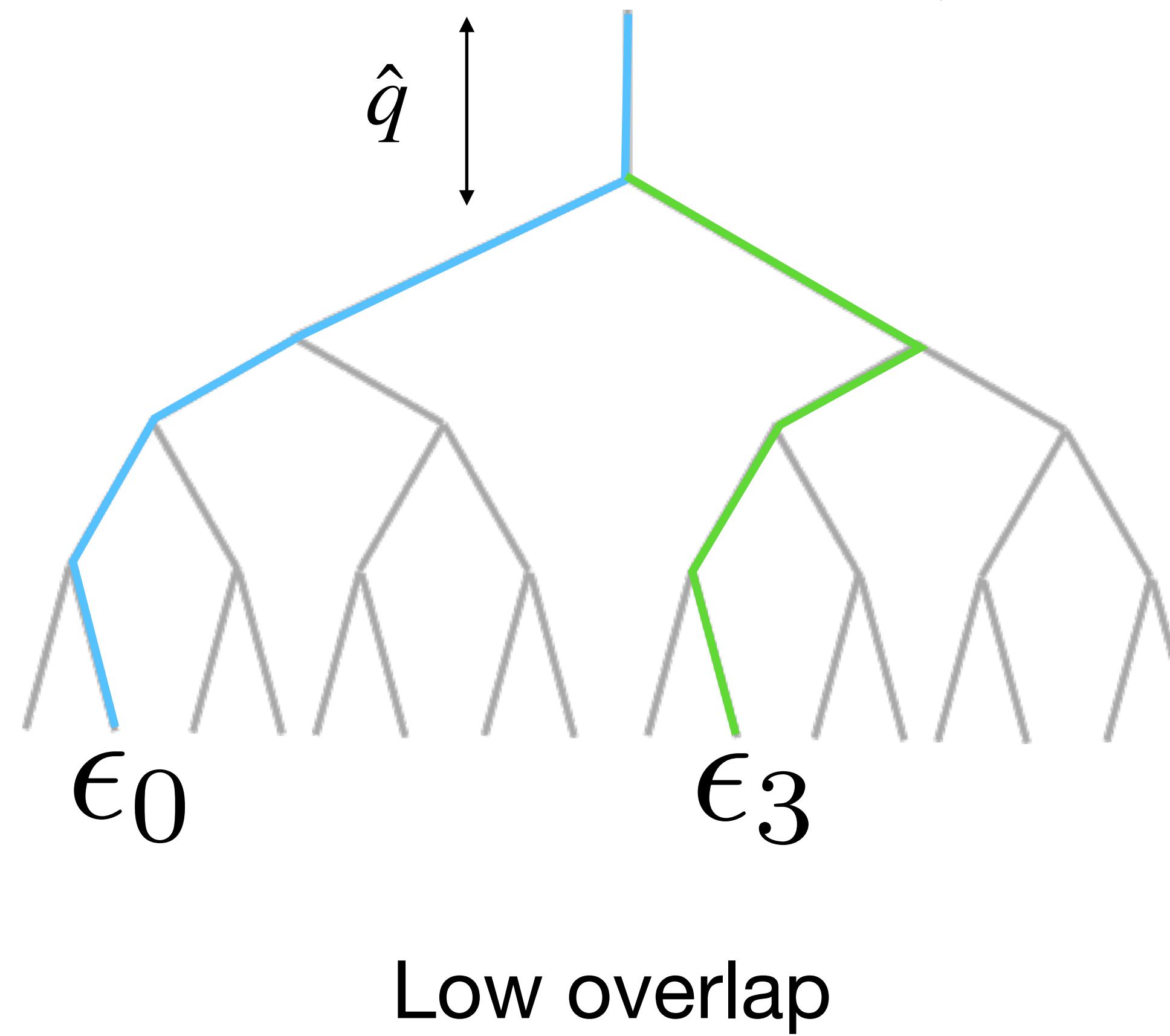


$$P_1 = \epsilon_0 + \min_{\alpha \neq 0} \frac{\epsilon_\alpha - \epsilon_0}{1 - \hat{q}_{0\alpha}/t} \sim \epsilon_0 + (\epsilon_{\alpha_1} - \epsilon_0) = \epsilon_{\alpha_1}$$

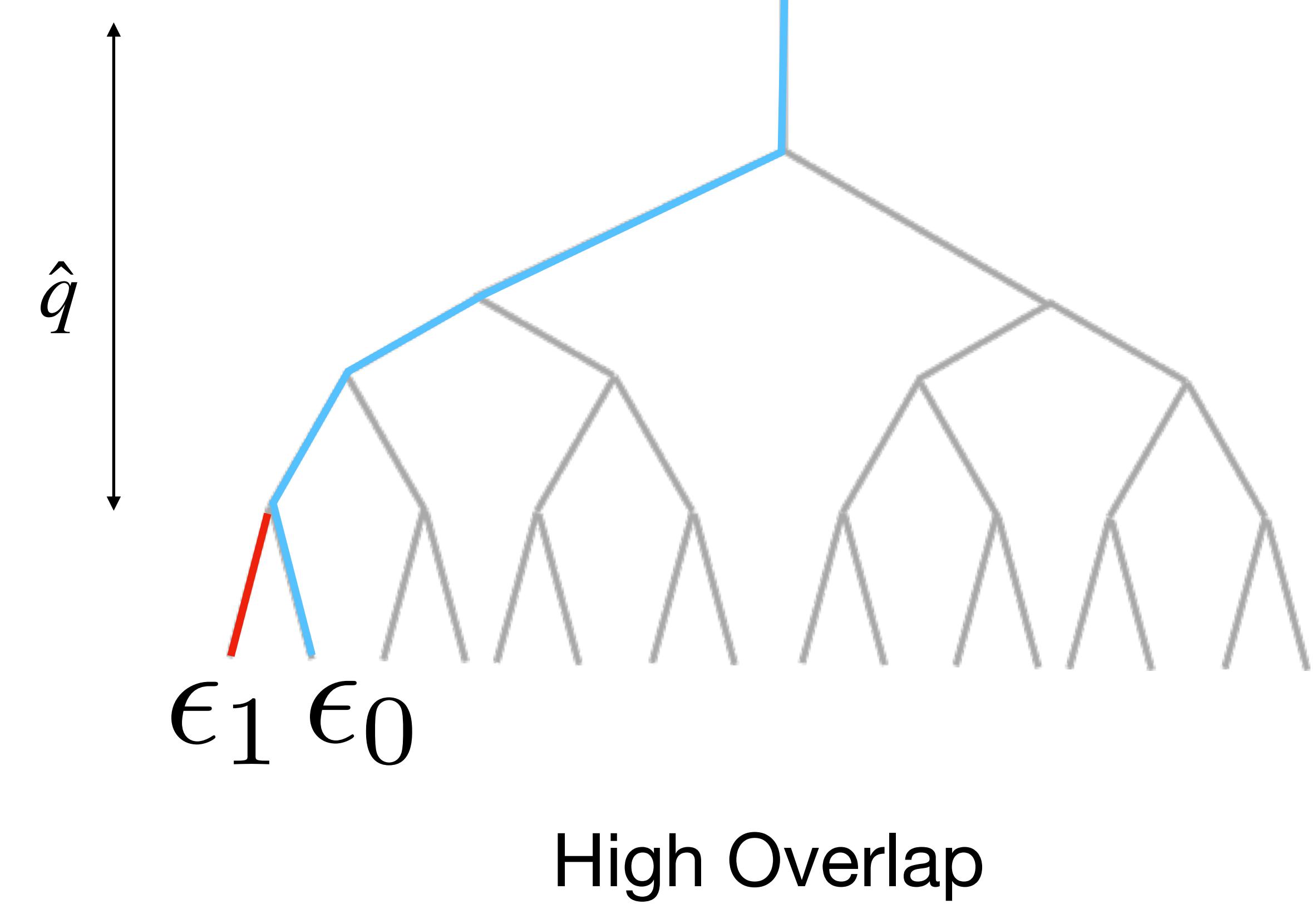


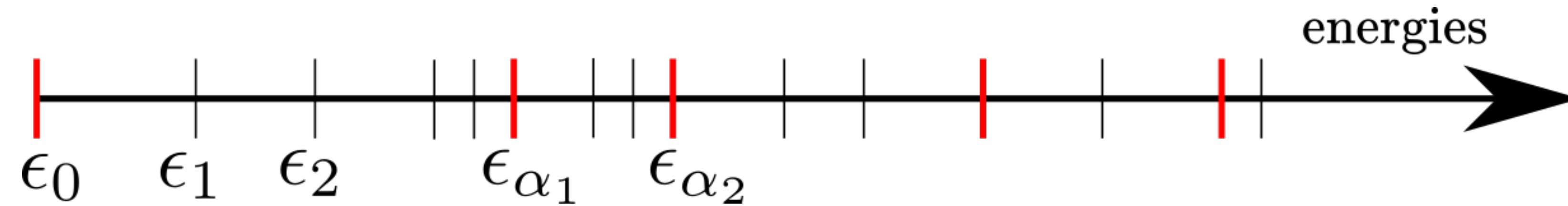
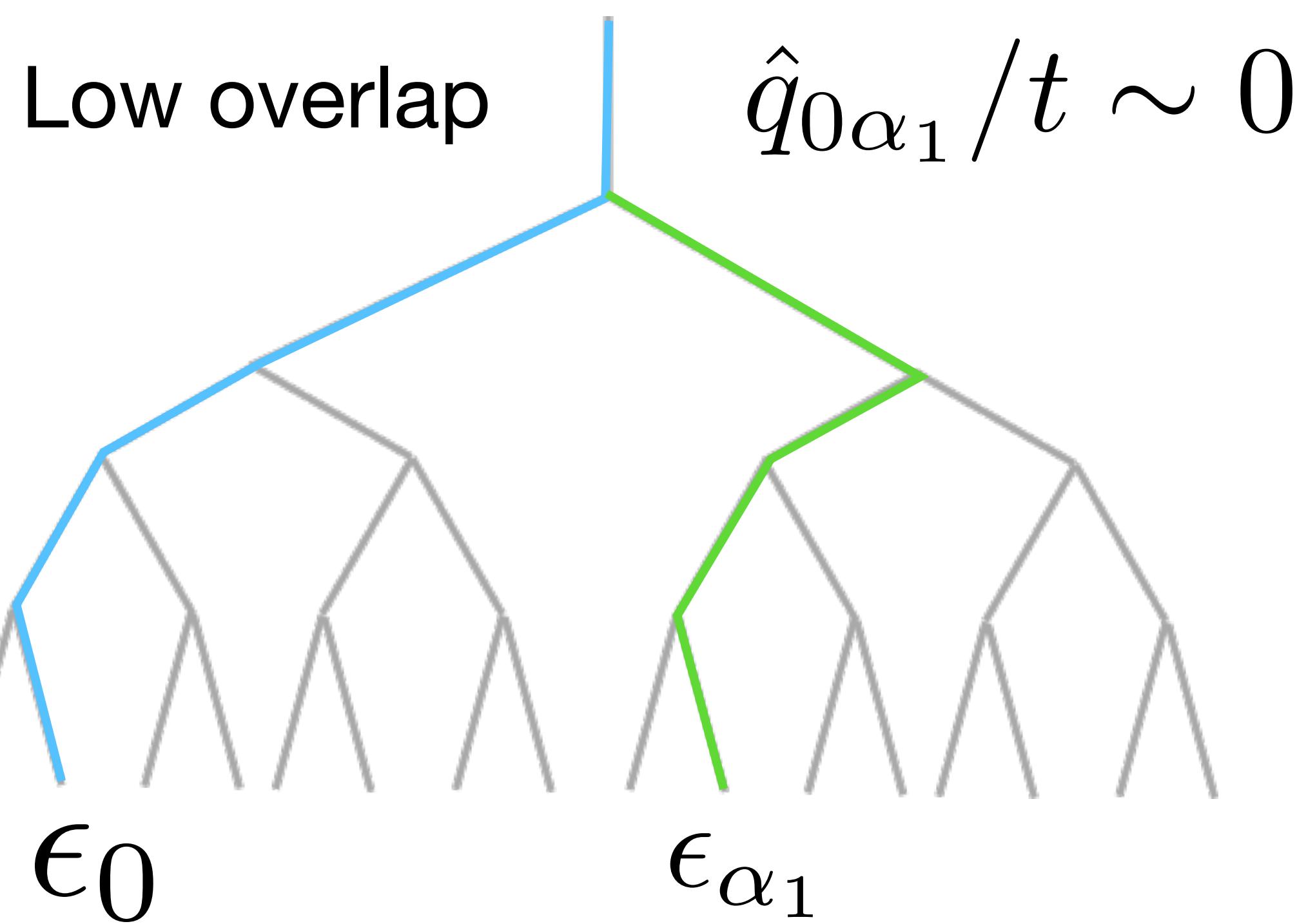
Exact results 2: the low energy excited states. One step RSB

$$P\left(q = \frac{\hat{q}}{t}\right) = Ty\delta(q) + (1 - Ty)\delta(q - 1)$$



Refs: Derrida, Spohn J Phys Stat (1988)





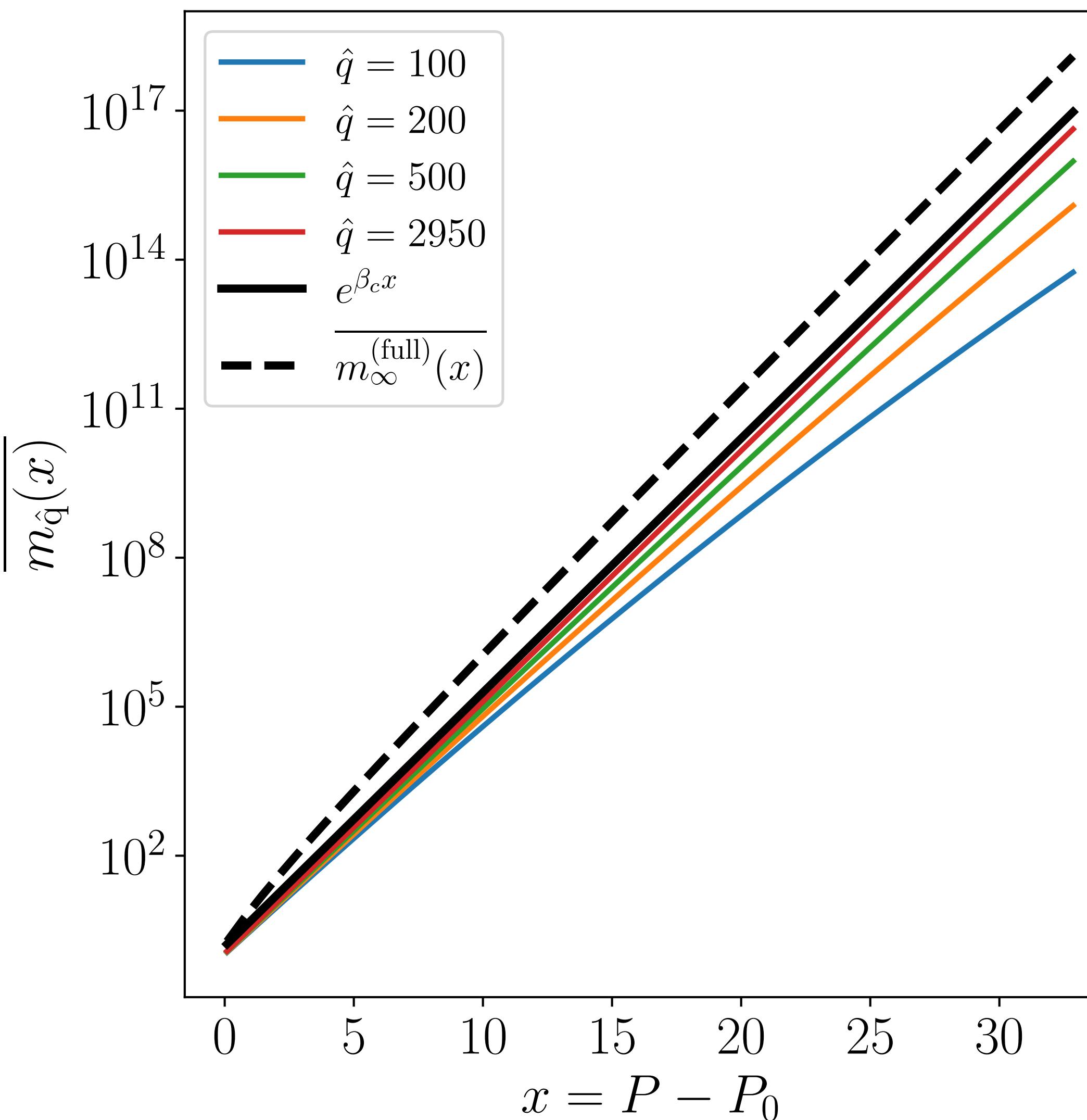
the sequence of channels are the low overlap and low energy excitations!

Pressures and flow in the low overlap limit (t very large)

$$P_1 = \epsilon_0 + \min_{\alpha \neq 0} \frac{\epsilon_\alpha - \epsilon_0}{1 - \hat{q}_{0\alpha}/t} = \epsilon_0 + \frac{\epsilon_{\alpha_1} - \epsilon_0}{1 - \hat{q}_{0\alpha_1}/t} = \epsilon_{\alpha_1}$$

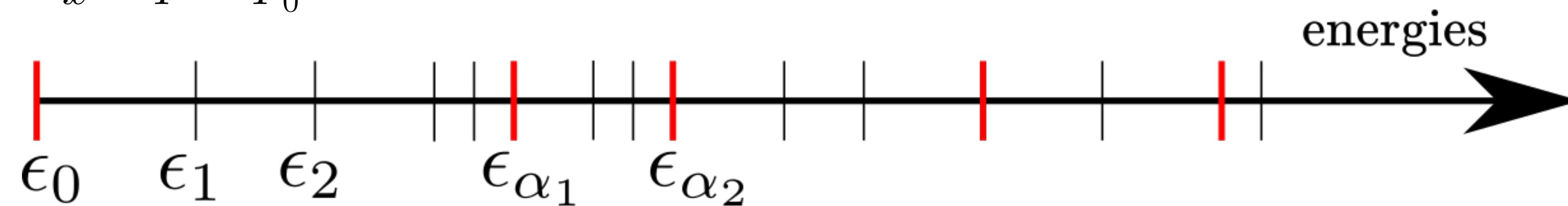
$$Q_1(P) \approx \frac{P - P_c}{t} + \frac{P - P_1}{t} \longrightarrow \text{Additive independent channels}$$

Adapting KPP approach by Deridda & Brunet



$$\overline{m}_\infty^{\text{full}}(P) \stackrel{P \gg P_0}{=} A(P - P_0) e^{\beta_c(P - P_0)}$$

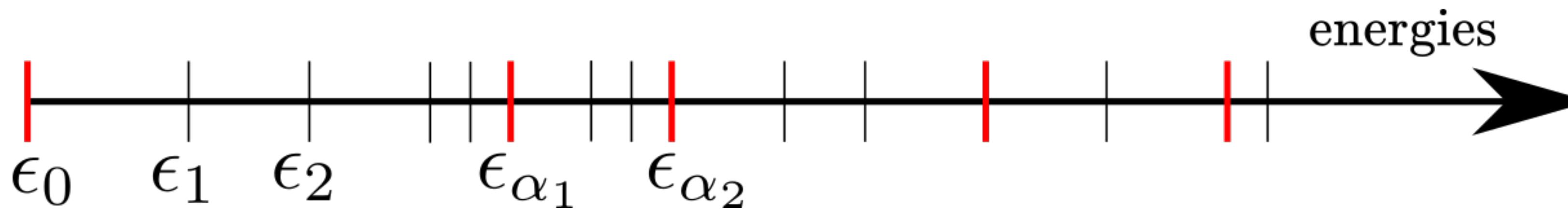
$$n_\infty^{\text{ch}}(P) = \lim_{\hat{q} \rightarrow \infty} \overline{m}_{\hat{q}}(P) = e^{\beta_c(P - P_0)}$$



Main results: first two flow regimes

$$n_\infty^{\text{ch}}(P) = \lim_{\hat{q} \rightarrow \infty} \overline{m_{\hat{q}}(P)} = e^{\beta_c(P - P_0)}$$

$$\overline{Q_t(P)} = \frac{e^{\beta_c(P - P_0)} - 1}{\beta_c t} = \frac{P - P_0}{t} + O((P - P_0)^2)$$



Main results: Large linear flow regime

$$Q_t(P) = \kappa(P - P^*) \quad \text{with } P \rightarrow \infty$$

$$\kappa = \frac{1}{2} \qquad \qquad P^* = \bar{\tau}t$$

Initial linear flow regime

$$\kappa_{\text{eff}}(P) = \frac{1}{t} \qquad \qquad P_{\text{eff}}^*(P) = -c(\beta_c)t$$

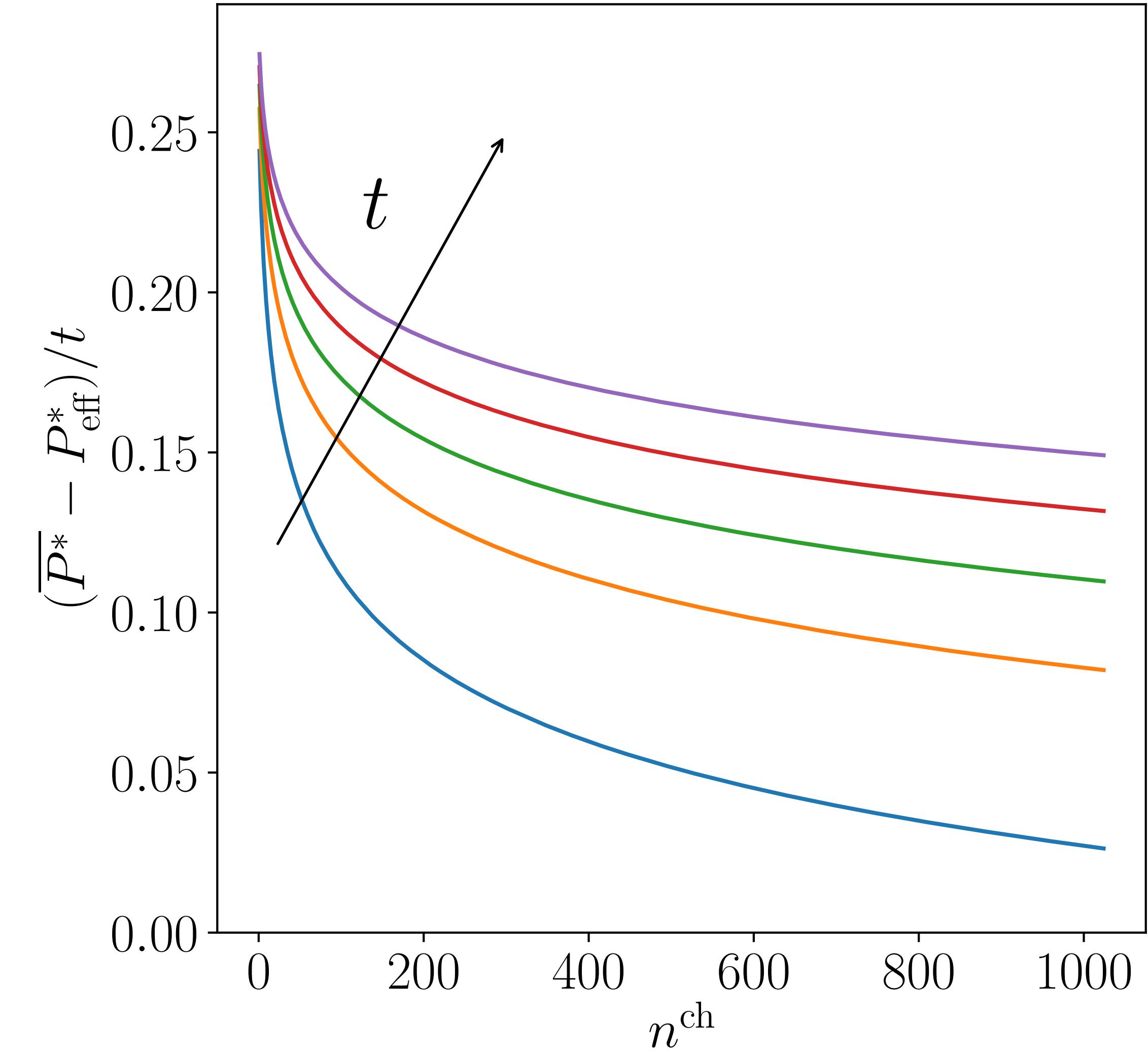
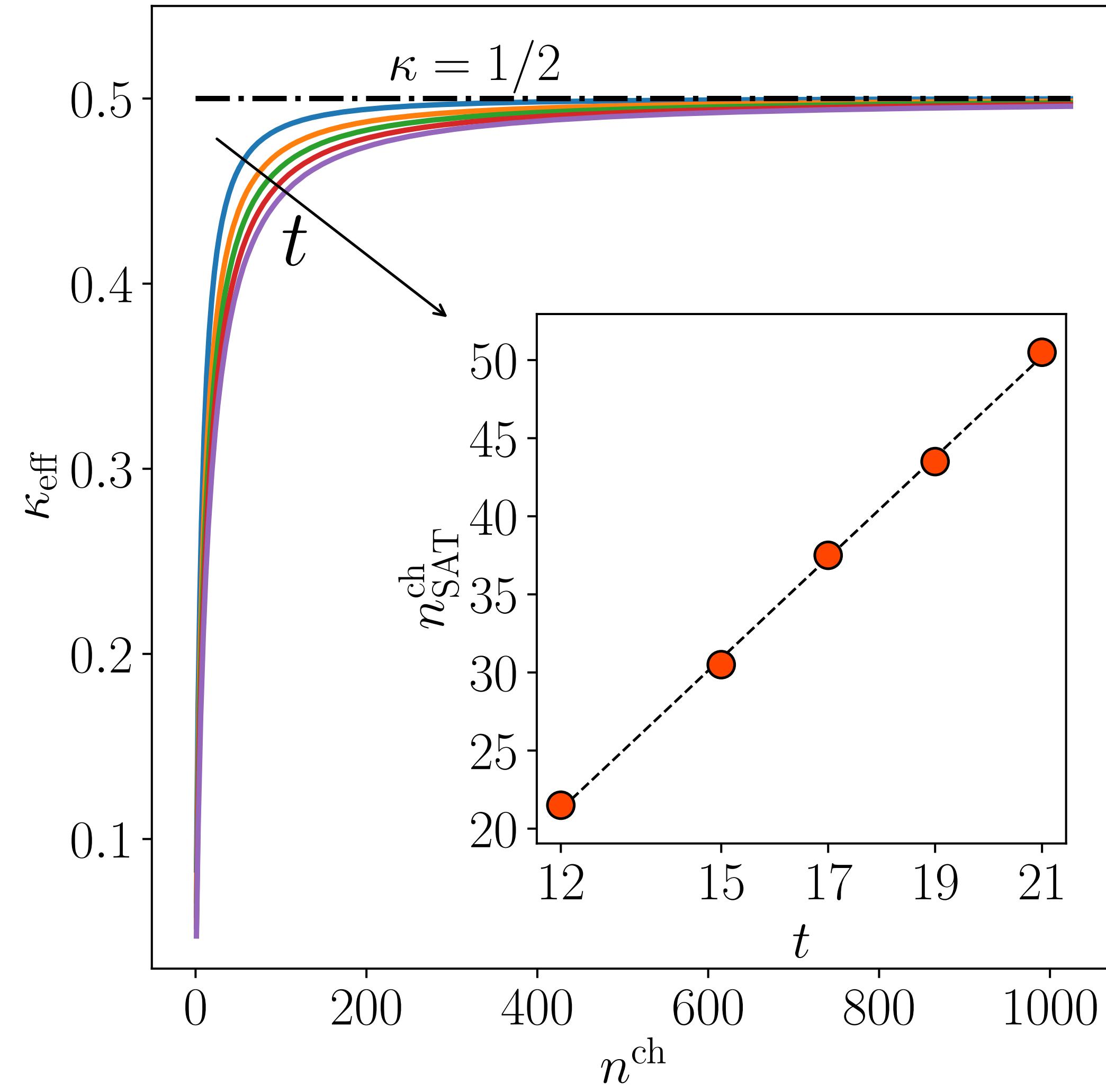
Saturation pressure by matching the two limits

$$Q(P) = \begin{cases} \frac{e^{\beta_c(P - P_0)} - 1}{t\beta_c} & P_0 < P < P_{\text{sat}} \\ \frac{1}{2}(P - P^*) & P \gg P_{\text{sat}} \end{cases}$$

$$P_{\text{sat}} = P_0 + (1/\beta_c) \ln t$$

$$n_{\text{sat}}^{ch} \sim t/2$$

Numerical simulation for $t=12, \dots, 21$



Conclusions and Perspectives

- Explicit determination of κ , P_0 , P^* and of the non-linear regime for the Bethe lattice
- P_{sat} is reached after the opening of the first t channels
- Darcy law in finite dimension (role of the low overlap excitations)

$$Q(P) = \begin{cases} \frac{e^{\beta_c(P - P_0)} - 1}{t\beta_c} & P_0 < P < P_{\text{sat}} \\ \frac{1}{2}(P - P^*) & P \gg P_{\text{sat}} \end{cases}$$