

Random energy model in a pure ferromagnet

Hajime Yoshino

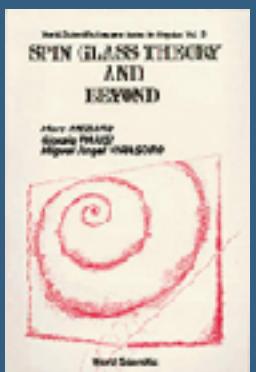
Cybermedia Center, Osaka Univ.

Thanks to K. Mitsumoto, A. G. Cavaliere, T. Rizzo, Y. R. Hamano, Y. Tomita

[Q] What is self-generated randomness?

Glass forming systems

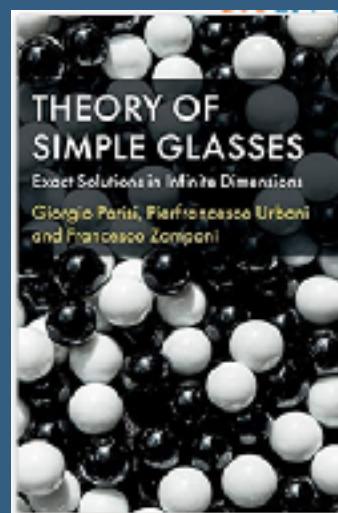
- I. glassy systems with “permanent” extrinsic disorder



spin-glass, vortex glass, pinned CDW, ...

2. glassy systems without permanent extrinsic disorder... disorder is self-generated (?)

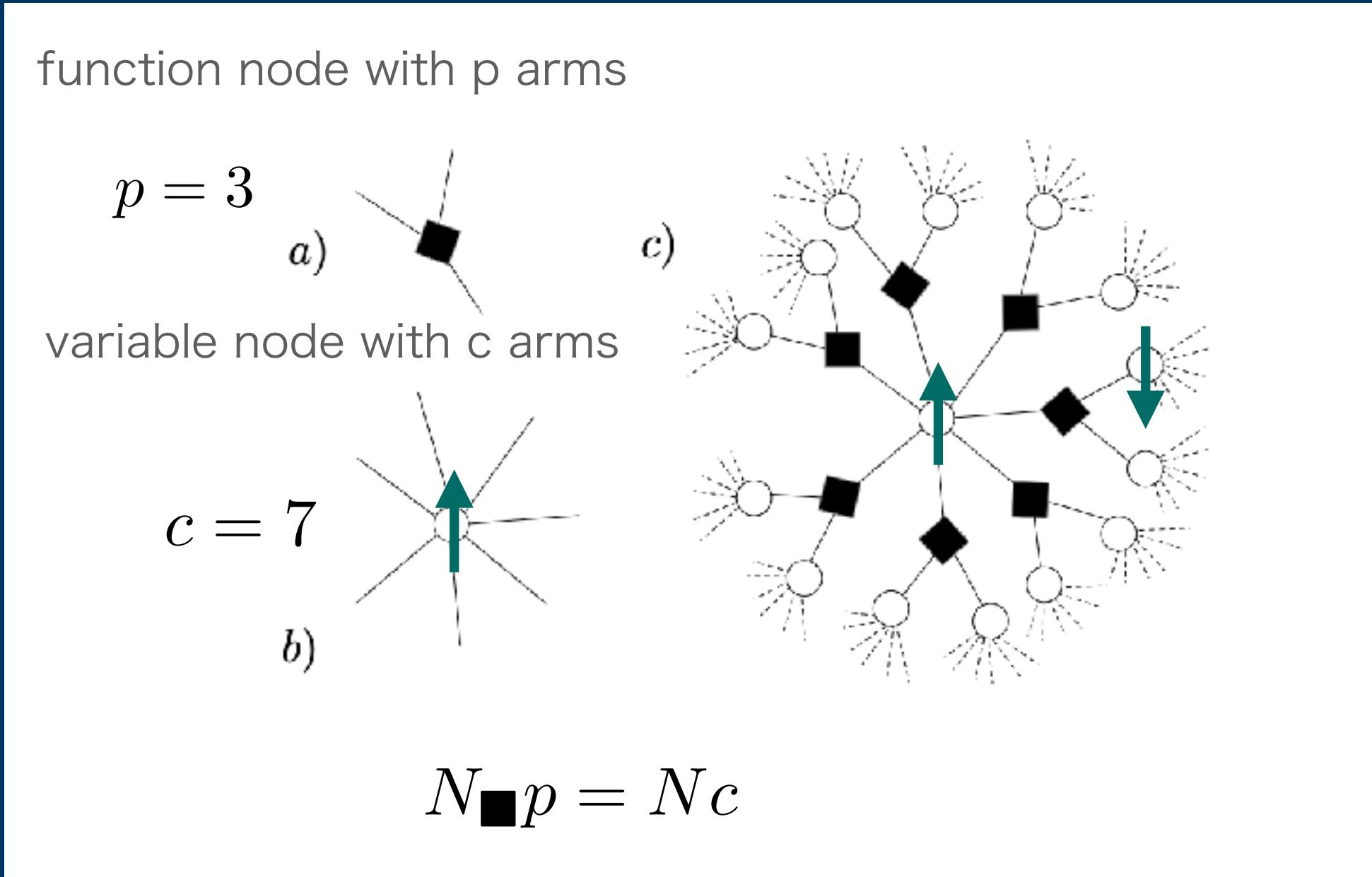
structural glasses



glass forming frustrated magnets
see K. Mitsumoto's talk

Introduction

p-spin model



a) Ferromagnet $J_{\blacksquare} = J$

b) Spinglass J_{\blacksquare} Gaussian with zero-mean and unit variance

..played very important roles in development of
1st principle mean-field theories for glasses
Kirkpatrick-Thirumalai-Wolynes,
Cugliandolo-Kurchan, Frans-Parisi, Monsasson,....

A) Dense limit

$$N \gg c \gg 1$$

contribution of loops can be neglected...Bethe lattice
saddle point computations become possible

1.“Random” graph

2.“Regular” graph

B) Global coupling

$$c \propto N^{p-1}$$

$$H = - \sum_{\blacksquare=1}^{N_{\blacksquare}} \frac{J_{\blacksquare}}{\sqrt{c/\alpha}} \prod_{i \in \partial \blacksquare} S_i$$

Ising $S_i = \pm 1$

$$\text{spherical} \quad \sum_{i=1}^N S_i^2 = N$$

Random Energy Model (REM) and p -spin Spin-Glass Models

B. Derrida, PRL 45, 79 (1980)

- b) Spinglass J_{\blacksquare} Gaussian with zero-mean and unit variance
- B) Global coupling $c \propto N^{p-1}$

$$\begin{aligned} P(E) &= \left\langle \delta \left(E + \frac{J_{\blacksquare}}{\sqrt{c/\alpha}} \sum_{\blacksquare} \prod_{j \in \partial \blacksquare} S_j \right) \right\rangle_{J_{\blacksquare}} = \int \frac{d\kappa}{2\pi} e^{i\kappa E} \left\langle e^{i\kappa \frac{J_{\blacksquare}}{\sqrt{c/\alpha}} \sum_{\blacksquare} \prod_{j \in \partial \blacksquare} S_j} \right\rangle_{J_{\blacksquare}} \\ &= \int \frac{d\kappa}{2\pi} e^{i\kappa E - \frac{\kappa^2}{2} \gamma J^2 N} = \frac{e^{-\frac{E^2}{2\gamma NJ^2}}}{\sqrt{2\pi\gamma NJ^2}} \end{aligned}$$

$$\langle \dots \rangle_J = \int dJ_{\blacksquare} \frac{e^{-J_{\blacksquare}^2/2}}{\sqrt{2\pi}} \dots$$

$$P(E_1, E_2; q) \rightarrow_{p \rightarrow \infty} P(E_1)P(E_2) \quad \text{for} \quad 0 \leq |q| < 1$$

Random Energy Model

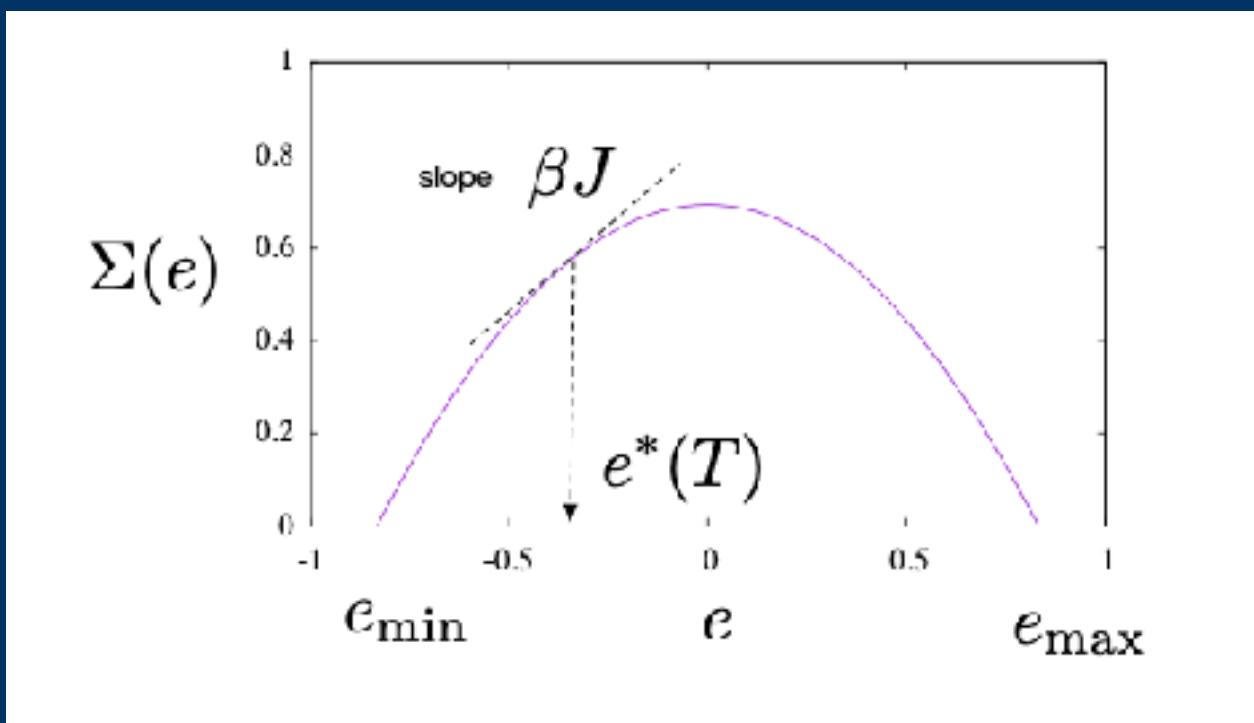
B. Derrida, PRL 45, 79 (1980)

Model

For $\alpha = 1, 2, \dots, 2^N$ choose e_α drawn from

$$p(e) = \frac{1}{\sqrt{2\gamma\pi N}} e^{-\frac{Ne^2}{2\gamma}} \quad \gamma = \alpha/p$$

Thermodynamics



$$e_{\min} = -\sqrt{2\gamma \ln 2} \quad e_{\max} = \sqrt{2\gamma \ln 2}$$

$$Z = \sum_{\alpha=1}^{2^N} e^{-N\beta J e_\alpha} = \int de e^{N(\Sigma(e) - (\beta J)e)} \sim e^{N(\Sigma(e^*) - (\beta J)e^*)}$$

complexity
(entropy)

$$\Sigma(e) = \frac{1}{N} \ln \sum_{\alpha} \delta(e - e_{\alpha}) = \frac{1}{N} \ln \left(2^N p(e) \right) \sim \ln 2 - \frac{e^2}{2\gamma}$$

spin-glass transition (or Kauzmann transition)

$$k_B T_K = J \sqrt{\frac{\gamma}{2 \ln 2}}$$

replica theory with 1RSB recovers these results

peculiarity: $q_{\text{EA}} = 1$ always jammed, no melting (dynamical transition)

REM in the ferromagnetic p-spin model

Ferromagnetic p-spin model with dense/global coupling

a) Ferromagnet

$$J_{\square} = J$$

A) Dense coupling / B) Global coupling

Ferromagnetic p-spin model

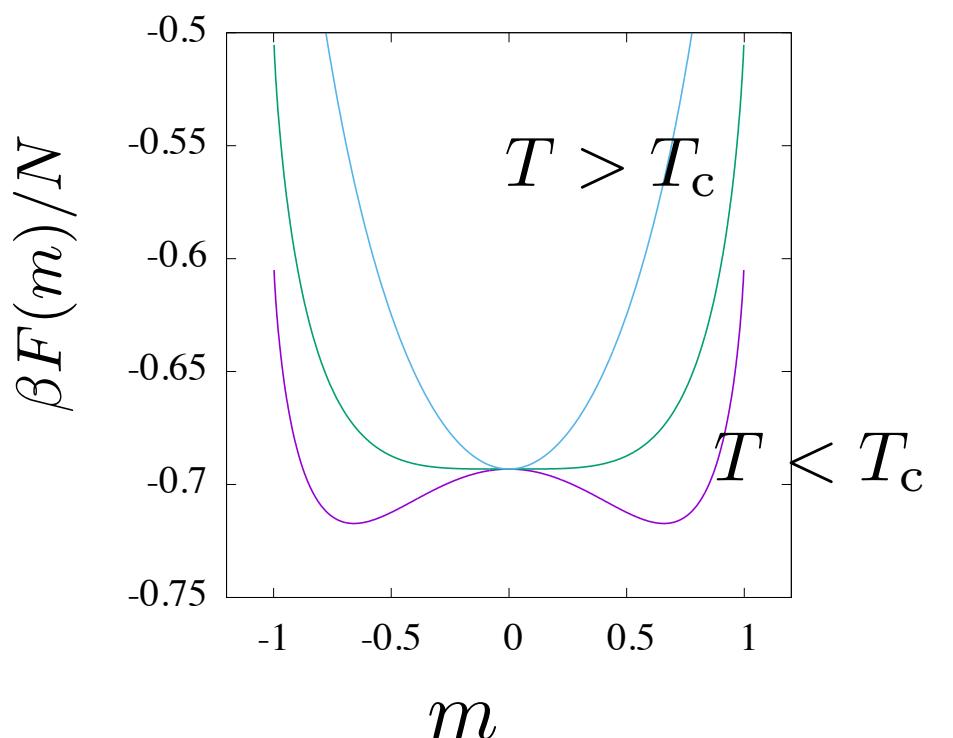
$$H = - \sum_{\square=1}^{N_{\square}} \frac{J}{\sqrt{c/\alpha}} \prod_{i \in \partial \square} S_i$$

ferromagnetic order parameter

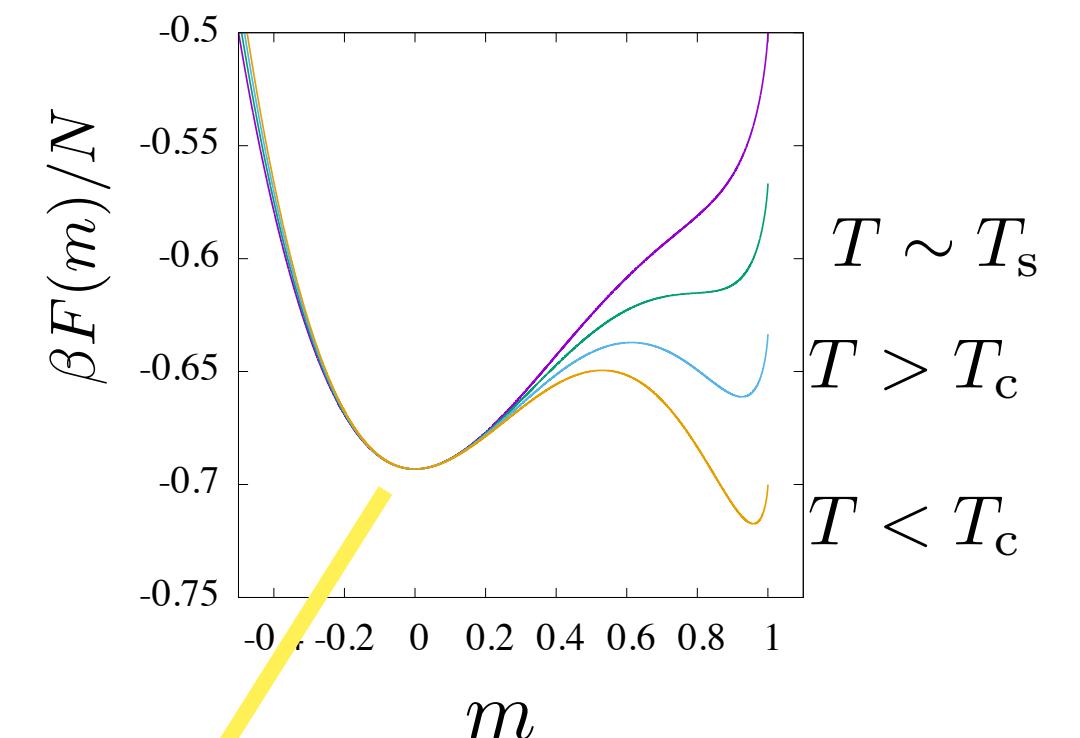
$$m = \frac{1}{N} \sum_{i=1}^N \langle S_i \rangle$$

Free-energy landscape with dense or global coupling

a) $p = 2$

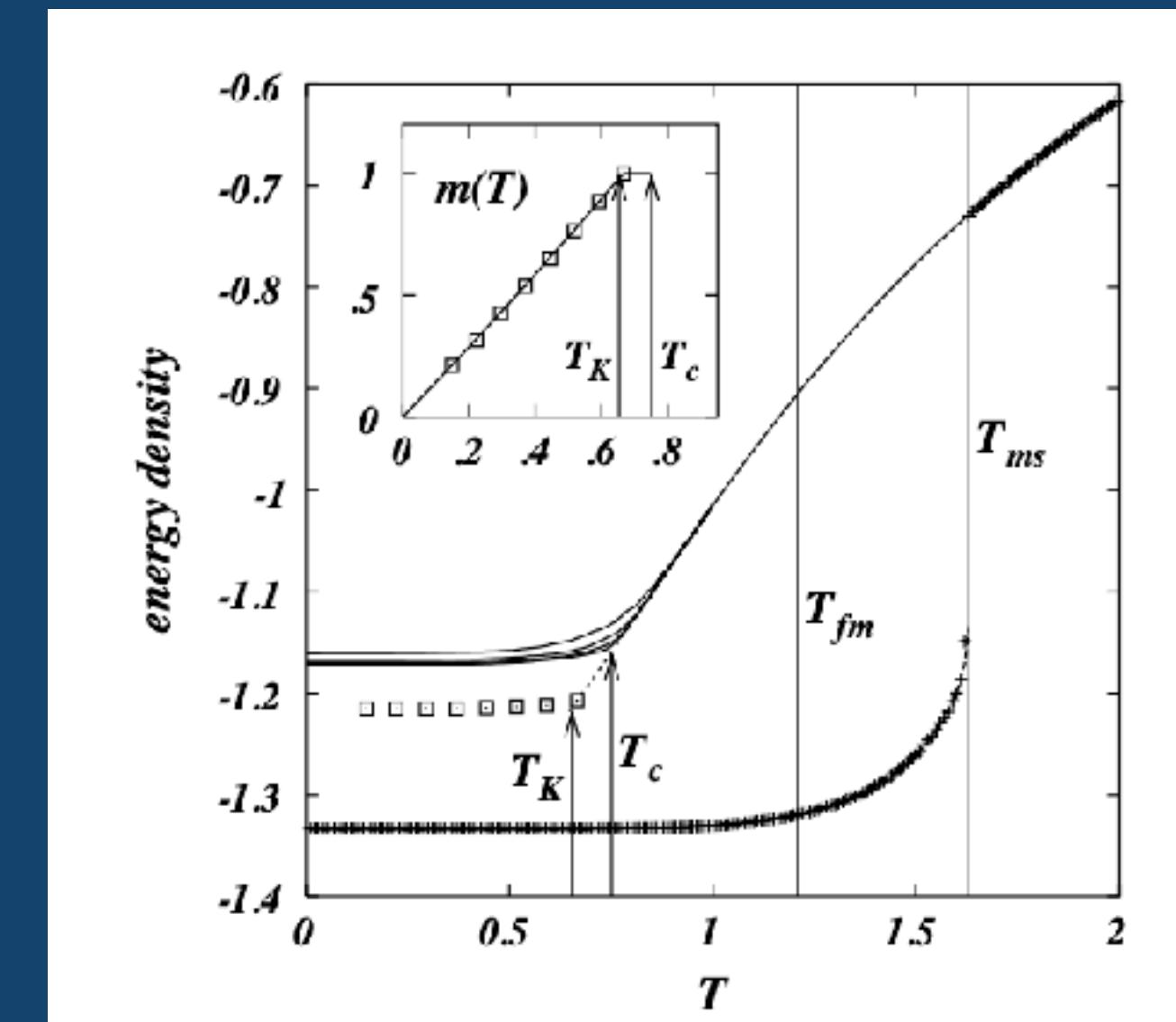


b) $p = 3$



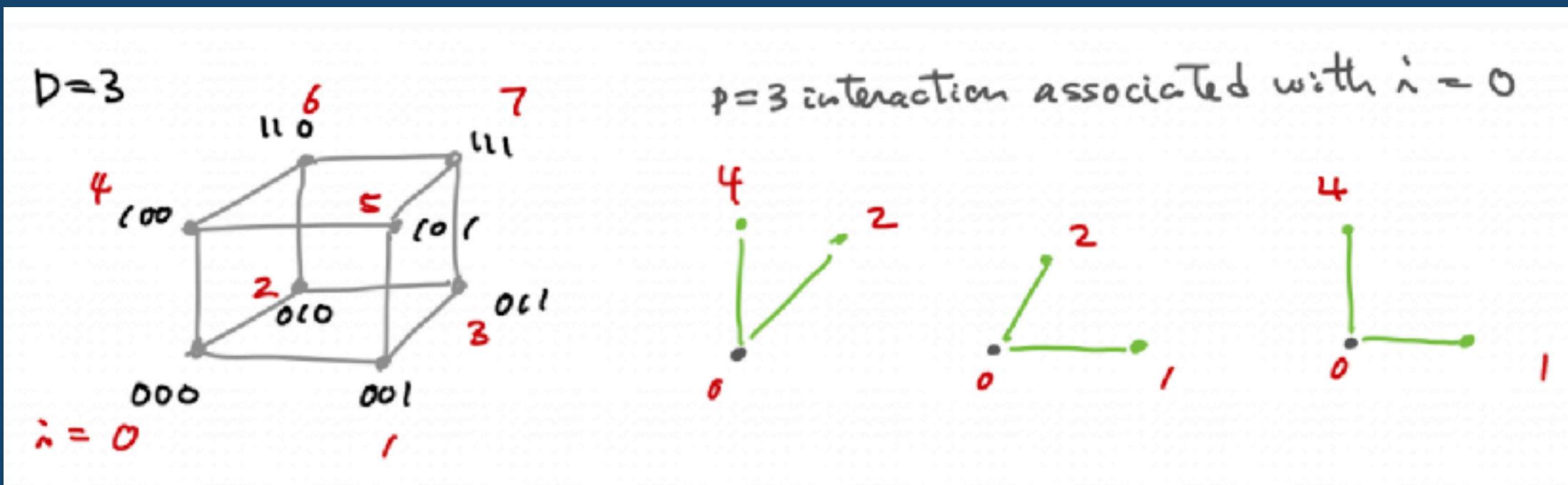
Glass transition in
super-cooled paramagnetic phase?

Glass transition has been found ferromagnetic p=3 model on sparse “random” graph



Can we get rid of “randomness” completely?

EX. “Regular” graphs on D-dim hyper-cube



A. G. Cavaliere, T. Rizzo and HY in progress

$$N = 2^D$$
$$c = \frac{3}{2}D(D - 1)$$

Distribution of energy in the ferromagnetic p-spin model

$$J_{\blacksquare} = J$$

HY, SciPostPhys 4, 6 (2018)

HY (in preparation)

$$\begin{aligned} P(E) &= \left\langle \delta \left(E + \frac{J}{\sqrt{c/\alpha}} \sum_{\blacksquare} \prod_{j \in \partial \blacksquare} S_j \right) \right\rangle_S = \int \frac{d\kappa}{2\pi} e^{i\kappa E} \left\langle e^{i\kappa \frac{J}{\sqrt{c/\alpha}} \sum_{\blacksquare} \prod_{j \in \partial \blacksquare} S_j} \right\rangle_S \\ &= \int \frac{d\kappa}{2\pi} e^{i\kappa E - \frac{\kappa^2}{2} \gamma J^2 N} = \frac{e^{-\frac{E^2}{2\gamma NJ^2}}}{\sqrt{2\pi\gamma NJ^2}} \end{aligned}$$

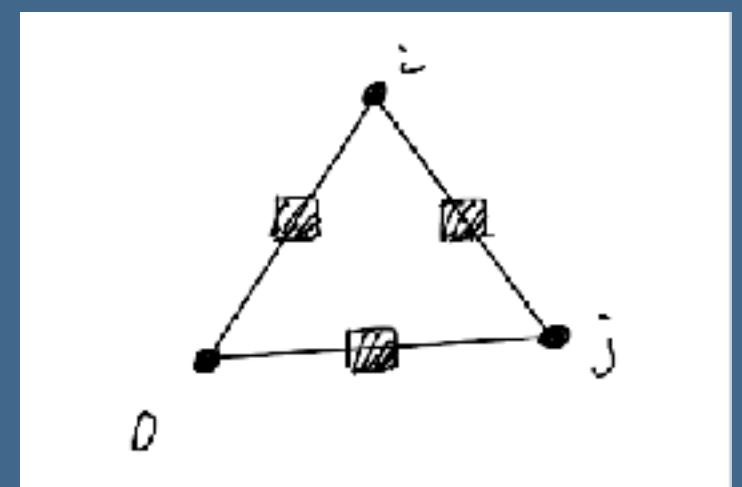
$$\langle \dots \rangle_S = \frac{\prod_{i=1}^N \sum_{S_i=\pm 1} \dots}{2^N}$$

$$\begin{aligned} \ln \left\langle e^{i\kappa \frac{J}{\sqrt{c/\alpha}} \sum_{\blacksquare} \prod_{j \in \partial \blacksquare} S_j} \right\rangle_S &= i\kappa \sum_{\blacksquare} \frac{J}{\sqrt{c/\alpha}} \left\langle \prod_{j \in \partial \blacksquare} S_j \right\rangle_S - \frac{\kappa^2}{2} \frac{J^2}{c/\alpha} \sum_{\blacksquare_1, \blacksquare_2} \left\langle \prod_{j \in \partial \blacksquare_1} \prod_{k \in \partial \blacksquare_2} S_j S_k \right\rangle_S^c \\ &\quad + \frac{\kappa^4}{3!} \frac{J^3}{(c/\alpha)^{3/2}} \sum_{\blacksquare_1, \blacksquare_2, \blacksquare_3} \left\langle \prod_{j_1 \in \partial \blacksquare_1} \prod_{j_2 \in \partial \blacksquare_2} \prod_{j_3 \in \partial \blacksquare_3} S_{j_1} S_{j_2} S_{j_3} \right\rangle_S^c + \dots \\ &= -\frac{\kappa^2}{2} \frac{J^2}{c/\alpha} \sum_{\blacksquare_1, \blacksquare_2} \delta_{\blacksquare_1, \blacksquare_2} + \frac{\kappa^4}{4!} \frac{J^4}{(c/\alpha)^2} \sum_{\blacksquare_1, \blacksquare_2, \blacksquare_3, \blacksquare_4} \delta_{\blacksquare_1, \blacksquare_2} \delta_{\blacksquare_2, \blacksquare_3} \delta_{\blacksquare_3, \blacksquare_4} + \dots \\ &= -\frac{\kappa^2}{2} \gamma J^2 N + \frac{\kappa^4}{4!} \gamma J^4 \frac{N}{c/\alpha} + \dots + \dots + \text{"loop corrections"} \end{aligned}$$

$$P(E_1, E_2; q) \rightarrow_{p \rightarrow \infty} P(E_1)P(E_2) \quad \text{for} \quad 0 \leq |q| < 1$$

$$P(q) = \left\langle \delta \left(q - \frac{1}{N} \sum_{i=1}^N S_i^a S_i^b \right) \right\rangle_{S^a, S^b} = \delta(q)$$

Loop corrections



global coupling

$$(\beta J / \sqrt{c/\alpha})^3 \times c(c-1) \sim (\beta J)^3 c^{1/2} \xrightarrow{N \rightarrow \infty} \infty \quad c \propto N^{p-1}$$

regular-random graph

$$(\beta J / \sqrt{c/\alpha})^3 \times c(c-1) \times (c-1)/N \sim (\beta J)^3 c^{3/2}/N \xrightarrow{N \rightarrow \infty} 0$$

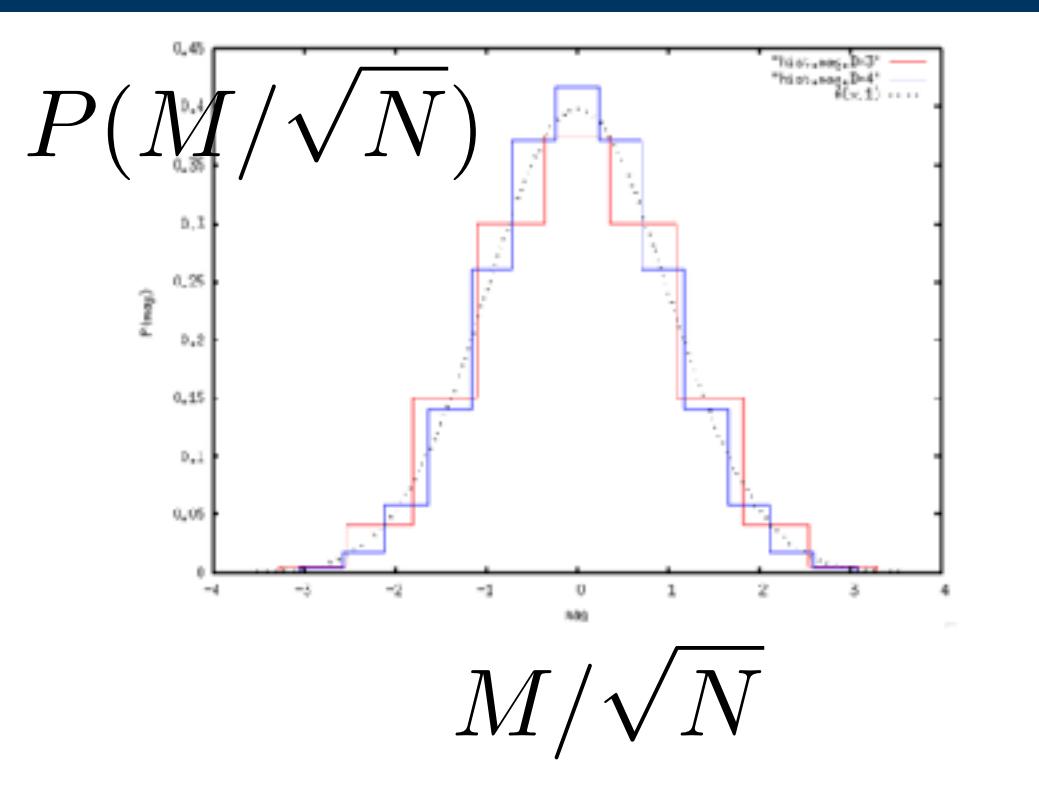
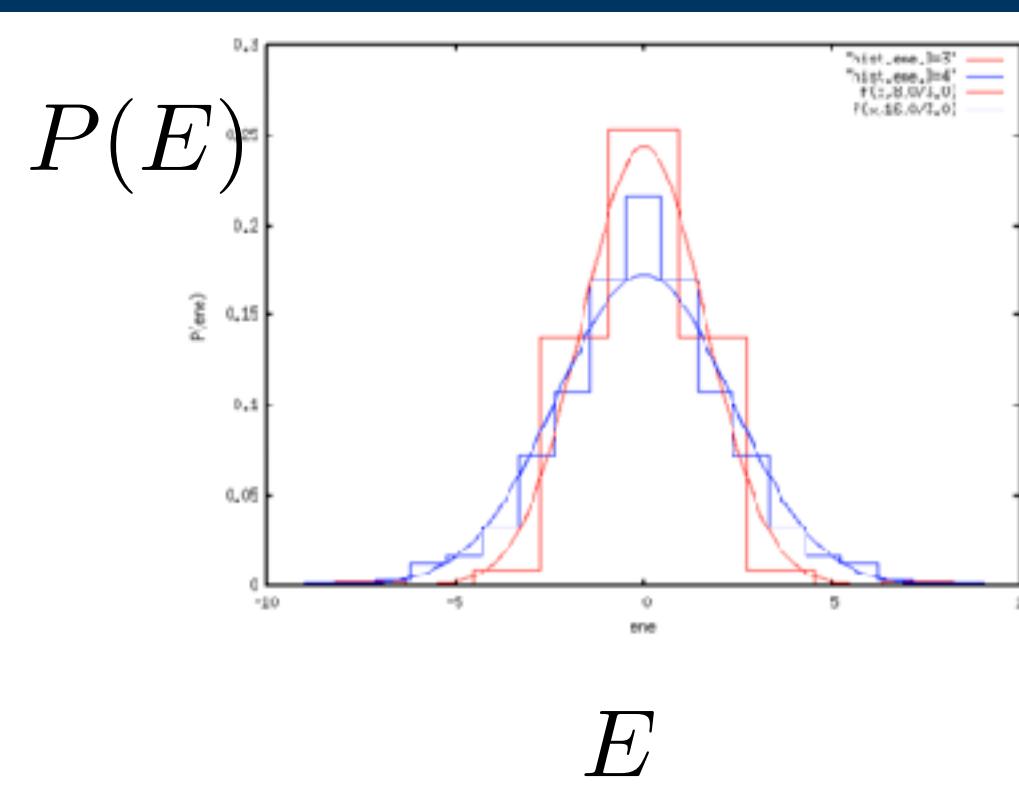
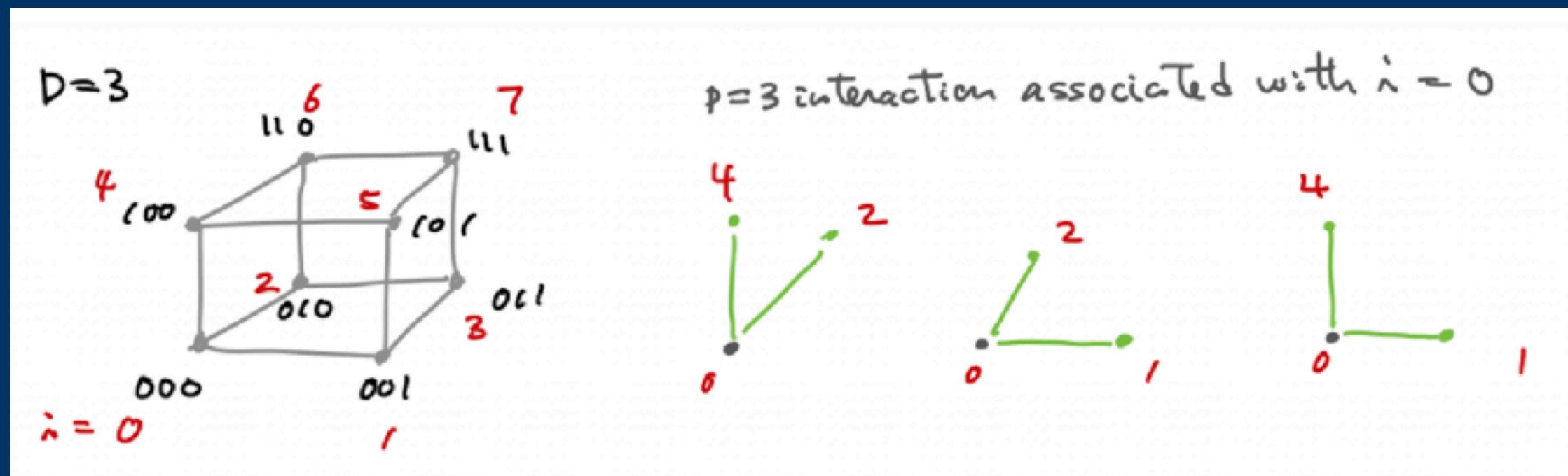
regular-regular graph

qualitatively similar to the above

Numerical check by exact enumeration

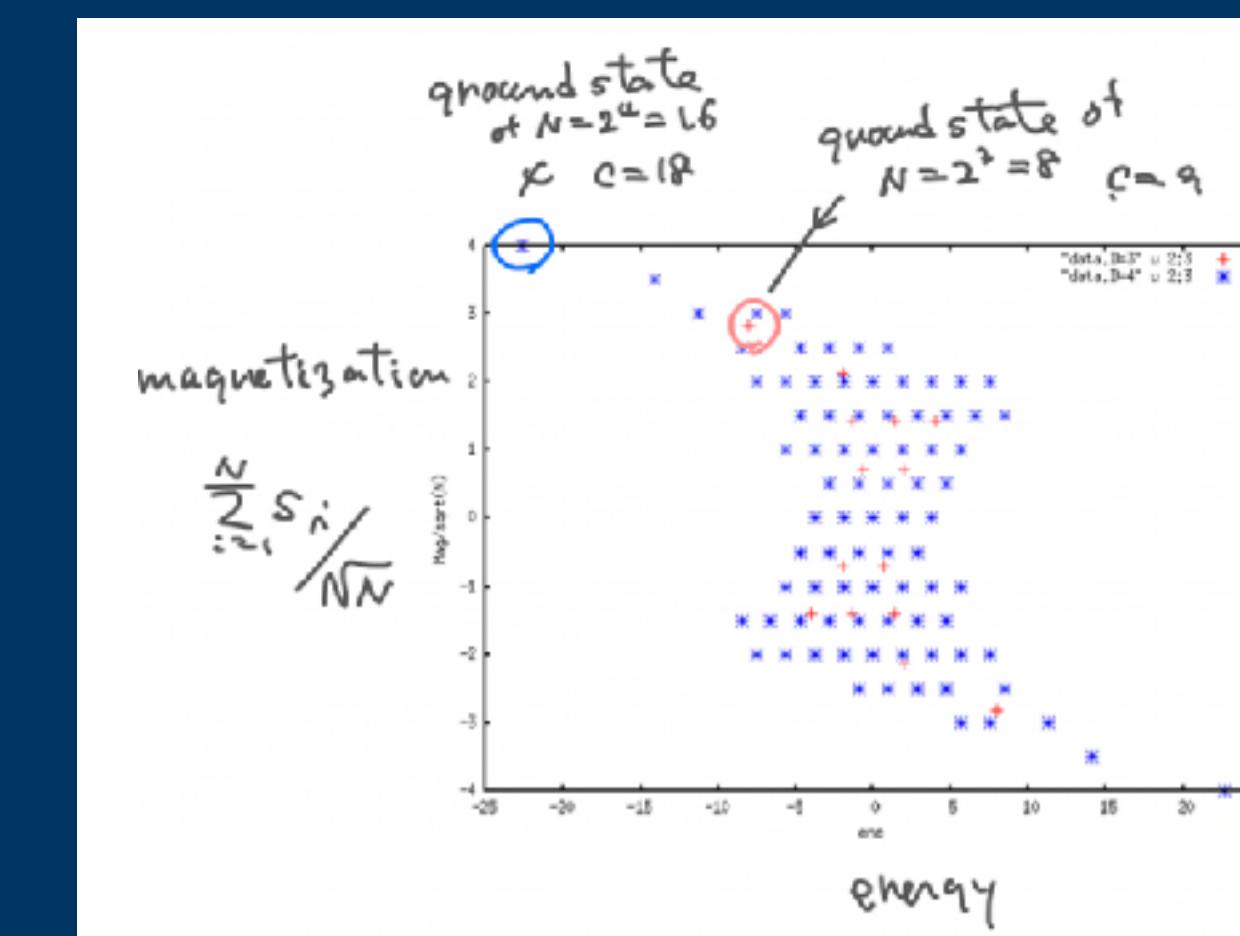
A. G. Cavalieri, T. Rizzo and HY in progress

p=3 ferromagnetic model on Regular dense graph

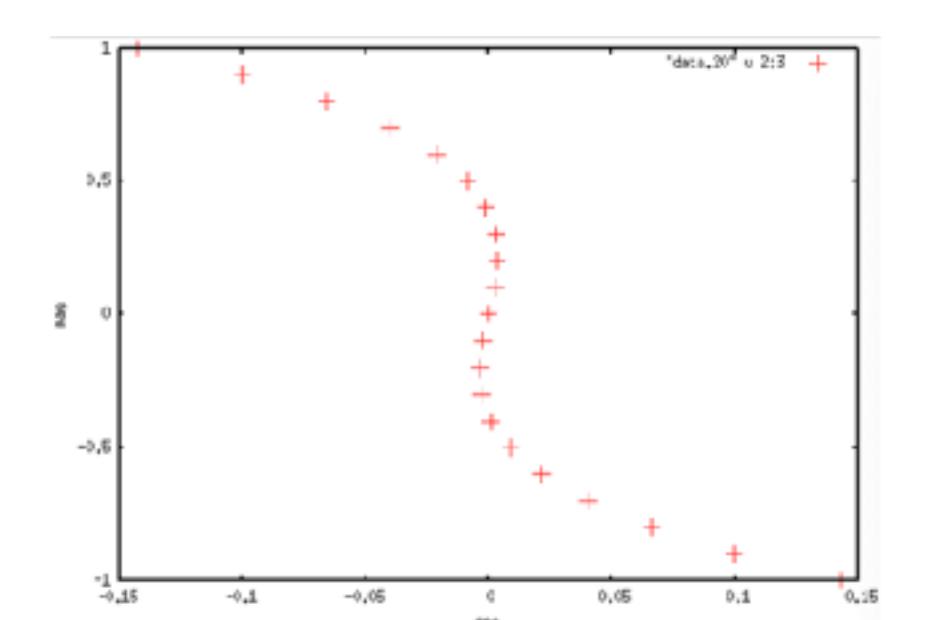


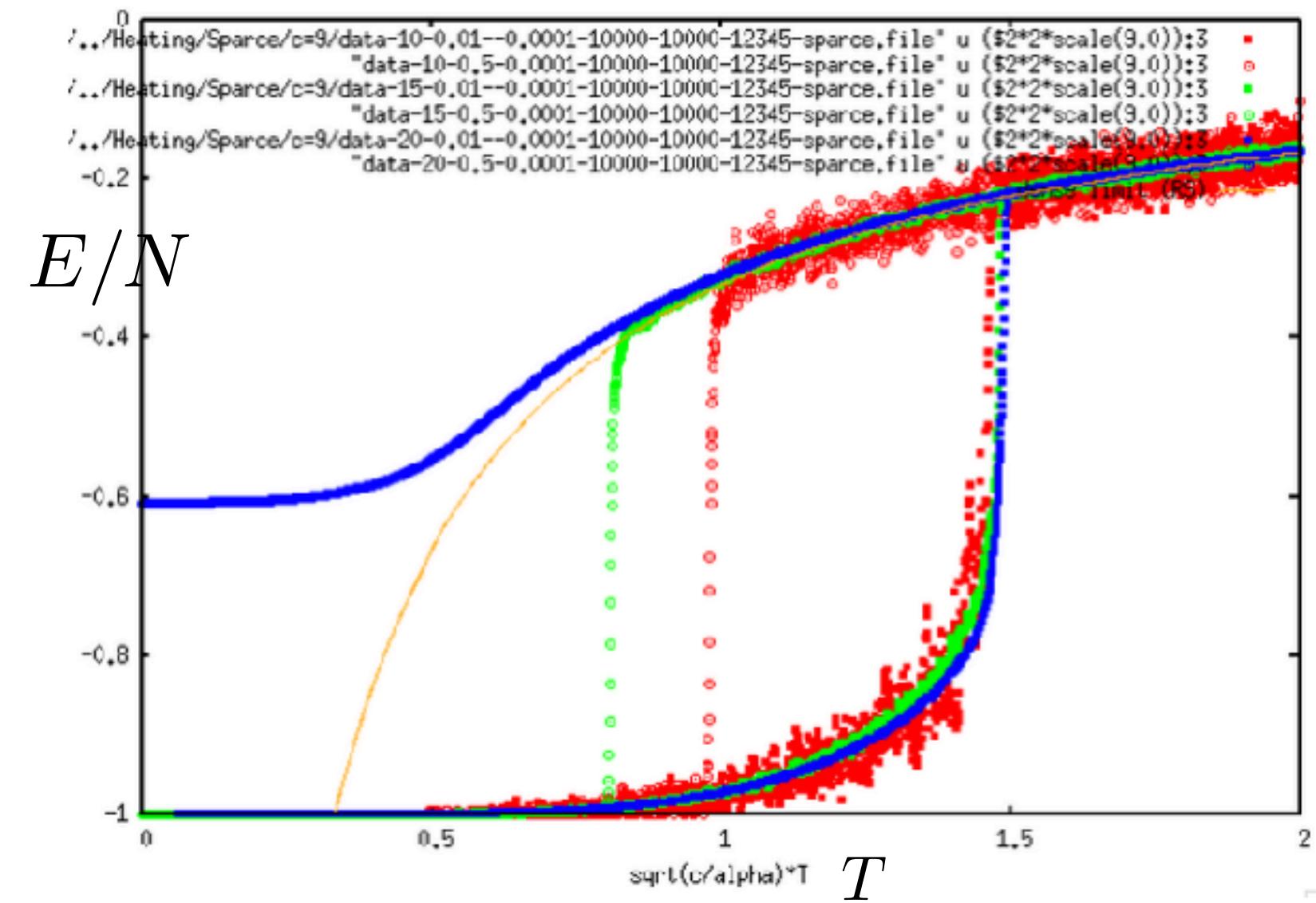
$$N = 2^D \quad c = \frac{3}{2} D(D - 1)$$

Dense coupling



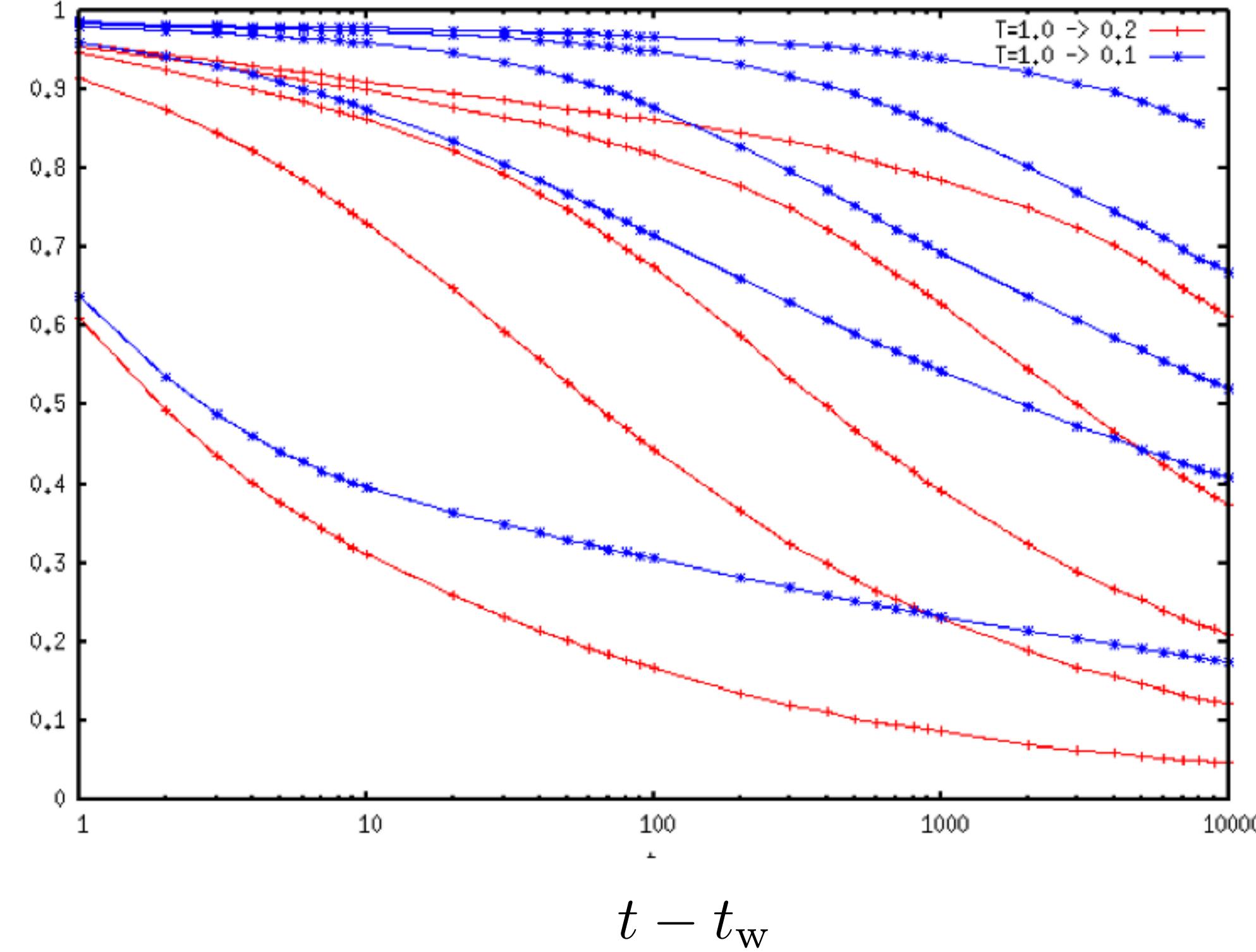
Global coupling





The orange line is the RS energy $e(T) = -\gamma/T$

$$C(t, t_w)$$



$$C(t, t_w) = \frac{1}{N} \sum_{i=1}^N \langle S_i(t) S_i(t_w) \rangle$$

Mean-field theory for the glassy supercooled-paramagnetic phases of the ferromagnetic p-spin model

Replica theory

How to obtain free-energy functional (with/without quenched disorder)

$$\overline{Z_n} = \prod_{a=1}^n \prod_{i=1}^N \text{Tr}_{S_i^a} e^{-\beta \sum_{a=1}^n H[S_i^a]}$$

$$1 = \int_{-\infty}^{\infty} \int_{-i\infty}^{i\infty} \prod_{a < b} \left(\frac{N}{2\pi i} \right) dq_{ab} d\varepsilon_{ab} e^{N \sum_{a < b} \varepsilon_{ab} (N^{-1} \sum_i^N (S_i)^a (S_i)^b - q_{ab})}$$

$$= \prod_{a < b} N \int dq_{ab} \int_{-i\infty}^{i\infty} \prod_{a < b} \frac{d\varepsilon_{ab}}{2\pi i} e^{-N \sum_{a < b} \varepsilon_{ab} q_{ab}} \prod_{i=1}^N \left(\prod_{a=1}^n \text{Tr}_{S_i^a} e^{\sum_{a < b} \varepsilon_{ab} S_i^a S_i^b} \right) e^{-\beta \sum_{a=1}^n H[\{S_i^a\}]}$$

$$= \prod_{a < b} N \int dq_{ab} \int_{-i\infty}^{i\infty} \frac{d\varepsilon_{ab}}{2\pi i} e^{-N \sum_{a < b} \varepsilon_{ab} q_{ab}} e^{-\beta G_{n,0}(\hat{\epsilon})} \langle e^{-\beta \sum_{a=1}^n H[\{S_i^a\}]} \rangle_{\epsilon,0}$$

$$= \prod_{a < b} N \int dq_{ab} \int_{-i\infty}^{i\infty} \frac{d\varepsilon_{ab}}{2\pi i} e^{-N \sum_{a < b} \varepsilon_{ab} q_{ab}} e^{-\beta G(\hat{\epsilon})}$$

Legendre transform

$$= \prod_{a < b} \int N dq_{ab} e^{-\beta F(\hat{q})}$$

“explicit RSB” G. Parisi and M. A. Virasoro,
Journal de Physique 50, 3317 (1989).

$$\langle \dots \rangle_{\epsilon,0} = \frac{\prod_{c=1}^n \prod_{i=1}^N \text{Tr}_{S_i^c} e^{\sum_{a < b} \varepsilon_{ab} S_i^a S_i^b} \dots}{\prod_{c=1}^n \prod_{i=1}^N \text{Tr}_{S_i^c} e^{\sum_{a < b} \varepsilon_{ab} S_i^a S_i^b}}$$

$$-\beta G_{n,0}(\hat{\epsilon}) = N \ln \prod_{c=1}^n \text{Tr}_{S^c} e^{\sum_{a < b} \varepsilon_{ab} S^a S^b} \quad -\beta G_n(\hat{\epsilon}) = -\beta G_{n,0}(\hat{\epsilon}) + \ln \langle e^{-\beta H} \rangle_{\epsilon,0}.$$

Organize a Plefka expansion

$$F_n = F_{n,0} + \lambda F_{n,1} + \frac{\lambda^2}{2} F_{n,2} + \dots$$

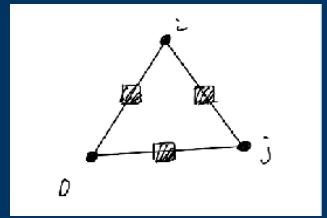
$$G_n = G_{n,0} + \lambda G_{n,1} + \frac{\lambda^2}{2} G_{n,2} + \dots$$

$$\varepsilon_{ab} = (\epsilon_0)_{ab} + \lambda (\epsilon_1)_{ab} + \frac{\lambda^2}{2} (\epsilon_2)_a$$

Evaluate this by cumulant
expansion

$$\begin{aligned} \frac{1}{N} \ln \langle e^{-\sqrt{\lambda} \beta \sum_{a=1}^n H[\{S_i^a\}]} \rangle_{\epsilon,0} &= \frac{\sqrt{\lambda}}{N} \sum_{a=1}^n \langle (-\beta H[\{S_a\}]) \rangle_{\epsilon,0} \\ &+ \frac{\lambda}{2N} \sum_{a,b=1}^n (\langle (\beta H_a)(\beta H_b) \rangle_{\epsilon,0} - \langle (\beta H_a) \rangle_{\epsilon,0} \langle (\beta H_b) \rangle_{\epsilon,0}) + \dots \end{aligned}$$

Diagrammatic expansion: (1) “G” does not contain disconnected diagrams (2) “F” does not contain 1PR (one-particle-reducible) diagrams



Contribution of higher-order terms including loops are negligible when we are lucky:
1) Gaussian spin-glass 2) disorder-free model in “dense limit” $N \gg c \gg 1$

$$-\beta F_n[\hat{q}]/N = -\frac{1}{2} \sum_{a,b} \epsilon_{ab}^* q_{ab} + \ln e^{\frac{1}{2} \sum_{a,b} \epsilon_{ab}^* \frac{\partial^2}{\partial h_a \partial h_b}} \prod_{c=1}^n 2 \cosh(h_c)|_{h=0} + \frac{\gamma(\beta J)^2}{2} \sum_{a,b} q_{ab}^p \quad \epsilon_{ab}^* = \gamma(\beta J)^2 p q_{ab}^{p-1} \quad a \neq b,$$

This strategy works well to obtain exact free-energy functional of disorder-free glassy systems in large-d limit

replicated liquid theory

Mezard-Parisi (1999), Kurchan-Parisi-Zamponi (2012)

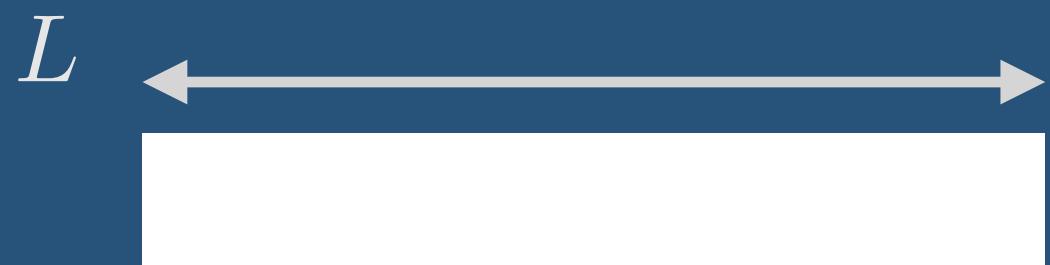
disorder-free spin models
in “dense” network

Yoshino, SciPostPhys 4,40 (2018) and in preparation

Mitsumoto-Yoshino(2023) (see K. Mitsumoto’s talk)

Cavaliere-Nagasawa-Yokoi-Obuchi-Yoshino(in preparation) (see A. G. Cavaliere’s poster)

1+d dim systems



Deep Neural Network Yoshino SciPostPhys Core 2, 5 (2020), arXiv:2302.07419

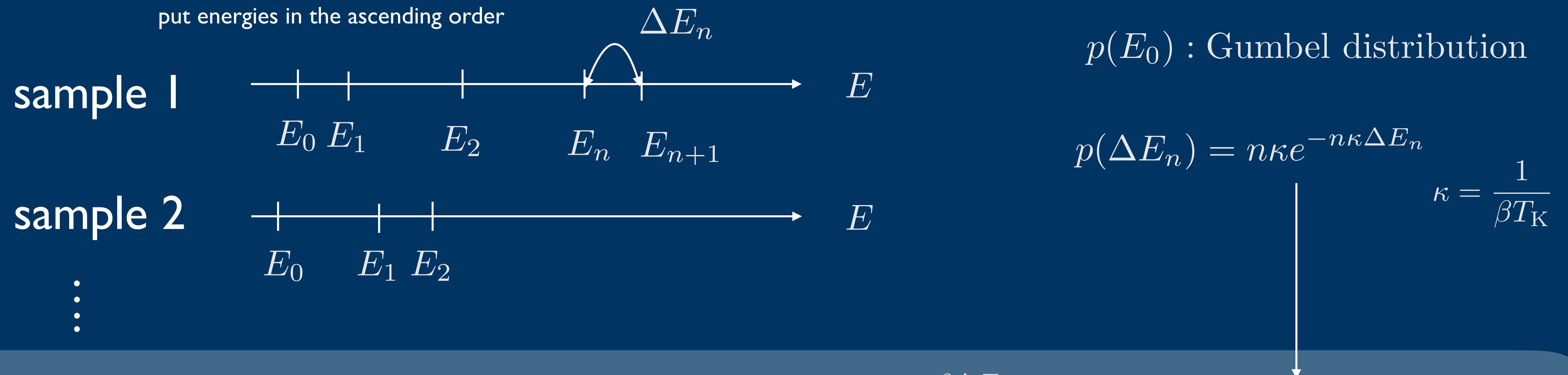
1+d dim replicated liquid Tomita-Yoshino (in preparation) (see Y. Tomita’s poster)

p-spin model put in layered geometry (see Y. R. Hamano’s poster)

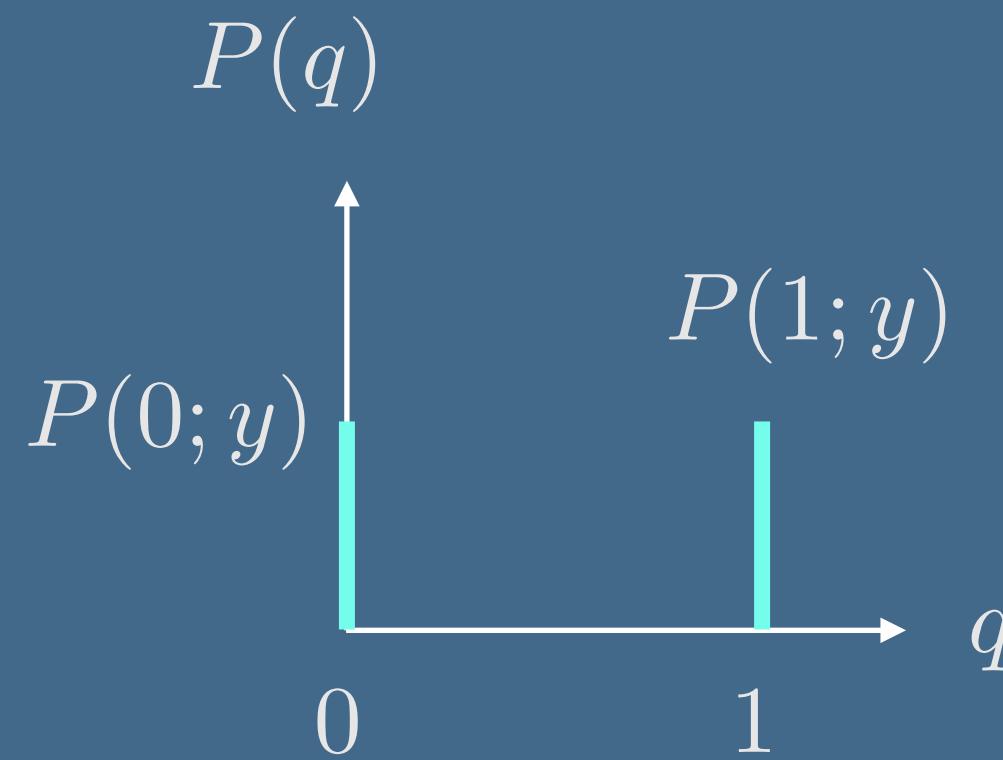
[Q] Really self-generated randomness?

Still something hidden under the carpet…

low lying states of REM (extreme value statistics)



distribution of overlaps



two level system

$$Z = 1 + y \quad y = e^{-\beta\Delta E_1} \quad P(1; y) = \frac{1 + y^2}{(1 + y)^2} \quad P(0; y) = 1 - P(1; y)$$

average over samples

$$\overline{P(1)} = \int_0^1 dx p(x) P(1; x) = \frac{T}{T_K} + O\left(\frac{T}{T_K}\right)^2$$

$$P(y) = y^{T/T_K} - 1$$

1 step replica symmetry breaking (RSB)
solution

$$\overline{P(1)} = x = \frac{T}{T_K}$$

∞ level system

...but in disorder-free system we just have 1 sample !!

Random pinning field hidden in the replica approach

$$1 = \int_{-\infty}^{\infty} \int_{-i\infty}^{i\infty} \prod_{a < b} \left(\frac{N}{2\pi i} \right) dq_{ab} d\varepsilon_{ab} e^{N \sum_{a < b} \varepsilon_{ab} (N^{-1} \sum_i^N (S_i)^a (S_i)^b - q_{ab})}$$

$$\begin{aligned} e^{-\beta G_n[\hat{\epsilon}]} &= \prod_a \prod_i \text{Tr}_{S_i^a} \overline{e^{-\beta H_n}} = \prod_a \prod_i Tr_{S_i^a} e^{-\beta \sum_a H[\{S_i^a\}] + \sum_i \sum_{a,b} \epsilon_{ab} S_i^a S_i^b} \\ &= \prod_a \prod_i Tr_{S_i^a} e^{-\beta \sum_a H[\{S_i^a\}]} \overline{e^{\sum_a \sum_i h_i^a S_i^a}} \end{aligned}$$

average over random pinning field

$$\overline{h_i^a h_j^b} = 2\epsilon_{ab} \delta_{ij}$$

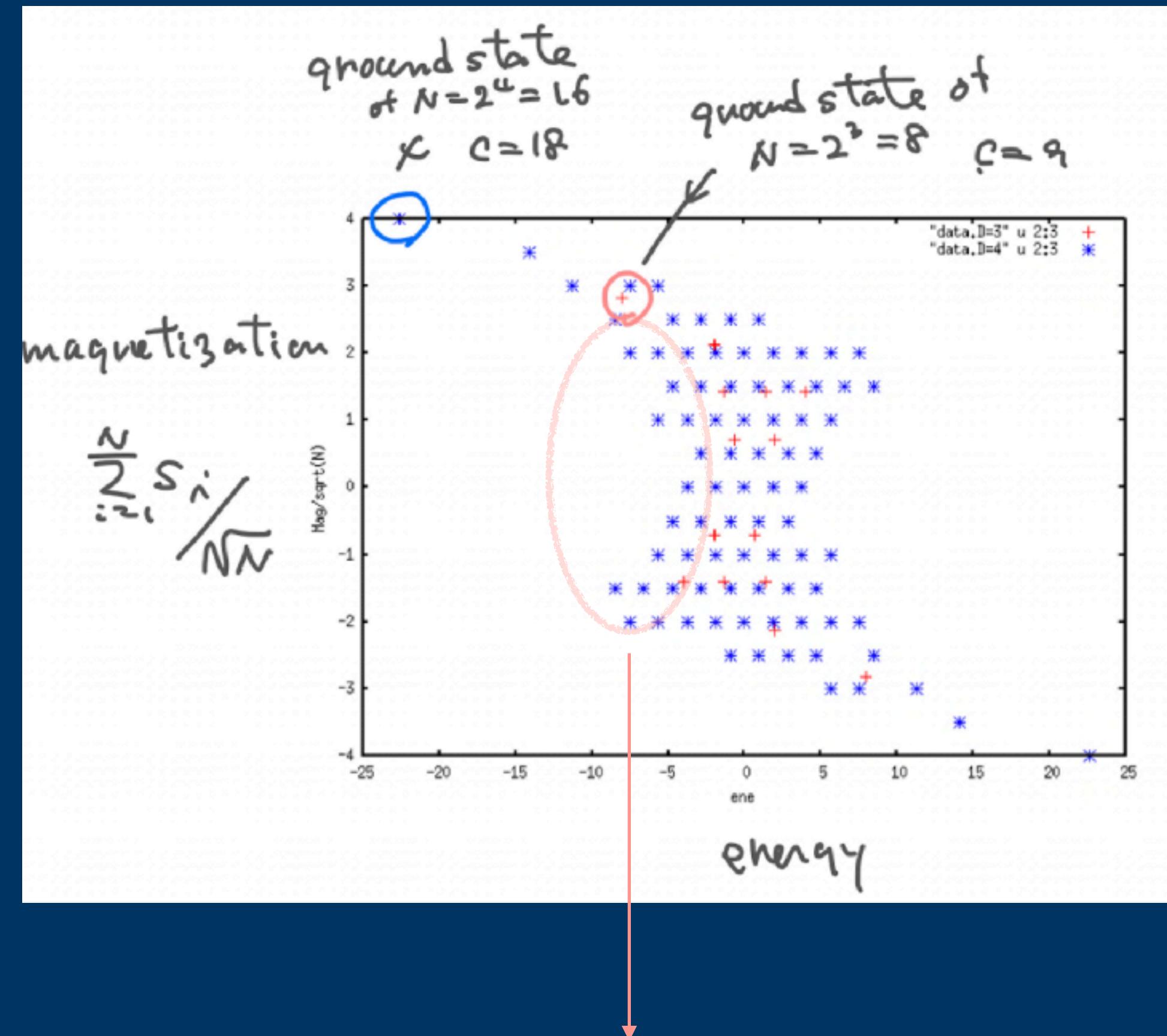
“explicit RSB” G. Parisi and M. A. Virasoro,
Journal de Physique 50, 3317 (1989).

In glass

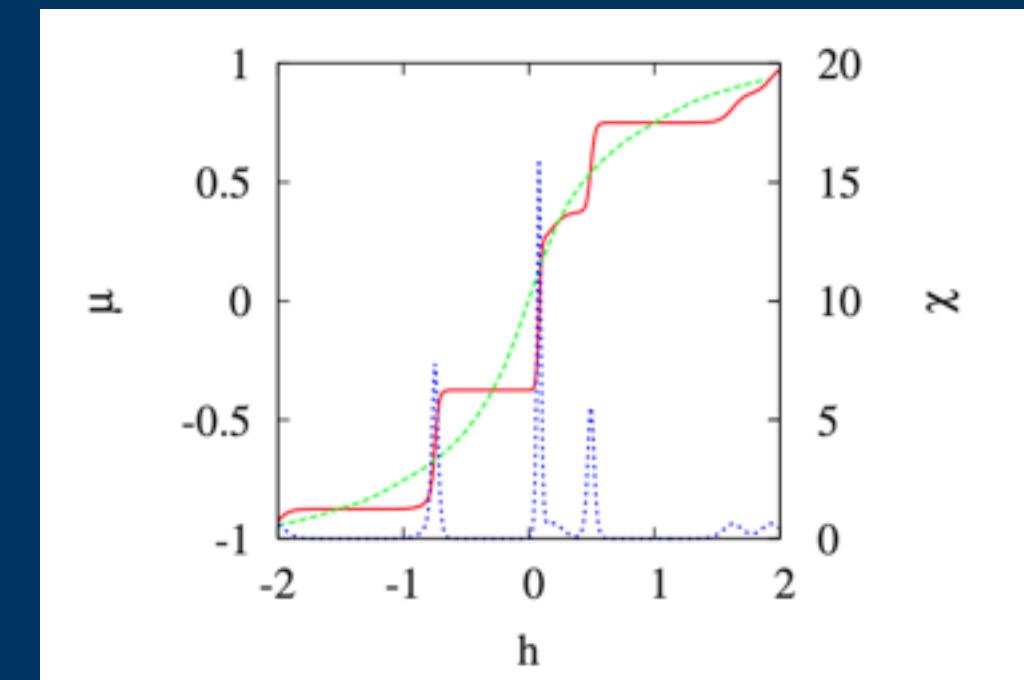
$$\lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} q_{ab}(\epsilon) = 0 \quad \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} q_{ab}(\epsilon) \neq 0$$

Sensitivity to infinitesimal random pinning field

low lying states of the p-spin ferromagnetic model on regular-regular graph



p-spin Ising spinglass model under magnetic field



H.Yoshino and T. Rizzo, PRB 77, 104429 (2008)

Chaotic reshuffling of energy levels by weak random pinning field?

Summary and Outlook

REM can be found in disorder-free $p \rightarrow \infty$ p-spin ferromagnetic model in **dense limit**

$$N \gg c \gg 1$$

All p-spin spin-glass results (replica/dynamic) can be recovered in the supercooled paramagnetic phase of the disorder-free ferromagnetic model

Glassy(?) ground state and low lying states in the super-cooled paramagnetic sector on the system on the “regular-regular” graph

1. What are they? (ideal glass states)
2. They can be chaotically reshuffled by infinitesimal pinning field?