New Aspects of the Bilinear Equations

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Summary

1. New Solutions.

   New solutions to the ultradiscrete soliton equations are obtained.

   (a) One is a "negative-soliton" which satisfies the *ultradiscrete* KdV equation(Box-Ball system).

   But there is not a corresponding traveling wave solution for the *discrete* KdV equation.

(b) The other one is a "static-soliton" which satisfies the *ultradiscrete* Toda equation.

But there is not a corresponding traveling wave solution to the *discrete* Toda equation.

Ryogo Hirota (2009).

2. Pfaffian Expressions

(a) We know the solutions to the discrete equations are expressed by pfaffians.

However pfaffians cannot be ultradiscretized because of negative problem.

(b) We have found that Casorati *permanents* play the same role as the Wronskian in the ultradiscrete equations.

D.Takahashi and R.Hirota,(2007).

H.Nagai,(2008).

3. Ultradiscrete Analogue of
Identities of Pfaffians (Determinants)

(a) The Casorati permanent solves the ultradiscrete 2-D Toda equation (up to $N = 4$).

"超離散プッリュカー関係式を用いたソリトン解の証明", 長井秀友、高橋大輔.

(b) Identities of *ultradiscrete pfaffians*

Ryogo Hirota, to appear.

4. Periodic Phase Solitons

(a) Shinya Nakamura (Waseda Univ.) has found that the ultradiscrete hungry Lotka-Volterra eq.

$$F_{n+1}^{m} + F_{n}^{m+1} = \max(F_{n}^{m} + F_{n+1}^{m+1}, F_{n-M}^{m} + F_{n+M+1}^{m+1} - 1)$$

exhibits "Periodic Phase Soliton" of the form

$$F_{n}^{m} = \max(0, pm - qn + \phi(n)),$$

where $\phi(n)$ is a periodic function of n with a period M.

But there is not a corresponding solution for the discrete hungry Lotka-Volterra equation.

(b) He has found $\tau-$function of N periodic phase soliton expressed by the Casorati *permanent.*

He has proved using "permanent technique" that the $\tau-$function solves the ultradiscrete hungry Lotka-Volterra equation for $M = 2$.

5. New Gauge Transformation

(a) The bilinear equations are invariant under the simple gauge transformation of the exponential type.

$$f \to f \exp(c_0 + c_1 l + c_2 m + c_3 n).$$

(b) Inspired by Nakamura's results I have found a discrete equation which is invariant under the new gauge transformation

$$f \to f\phi(n),$$

where $\phi(n)$ is a periodic function of n with a period M.

(c) The new gauge changes the interaction (phase shifts) of solitons drastically.

1. New Solutions

Solutions to the ultradiscrete soliton equations have been obtained by ultradiscretizing the known solutions to the discrete equations.

I review "new solutions" to the ultradiscrete soliton equations,which have no corresponding solutions to the discrete soliton equations.

1. Negative solutions to the ultradiscrete KdV eq.

   A discrete KdV eq.(Box and Ball system)

$$\frac{1}{u_{n+1}^{m+1}} - \frac{1}{u_n^m} = \delta(u_{n+1}^m - u_n^{m+1})$$

   is transformed,through the dependent variable transformation

$$u_n^m = \frac{f_{n+1}^m f_n^{m+1}}{f_n^m f_{n+1}^{m+1}},$$

into the bilinear eq.

$$f_n^{m-1} f_{n+1}^{m+1} = \delta f_{n+1}^{m-1} f_n^{m+1} + (1-\delta) f_n^m f_{n+1}^m. \quad (1)$$

We look for a "negative-soliton" traveling with the speed 1

$$u_n^m = \frac{f_{n+1}^m f_n^{m+1}}{f_n^m f_{n+1}^{m+1}} \leq 1,$$
$$f_n^m = f(n-m),$$

which give the following relations

$$\frac{f_{n+1}^{m-1} f_n^{m+1}}{f_{n+1}^m f_n^m} = u_{n+1}^m u_n^m. \quad (2)$$

$$\frac{f_{n+1}^{m+1} f_n^{m-1}}{f_{n+1}^m f_n^m} = 1, \quad (3)$$

Equation (1) is rearranged as

$$\frac{f_n^{m-1} f_{n+1}^{m+1}}{f_n^m f_{n+1}^m} = \delta \frac{f_{n+1}^{m-1} f_n^{m+1}}{f_n^m f_{n+1}^m} + 1 - \delta \quad (4)$$

which is reduced using the relations (2) and (3) to

$$1 = \delta u_{n+1}^m u_n^m + 1 - \delta,$$

which is not satisfied by a negative-soliton $u_n^m \leq 1$.

However, the above equation is reduced, in the ultradiscrete limit, to the following form,

$$0 = \max(\hat{U}_n^m + \hat{U}_{n+1}^m - 1, 0),$$

which is satisfied by the negative-soliton

$$\hat{U}_n^m \leq 0$$

.

The negative-soliton plays an important role in the initial value problem of the Box-Ball system.

It generates many balls in a box over the capacity of the box after colliding with a soliton as is shown below.

```
m=0 {0,1,1,1,1,1,0,0,0,0,-2,0,0,0,0,0,0,0,0,0,0,0,0}

m=1 {0,0,0,0,0,0,1,1,1,1,1,-2,0,0,0,0,0,0,0,0,0,0,0}

m=2 {0,0,0,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,0,0,0}

m=3 {0,0,0,0,0,0,0,0,0,0,0,-2,1,1,1,1,1,0,0,0,0,0,0}

m=4 {0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0,0,1,1,1,1,1,0}


Three balls in a box of capacity 1.
```

2. Static solutions to the ultradiscrete Toda eq.

We have the discrete Toda equation in the bilinear form

$$f_n^{m+1} f_n^{m-1} - (f_n^m)^2 = \delta^2 [f_{n+1}^m f_{n-1}^m - (f_n^m)^2] \quad (5)$$

which is transformed into the discrete Toda equation

$$\frac{V_n^{m+1} V_n^{m-1}}{(V_n^m)^2} = \frac{(1+\tilde{\delta}^2 V_{n+1}^m)(1+\tilde{\delta}^2 V_{n-1}^m)}{(1+\tilde{\delta}^2 V_n^m)^2},$$

$$\tilde{\delta}^2 = \frac{\delta^2}{1-\delta^2}, \quad (6)$$

through the transformation

$$V_n^m = \frac{f_{n+1}^m f_{n-1}^m}{(f_n^m)^2}.$$

Let

$$V_n^m = \exp(x_n^m/\epsilon), \quad \delta = \exp(-L/\epsilon).$$

Then we obtain an nonlinear discrete equation of x_n^m,

$$x_n^{m+1} - 2x_n^m + x_n^{m-1} = \epsilon \log[\frac{(1+\hat{\delta}^2 \exp(x_{n+1}^m/\epsilon))(1+\hat{\delta}^2 \exp(x_{n-1}^m/\epsilon)}{(1+\hat{\delta}^2 \exp(x_n^m/\epsilon))^2}],$$

which is reduced, in the small limit of ϵ, to the ultradiscrete Toda equation,

$$\begin{aligned} x_n^{m+1} - 2x_n^m + x_n^{m-1} &= \max(0, x_{n+1}^m - 2L) - 2\max(0, x_n^m - 2L) \\ &\quad + \max(0, x_{n-1}^m - 2L). \end{aligned}$$

We look for a static solution $Vs(n)$,

$$Vs(n) = \frac{f^m_{n+1} f^m_{n-1}}{(f^m_n)^2},$$
$$f^m_n = fs(n),$$

to the discrete Toda equation.

The bilinear equation (5) is rearranged as

$$\frac{f^{m+1}_n f^{m-1}_n}{(f^m_n)^2} + \delta^2 = 1 + \delta^2 \frac{f^m_{n+1} f^m_{n-1}}{(f^m_n)^2},$$

which is reduced, for a static solution, to

$$1 + \delta^2 = 1 + \delta^2 Vs(n).$$

Obviously $Vs(n)$ does not solve it except a trivial case $Vs(n) = 1$.

However the above equation is reduce, in the ultradiscrete limit, to

$$\max(0, -2L) = \max(0, x_s(n) - 2L)$$

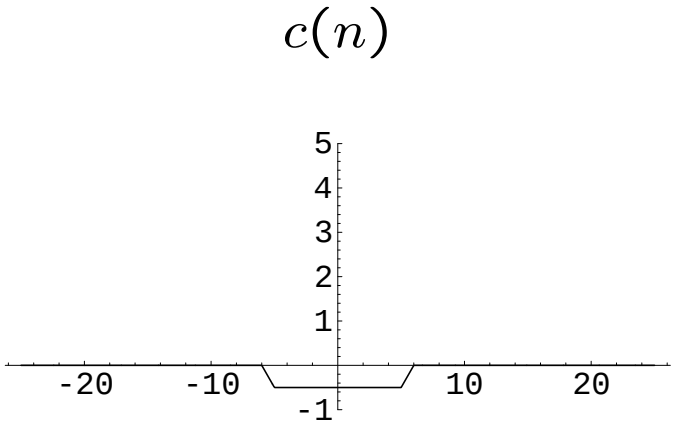
which is satisfied by $x_s(n)$ if

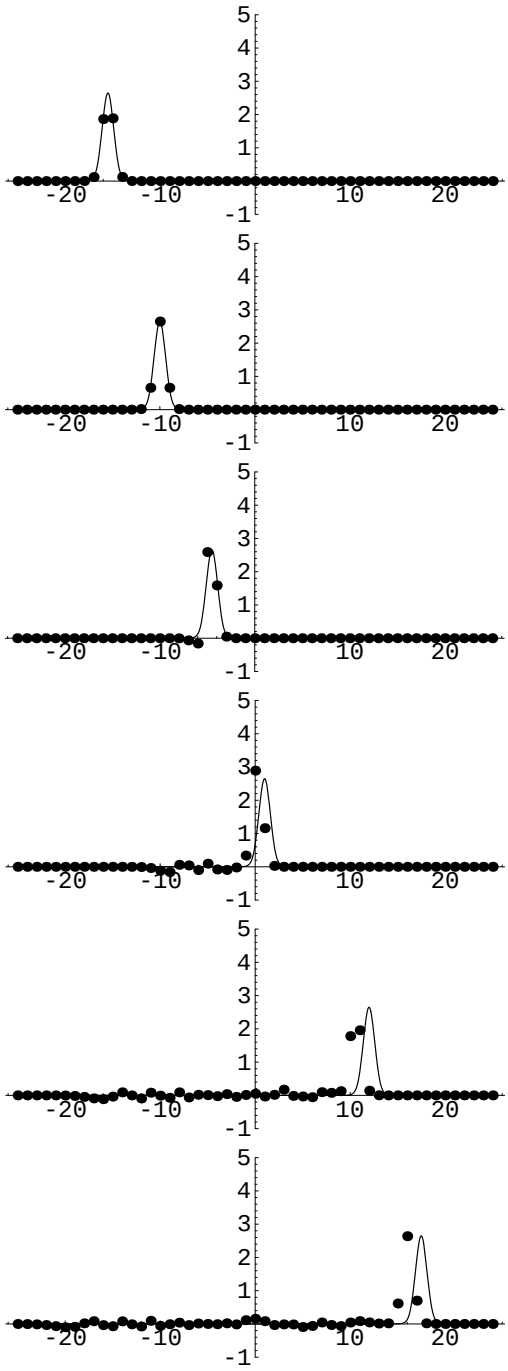
$$x_s(n) \leq 2L, \quad \text{for all } n.$$

The static solution plays an important role in the *ultradiscrete nonuniform* Toda equation.

We have calculated a soliton y_n^m passing through junctions in the *nonuniform* Toda lattice.

The figure shows the non-uniformity $c(n)$ introduced to the discrete Toda lattice,where the atoms located at $-5 \leq n \leq 5$ are different from others.

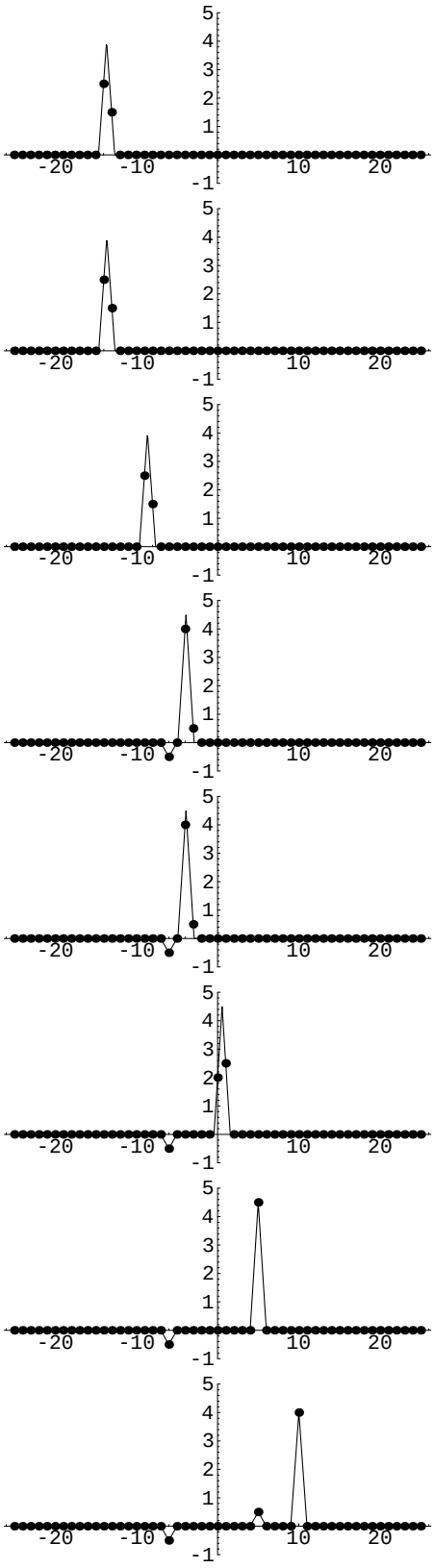




We observe a soliton passing through junctions generates ripples at the junctions.

We have calculated a soliton passing through junctions of the *ultradiscrete nonuniform* Toda equation,

$$
\begin{aligned}
& y_n^{m+1} - 2y_n^m + y_n^{m-1} \\
&\quad = \max[0, y_{n+1}^m - 2L + c(n+1)] \\
&\qquad -2\max[0, y_n^m - 2L + c(n)] \\
&\qquad + \max[0, y_{n-1}^m - 2L + c(n-1)].
\end{aligned}
$$


5
4
3
2
1
-20
-10
10
20
-1

In the figures the solid lines express theoretical values of $y(m,n)$ as a function of n, while the dots indicates numerical values of y_n^m.

All dots are on the solid lines.

Ryogo Hirota, "New Solutions to the Ultradiscrete Soliton Equations",
STUDIES IN APPLIED MATHEMATICS 122:361-376(2009).

2. Pfaffian Expressions

Multi-soliton solution to the soliton equation is expressed by the pfaffian and the bilinear form of the soliton equation is reduced to the identity of pfaffians.

Multi-soliton to a soliton equation has two types of expression.

1. One is expressed by a sum of exponential functions which is obtained by a perturbational method.

2. Another is expressed by a pfaffian (determinant).

The perturbational method of finding soliton solution is very powerful but difficulty of finding solution increases very rapidly as increasing number of solitons included in the solution.

However we may assume an algebraic structure of solution by the perturbational method and find a pfaffian expression for solution.

The $\tau-$function f_n^m in the perturbed form has the following form in general

$$f_n^m = 1 + e^{\eta_1(m,n)} + e^{\eta_2(m,n)} + a_{12} e^{\eta_1(m,n)+\eta_2(m,n)},$$

and is easily ultradiscretized.

However pfaffians (determinants) can not be ultradiscretized due to negative terms.

A remedy for the problem was found by Takahashi and Hirota.

D.Takahashi and R.Hirota:
“ Ultradiscrete Soliton Solution of Permanent Type”,
JPSJ 76 (2007) 104007,

We have expressed the multi-soliton solutions to an ultradiscrete soliton equation called “Box and Ball system” by ultradiscretized permanents instead of determinants.

A permanent is a signature free determinant.

Nagai has shown that soliton solutions to the ultradiscrete Toda equation are expressed by the ultradiscretized permanents.

H.Nagai:
“ A New Expression of Soliton Solution to the Ultradiscrete Toda Equation”,
J.Phys. A 41 (2008) 235204,

These facts suggest that there must be an idenity of ultradiscretized permanents instead of determinants.

More generally we expect an identity of ultra-discretized hafnians instead of pfaffians.

A hafnian is a signature free pfaffian introduced by Caieniello.

3. Ultradiscrete Analogue of Identities of Pfaffians

(a) Plücker relation:

We look for an ultradiscrete analogue of the following simple identity of determinants

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0,$$

which is one of the Plücker relations.

We replace the determinants by the corresponding permanents

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}_+ \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}_+ - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}_+ + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}_+ = 0, \quad (7)$$

Let each term in Eq.(7) be q_1, q_2 and q_3,namely

$$q_1 = \left| \begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array} \right|_+ \left| \begin{array}{cc} a_3 & a_4 \\ b_3 & b_4 \end{array} \right|_+$$

$$= a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3,$$

$$q_2 = \left| \begin{array}{cc} a_1 & a_3 \\ b_1 & b_3 \end{array} \right|_+ \left| \begin{array}{cc} a_2 & a_4 \\ b_2 & b_4 \end{array} \right|_+$$

$$= a_1a_2b_3b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_3a_4b_1b_2,$$

$$q_3 = \left| \begin{array}{cc} a_1 & a_4 \\ b_1 & b_4 \end{array} \right|_+ \left| \begin{array}{cc} a_2 & a_3 \\ b_2 & b_3 \end{array} \right|_+$$

$$= a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2,$$

where q_1, q_2 and q_3 have no negative terms and can be ultradiscretized.

However the corresponding Plücker relation does not hold,

$$q_1 - q_2 + q_3 = 2(a_1a_3b_2b_4 + a_2a_4b_1b_3) \neq 0. \quad (8)$$

We notice that the products of the permanents, q_1, q_2 and q_3 are decomposed into a sum of common terms q_{12}, q_{13} and q_{23},where q_{ij} is the common term of q_i and q_j for $i, j = 1, 2, 3$,

$$\begin{aligned} q_1 &= q_{12} + q_{13}, \quad q_2 = q_{12} + q_{23}, \\ q_3 &= q_{13} + q_{23}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} q_{12} &= a_1a_4b_2b_3 + a_2a_3b_1b_4, \\ q_{13} &= a_1a_3b_2b_4 + a_2a_4b_1b_3, \\ q_{23} &= a_1a_2b_3b_4 + a_3a_4b_1b_2. \end{aligned}$$

An ultradiscrete analogue of the Plücker relation is obtained as follows.

Replacing the determinants by the correspoding permanents we have

$$q_1 + q_3 = q_2. \tag{10}$$

Let

$$q_i = \exp(Q_i/\epsilon) \text{ for } i = 1, 2, 3,$$
$$q_{ij} = \exp(Q_{ij}/\epsilon) \text{ for } i, j = 1, 2, 3,.$$

In the small limit of ϵ we have an ultradiscrete analogue of the Plücker relation , Eq.(10),

$$Q_2 = \max(Q_1, Q_3), \tag{11}$$

which does not hold in general.
We investigate under what conditions on Q_1, Q_2 and Q_3 Eq.(11) does hold. The ultradiscrete form of Eq.(9) are

$$\begin{aligned} Q_1 &= \max(Q_{12}, Q_{13}), \\ Q_2 &= \max(Q_{12}, Q_{23}), \\ Q_3 &= \max(Q_{13}, Q_{23}). \end{aligned} \tag{12}$$

Substituting these expressions into Eq.(11) we obtain

$$\max(Q_{12}, Q_{23}) = \max(Q_{12}, Q_{13}, Q_{23}). \tag{13}$$

Obviously Eq.(13) does hold if

$$Q_{13} \leq \max(Q_{12}, Q_{23}).$$

But it does not hold if

$$Q_{13} > \max(Q_{12}, Q_{23}).$$

However if $Q_{13} > \max(Q_{12}, Q_{23})$ we find, using Eq.(12)

$$Q_1 = Q_3.$$

Hence we obtain the following algebraic identity of the ultradiscretized permanents,

$$[Q_2 - \max(Q_1, Q_3)](Q_1 - Q_3) = 0, \quad (14)$$

which we call "ultradiscrete analogue of the Plücker relation".

(b) Identities of pfaffians:

It is known that a variety of soliton equations exhibiting multi-soliton solutions expressed by pfaffians give rise to the following identity of pfaffians,

$$\begin{aligned}
&\mathrm{pf}(1,2,3,4,5,6,\cdots,2n)\mathrm{pf}(5,6,\cdots,2n)\\
&=\mathrm{pf}(1,2,5,6,\cdots,2n)\mathrm{pf}(3,4,5,6,\cdots,2n)\\
&-\mathrm{pf}(1,3,5,6,\cdots,2n)\mathrm{pf}(2,4,5,6,\cdots,2n)\\
&+\mathrm{pf}(1,4,5,6,\cdots,2n)\mathrm{pf}(2,3,5,6,\cdots,2n).
\end{aligned}$$

I replace the above pfaffians by the corresponding hafnians.
Let the products of hafnians be

$$\begin{aligned}
f_0 &= (1,2,3,4,5,6,\cdots,2n)(5,6,\cdots,2n),\\
f_1 &= (1,2,5,6,\cdots,2n)(3,4,5,6,\cdots,2n),\\
f_2 &= (1,3,5,6,\cdots,2n)(2,4,5,6,\cdots,2n),\\
f_3 &= (1,4,5,6,\cdots,2n)(2,3,5,6,\cdots,2n).
\end{aligned}$$

I have proved by induction that the products of the hafnians are decomposed into the following forms

$$
\begin{aligned}
f_0 &= f_{01} + f_{02} + f_{03}, \\
f_1 &= f_{01} + f_{12} + f_{13}, \\
f_2 &= f_{02} + f_{12} + f_{23}, \\
f_3 &= f_{03} + f_{13} + f_{23}.
\end{aligned}
$$

Consider a relation,

$$
f_0 + f_2 = f_1 + f_3, \tag{15}
$$

which does hold for pfaffians but not for hafnians.

Following the same procedure as the one used before I find the algebraic idenity of the ultradiscretized hafnians,

$$(\max(F_0, F_2) - \max(F_1, F_3))(F_0 - F_2)(F_1 - F_3) = 0,$$

where F_0, F_1, F_2 and F_3 are the ultradiscrete form of f_0, f_1, f_2 and f_3,respectively.

We call it the ultradiscrete analogue of the identity of the pfaffians.

Ryogo Hirota, "Ultradiscrete analogue of the Identity of Pfaffians" ,to appear in RIMS Kôkyûroku Bessatsu.

4. Periodic Phase Solitons

We know that the hungry Lotka-Volterra eq.

$$(1+\delta_1) f_{n+1}^m f_n^{m+1} = f_n^m f_{n+1}^{m+1} + \delta_1 f_{n-M}^m f_{n+M+1}^{m+1},$$

exhibits $1-$soliton solution for an integer M,

$$f_n^m = 1 + r_1(m,n), \quad r_1(m,n) = \omega_1^m k_1^{(n-n_1)},$$

$$\omega_1 = \frac{1+\delta_1(1+k_1^{-1}+k_1^{-2}+\cdots+k_1^{-M})}{1+\delta_1(1+k_1+k_1^2+\cdots+k_1^M)}.$$

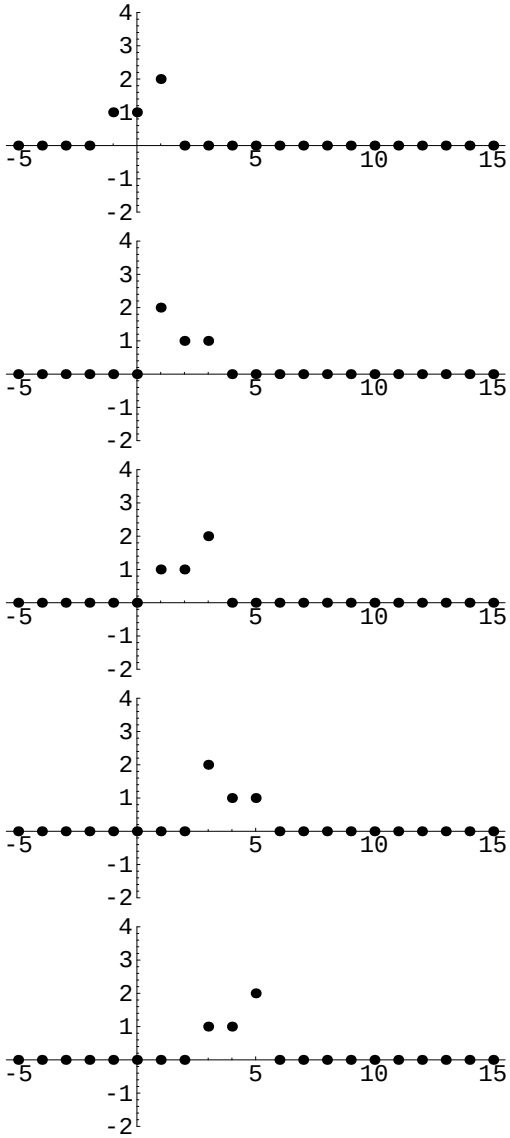
The ultradiscrete hungry Lotka-Volterra eq.

$$F_{n+1}^{m} + F_{n}^{m+1} = \max(F_{n}^{m} + F_{n+1}^{m+1}, F_{n-M}^{m} + F_{n+M+1}^{m+1} - 1)$$

is known to describes an extended "Box and Ball system".
In this system all balls are numbered and the balls with the smaller number moves earlier.

D.Takahashi "On some soliton systems defined by using boxes and balls",1993 International Symposium on Nonlinear Theory and its Applications(NOLTA'93) Hawaii,U.S.A., December 5-10,1993,

I have found numerically that the ultradiscrete hungry Lotka-Volterra equation exhibits the following soliton solutions for $M = 2$.



I called it "Wiggler".

(a) Shinya Nakamura (Waseda Univ.) discovered that "Wiggler" is expressed by the following $\tau-$ function,

$$
\begin{aligned}
&F_n^m = \max(0, s_1(m,n) + \phi_1(n)),\\
&s_1(m,n) = p_1 m - q_1(n - n_1),\\
&p_1 = M q_1 - 1 > 0,
\end{aligned}
$$

under the condition

$$q_1 > \phi_1(n+1) - \phi_1(n), \quad \text{for all } n$$

where $\phi_1(n)$ is periodic function of n of period M,

$$\phi_1(n+M) = \phi_1(n), \text{ for all } n.$$

We now call it " periodic phase soliton" because of the periodic phase factor $\phi_1(n)$.

We have found that there is not a corresponding solution for the discrete hungry Lotka-Volterra equation.

(b) He has also found that $\tau-$function of N periodic phase soliton expressed by the Casorati permanent,

$$F_n^m = \frac{1}{2}\max \begin{bmatrix} |s_1+\phi_1(n)|+\phi_1(n), & |s_1+3q_1+\phi_1(n+1)|+\phi_1(n+1), & \cdots \\ |s_2+\phi_2(n)|+\phi_2(n), & |s_1+3q_2+\phi_2(n+1)|+\phi_2(n+1), & \cdots \\ \cdots & \cdots & \cdots \\ |s_N+\phi_N(n)|+\phi_N(n), & |s_N+3q_N+\phi_N(n+1)|+\phi_N(n+1), & \cdots \end{bmatrix}$$

He has proved using "permanent technique" that the $\tau-$function solves the ultradiscrete hungry Lotka-Volterra eq. for $M=2$.

5. Gauge Transformations

The bilinear equations are known to be invariant under the simple gauge transformation of the exponential type,

$$f \to f \exp(c_0 + c_1 l + c_2 m + c_3 n).$$

Inspired by Nakamura's results I have found that a discrete equation

$$f_{n+1}^{m} f_{n}^{m+1} = f_{n}^{m} f_{n+1}^{m+1} + \delta(f_{n-M}^{m} f_{n+M+1}^{m+1} - f_{n-M+1}^{m} f_{n+M}^{m+1}), \quad (16)$$

is invariant under the new gauge transformation,

$$f \to f\phi(n),$$

where $\phi(n)$ is a periodic function of n with a period M.

The new gauge transforms Eq.(16) into

$$f_{n+1}^{m}\phi(n+1)f_{n}^{m+1}\phi(n) = f_{n}^{m}\phi(n)f_{n+1}^{m+1}\phi(n+1)$$
$$+\delta[f_{n-M}^{m}\phi(n-M)f_{n+M+1}^{m+1}\phi(n+M+1)$$
$$-f_{n-M+1}^{m}\phi(n-M+1)f_{n+M}^{m+1}\phi(n+M)],$$

which is reduced , by the periodicity of $\phi(n) = \phi(n+M)$, to Eq.(16).

I call Eq.(16) "Discrete Hungry Lotka-Voltera equation of BKP type" for an integer M, which was, for $M = 2$, called "Discrete Sawada-Kotera equation.

Let

$$w_n^m = \frac{f_{n-M+1}^m f_{n+M}^{m+1}}{f_n^m f_{n+1}^{m+1}},$$

$$x_n^m = \frac{f_{n-M}^m f_{n+M+1}^{m+1}}{f_n^m f_{n+1}^{m+1}}.$$

Then Eq.(16) is transformed into a coupled nonlinear discrete equations,

$$w_n^{m+1} = w_n^m \prod_{j=1}^{M-1} \frac{1+\delta(x_{n-j}^m - w_{n-j}^m)}{1+\delta(x_{n+j}^{m+1} - w_{n+j}^{m+1})},$$

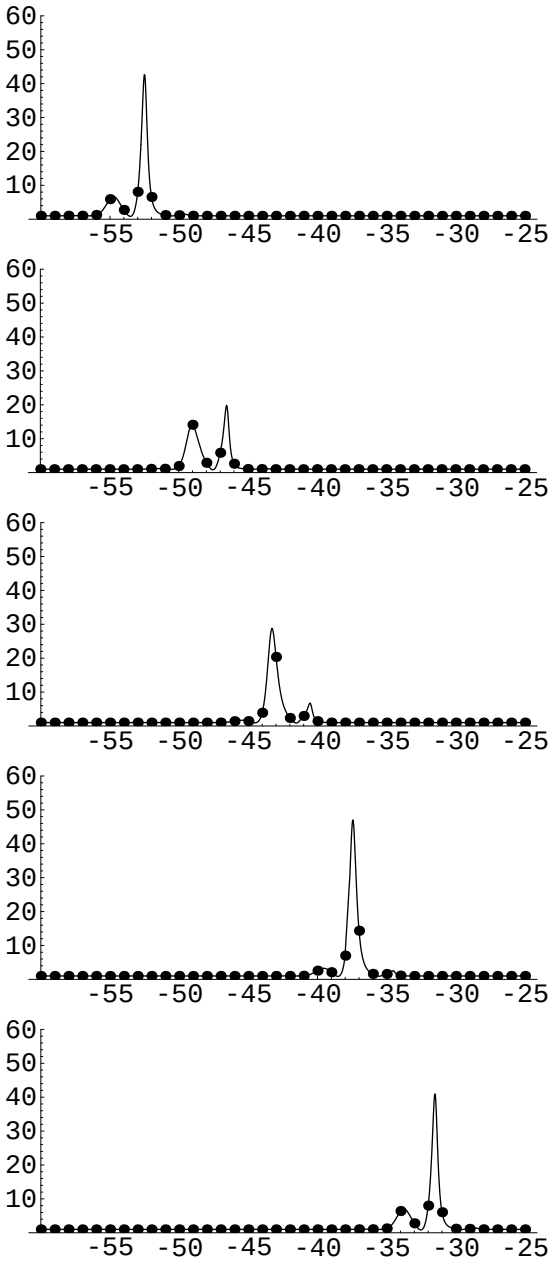
$$x_n^{m+1} = x_n^m (w_n^{m+1}/w_n^m) \frac{1+\delta(x_{n-M}^m - w_{n-M}^m)}{1+\delta(x_{n+M}^{m+1} - w_{n+M}^{m+1})}.$$

$\tau-$function of one periodic phase soliton is given by

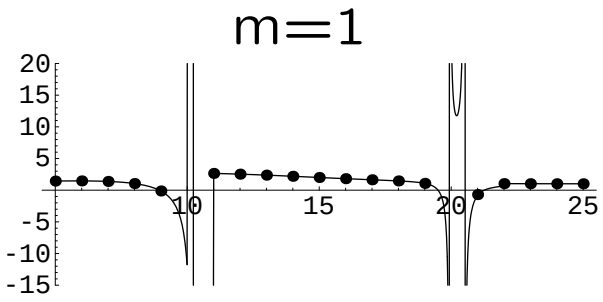
$$f_n^m = 1 + r_1(m,n),$$

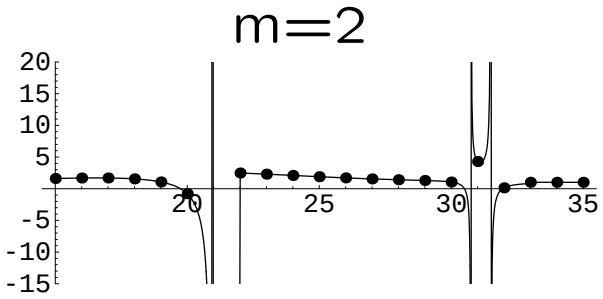
$$r_1(m,n) = \omega_1^m k_1^{(n-n_1)} \phi(n),$$

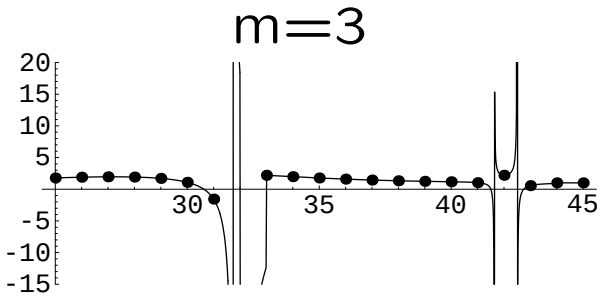
$$\omega_1 = \frac{1+\delta/k_1^M}{1+\delta k_1^M}.$$

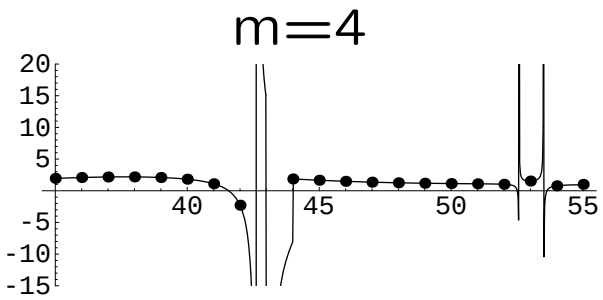


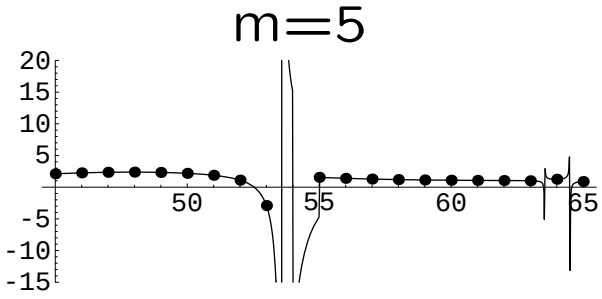
Periodic phase soliton of normal type
($\phi(n) > 0$ for all n).

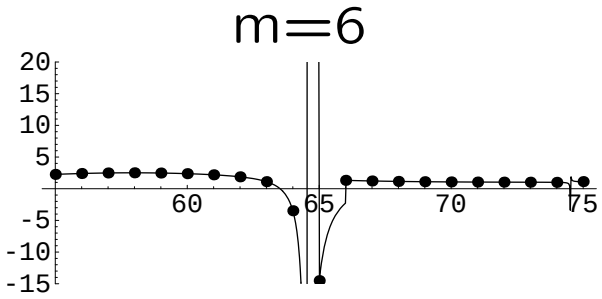

m=1
20
15
10
5
-5
-10
-15
10
15
20
25

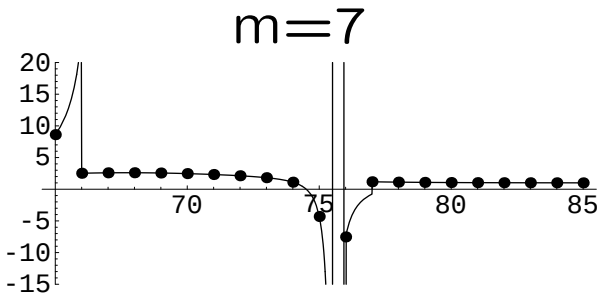

m=2
20
15
10
5
-5
-10
-15
20
25
30
35

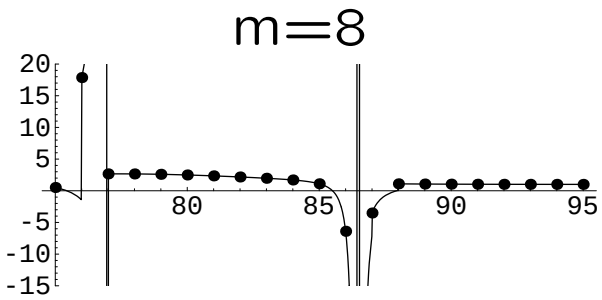

m=3
20
15
10
5
-5
-10
-15
30
35
40
45

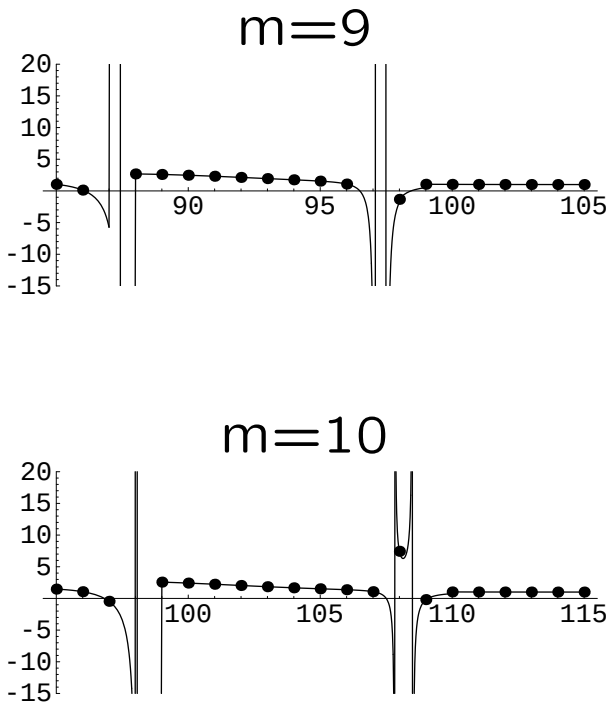

m=4
20
15
10
5
-5
-10
-15
40
45
50
55


m=5
20
15
10
5
-5
-10
-15
50
55
60
65


m=6
20
15
10
5
-5
-10
-15
60
65
70
75


m=7
20
15
10
5
-5
-10
-15
70
75
80
85


m=8
20
15
10
5
-5
-10
-15
80
85
90
95



Periodic phase soliton of singular type
($\phi(n) < 0$ for some n).

In the figures the solid lines express theoretical values of $x(m, n)$ as a function of n, while the dots indicates numerical values of x_n^m.

All dots are on the solid lines.

The new gauge changes the interaction (phase shifts) of solitons drastically.

The usual 2-soliton to Eq.(16) is given by

$$f_2(m,n) = 1 + r_1(m,n) + r_2(m,n) + a_{12} r_1(m,n) r_2(m,n),$$

where

$$r_j(m,n) = \omega_j^m k_j^{(n-n_j)},$$
$$\omega_j = \frac{1+\delta/k_j^M}{1+\delta k_j^M},$$
$$a_{ij} = \frac{k_i^M - k_j^M}{(k_i k_j)^M - 1} \frac{k_i - k_j}{k_i k_j - 1}.$$

for $i, j = 1, 2$.

While 2-periodic phase soliton solution is given by

$$f_2(m,n) = 1 + r_1(m,n) + r_2(m,n) + a_{12}(n) r_1(m,n) r_2(m,n), \qquad (17)$$

where

$$
\begin{aligned}
r_j(m,n) &= \omega_j^m k_j^{(n-n_j)} \phi_j(n), \\
\omega_j &= \frac{1+\delta/k_j^M}{1+\delta k_j^M}, \\
c_{ij} &= \frac{k_i^M - k_j^M}{(k_i k_j)^M - 1}, \\
a_{ij}(n) &= -(1/\Delta_{ij}) \\
&\qquad \times [\sum_{n_1=1}^{M} b_{ij}(n_1+n) \prod_{n_2=1}^{n_1-1} h_{ij}(n_2+n)], \\
\Delta_{ij} &= [\prod_{n=1}^{M} h_{ij}(n)] - 1, \\
b_{ij}(n) &= -[h_i(n) - h_j(n)] c_{ij}, \\
h_{ij}(n) &= h_i(n) h_j(n), \\
h_i(n) &= k_i \phi_i(n)/\phi_i(n-1), \quad \text{for } i,j = 1,2,3.
\end{aligned}
$$

The usual 3-soliton to Eq.(16) is given by

$$\begin{aligned} f_3(m,n) = 1 &+ r_1(m,n) + r_2(m,n) + r_3(m,n) \\ &+ a_{12} r_1(m,n) r_2(m,n) \\ &+ a_{13} r_1(m,n) r_3(m,n) \\ &+ a_{23} r_2(m,n) r_3(m,n) \\ &+ a_{123} r_1(m,n) r_2(m,n) r_3(m,n), \end{aligned}$$

where

$$\begin{aligned} r_j(m,n) &= \omega_j^m k_j^{(n-n_j)}, \\ \omega_j &= \frac{1+\delta/k_j^M}{1+\delta k_j^M}, \\ a_{ij} &= \frac{k_i^M - k_j^M}{(k_i k_j)^M - 1} \frac{k_i - k_j}{k_i k_j - 1}. \\ a_{123} &= a_{12} a_{13} a_{23}, \end{aligned}$$

for $i, j = 1, 2, 3$.

While 3-periodic phase soliton solution is given by

$$
\begin{aligned}
f_3(m,n) = 1 &+ r_1(m,n) + r_2(m,n) + r_3(m,n) \\
&+ a_{12}(n) r_1(m,n) r_2(m,n) \\
&+ a_{13}(n) r_1(m,n) r_3(m,n) \\
&+ a_{23}(n) r_2(m,n) r_3(m,n) \\
&+ a_{123}(n) r_1(m,n) r_2(m,n) r_3(m,n),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{123}(n) &= -(1/\Delta_{123}) \\
&\quad \times [\sum_{n_1=1}^{M} b_{123}(n_1+n) \prod_{n_2=1}^{n_1-1} h_{123}(n_2+n)], \\
\Delta_{123} &= [\prod_{n=1}^{M} h_{123}(n)] - 1, \\
b_{123} &= \bar{b}_{12}(n) - \bar{b}_{13}(n) + \bar{b}_{23}(n), \\
h_{123} &= h_1(n) h_2(n) h_3(n), \\
\bar{b}_{12}(n) &= [a_{12}(n) h_{12}(n) - a_{12}(n-1) h_3(n)] c_{13} c_{23}, \\
\bar{b}_{13}(n) &= [a_{13}(n) h_{13}(n) - a_{13}(n-1) h_2(n)] c_{12} c_{23}, \\
\bar{b}_{23}(n) &= [a_{23}(n) h_{23}(n) - a_{23}(n-1) h_1(n)] c_{12} c_{13}.
\end{aligned}
$$

What we get,substituting the conjectured $\tau-$function (17) into the bilinear form (16), is not an explicit form of $a_{12}(n)$ nor $a_{12}(n+1)$, but a relation between $a_{12}(n)$ and $a_{12}(n+1)$.

We have totally M such relations,which determine an individual $a_{12}(n)$. a_{12} is not a scalar but a vector whose elements are $a_{12}(n)$, for $n = 1, 2, \cdots, M$.