一般大久保型方程式と middle convolutionの拡張について

(Generalized Okubo systems and the middle convolution for non-Fuchsian systems)

川上 拓志（東京大学大学院数理科学研究科）

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Outline

- Introduction to Katz theory and Yokoyama theory
- Generalized Okubo systems
- Definition of π
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For the sake of simplicity, we represent a Fuchsian system of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \left(\frac{A_1}{x-t_1} + \cdots + \frac{A_p}{x-t_p}\right) Y \qquad (m \times m)$$

as $\mathcal{A} = (A_1, \ldots, A_p)$.

number of accessory parameters N:

$$N := 2 + (p-1)m^2 - \sum_{\nu=0}^{p} \dim Z(A_\nu) \qquad \left(A_0 := -\sum_{\nu=1}^{p} A_\nu\right).$$

$Z(A_\nu)$: centralizer of A_ν. For example,

$$M = \begin{pmatrix} \lambda_1 I_{l_1} & & \\ & \lambda_2 I_{l_2} & \\ & & \lambda_3 I_{l_3} \end{pmatrix} \Rightarrow \dim Z(M) = (l_1)^2 + (l_2)^2 + (l_3)^2.$$

Example. $p = 2$, $m = 2$ (Gauss' hypergeometric equation)

$N = 2 + (2-1) \times 2^2 - \sum_{\nu=0}^{2}(1^2 + 1^2) = 2 + 4 - 6 = 0$

: rigid ($\Leftrightarrow$ accessory parameter free)

Example. $p = 3$, $m = 2$

$N = 2 + (3-1) \times 2^2 - \sum_{\nu=0}^{3}(1^2 + 1^2) = 2 + 8 - 8 = 2.$

We can regard rigid Fuchsian systems as generalizations of the Gauss hypergeometric equation.

How do we get all rigid Fuchsian systems?

$\longleftarrow$ Katz theory and Yokoyama theory.

Katz theory

Katz introduced the operations, called *addition* and *middle convolution*, and he showed the theorem:

Theorem (Katz) . *Every irreducible rigid Fuchsian system is obtained from rank 1 Fuchsian system by a finite iteration of the two operations.*

We explain here the Katz's operations which are reformulated by Dettweiler and Reiter in terms of linear algebra.

Definition (addition) . For $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$, an operation

$$\mathcal{A} \mapsto (A_1 + \alpha_1 I_m, \ldots, A_p + \alpha_p I_m)$$

is called *addition.*

Fix $\lambda \in \mathbb{C}$.

We put a $pm \times pm$ matrix G_ν as follows:

$$G_\nu = \begin{pmatrix} O_m & & \ldots & & O_m \\ \vdots & \ddots & & & \vdots \\ A_1 & \ldots & A_\nu + \lambda I_m & \ldots & A_p \\ \vdots & & & \ddots & \vdots \\ O_m & & \ldots & & O_m \end{pmatrix} \quad (\nu = 1, \ldots, p).$$

Definition (convolution) . The system $(G_1, \ldots, G_p)$ is called *convolution with λ of $\mathcal{A}$.* We denote this system by $c_\lambda(\mathcal{A})$.

Let $\mathcal{K}, \mathcal{L}_\lambda$ be the linear subspaces of $\mathbb{C}^{pm}$:

$$\mathcal{K} := \begin{pmatrix} \mathrm{Ker}(A_1) \\ \vdots \\ \mathrm{Ker}(A_p) \end{pmatrix},$$

$$\mathcal{L}_\lambda := \mathrm{Ker}(G_1 + \cdots + G_p).$$

$\mathcal{K}, \mathcal{L}_\lambda$ are $G_1, \ldots, G_p$-invariant subspaces.

Let $\bar{G}_\nu$ be an endomorphism of $\mathbb{C}^{pm}/(\mathcal{K} + \mathcal{L}_\lambda)$ induced by G_ν.
Definition (middle convolution) . We call the system $(\bar{G}_1, \ldots, \bar{G}_p)$ *middle convolution with* λ *of* $\mathcal{A}$ and denote by $mc_\lambda(\mathcal{A})$.

Correspondence of solutions

Let Y be a solution of $\mathcal{A} = (A_1, \ldots, A_p)$.

addition

$\prod (x - t_\nu)^{\alpha_\nu} Y$: solution of $(A_1 + \alpha_1 I_m, \ldots, A_p + \alpha_p I_m)$.

convolution

Put $F(x) := \begin{pmatrix} \frac{Y(x)}{x - t_1} \\ \vdots \\ \frac{Y(x)}{x - t_p} \end{pmatrix}$. Then

$\int_\gamma (x - t)^\lambda F(t) \mathrm{d}t$: solution of $c_\lambda(\mathcal{A}) = (G_1, \ldots, G_p)$.

Yokoyama theory

Yokoyama introduced the operations, called *extension* and *restriction* for Okubo systems, and he showed the theorem:

Theorem (Yokoyama) . *Every irreducible rigid semisimple Okubo system is obtained from rank 1 Okubo system by a finite iteration of the two operations.*

Here Okubo system means a system of linear differential equations of the form:

$$(xI_n - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = A\Psi.$$

T is an $n \times n$ constant diagonal matrix, A is an $n \times n$ constant matrix.

Yokoyama theory is a theory for Okubo systems.

On the other hand, Katz's middle convolution is closely related to transform a given equation into Okubo system. Thus the Okubo system also plays an important role in Katz theory.

Then, it is natural to focus on the Okubo systems when we want to generalize the theory by Katz and Yokoyama to deal with non-Fuchsian systems.

Recently, Oshima gave a concrete relation between Katz's middle convolution and Yokoyama's extension and showed the equivalence of both algorithms.

In what follows, we mainly consider a generalization of the middle convolution.

Generalized Okubo system

A system of linear differential equations of the form

$$(xI_n - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = A\Psi \tag{1}$$

is called an Okubo system.

T : $n \times n$ constant diagonal matrix, A : $n \times n$ constant matrix.

When T is of the form

$$T = \begin{pmatrix} t_1 I_{l_1} & & \\ & \ddots & \\ & & t_p I_{l_p} \end{pmatrix},$$

(1) has regular singularities at $x = t_1, \ldots, t_p$ and $x = \infty$.

When the matrix T is not semisimple, the system (1) may have irregular singularities.

In the case when T is a Jordan matrix, non-semisimple, call (1) *generalized Okubo system*.

Example. $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (1) $\Longleftrightarrow$

$$\begin{aligned} \frac{\mathrm{d}\Psi}{\mathrm{d}x} &= (xI - T)^{-1} A \Psi \\ &= \left\{ \frac{1}{x^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A + \frac{1}{x} A \right\} \Psi. \end{aligned}$$

We assume that the matrix A is semisimple and denote its non-zero eigenvalues by $-\rho_1, \ldots, -\rho_m$, namely, we put

$$A = -GRG^{-1}, \ R = \mathrm{diag}(\rho_1, \ldots, \rho_m, 0, \ldots, 0).$$

We represent the following (generalized) Okubo system

$$(xI - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = -GRG^{-1}\Psi$$

as (T, R, G).

Let $\mathsf{Stab}(M)$ be the stabilizer of $M \in M(n, \mathbb{C})$, i.e.

$$\mathsf{Stab}(M) = \{g \in GL(n, \mathbb{C}) \mid gM = Mg\}.$$

For a Jordan matrix T and a diagonal matrix

$$R = \mathsf{diag}(\rho_1, \ldots, \rho_m, 0, \ldots, 0), \tag{2}$$

let $\mathcal{O}(T, R)$ be the following set of systems:

$$\mathcal{O}(T, R) := \{(T, R, G)\} / \underset{\mathcal{O}}{\sim}\ .$$

Here the equivalent relation $\underset{\mathcal{O}}{\sim}$ is defined by

$$G \underset{\mathcal{O}}{\sim} hGg \quad (h \in \mathsf{Stab}(T),\ g \in \mathsf{Stab}(R)).$$

We write the set of all generalized Okubo systems as

$$\mathcal{GO} := \coprod_{T,R} \mathcal{O}(T,R),$$

where T runs over all Jordan matrix, including diagonal matrices, and R runs over all diagonal matrices of the form (2).

Similarly, we denote the set of all Okubo systems, a subset of $\mathcal{GO}$, by

$$\mathcal{O} := \coprod_{T,R} \mathcal{O}(T,R)$$

where T runs over all diagonal matrices.

Let X_p be the following set:

$$X_p := \{(t_1, \ldots, t_p) \in \mathbb{C}^p \mid t_i \neq t_j \ (i \neq j)\}.$$

We put $\Gamma_{(m,p)}$ and $\Gamma^*_{(m,p)}$ as

$$\begin{aligned}\Gamma_{(m,p)} &= X_p \times (\mathbb{Z}_{\geq 0})^p \times (\mathbb{C}^\times)^m,\\ \Gamma^*_{(m,p)} &= X_p \times (\mathbb{C}^\times)^m.\end{aligned}$$

We regard the set $\Gamma^*_{(m,p)}$ as a subset of $\Gamma_{(m,p)}$ through the inclusion mapping

$$\begin{aligned}\Gamma^*_{(m,p)} &\hookrightarrow \Gamma_{(m,p)}\\ (t_1, \ldots, t_p, \rho_1, \ldots, \rho_m) &\mapsto (t_1, \ldots, t_p, \overbrace{0, \ldots, 0}^{p}, \rho_1, \ldots, \rho_m).\end{aligned}$$

For every element

$$\gamma = (t_1, \ldots, t_p, r_1, \ldots, r_p, \rho_1, \ldots, \rho_m) \in \Gamma_{(m,p)},$$

we denote by $\tilde{R}_\gamma$ the $m \times m$ diagonal matrix

$$\mathrm{diag}(\rho_1, \ldots, \rho_m).$$

Then we define $\mathcal{E}_\gamma$ by

$$\mathcal{E}_\gamma = \left\{ A(x) = \sum_{\nu=1}^{p} \sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x-t_\nu)^{k+1}} \;\middle|\; A_\nu^{(-k)} \in M(m,\mathbb{C}),\ A_\nu^{(-r_\nu)} \neq O,\ -\sum_{\nu=1}^{p} A_\nu^{(0)} = \tilde{R}_\gamma \right\} \Big/ \underset{\mathcal{E}_\gamma}{\sim} .$$

Here equivalent relation $\underset{\mathcal{E}_\gamma}{\sim}$ is defined by

$$A(x) \underset{\mathcal{E}_\gamma}{\sim} gA(x)g^{-1} \quad (g \in \mathrm{Stab}(\tilde{R}_\gamma)).$$

We identify an element $A(x)$ of $\mathcal{E}_\gamma$ with the system

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = A(x)Y.$$

We put

$$\mathcal{E} := \coprod_{m,p\in\mathbb{Z}_{\geq 1}} \coprod_{\gamma\in\Gamma_{(m,p)}} \mathcal{E}_\gamma,$$
$$\mathcal{F} := \coprod_{m,p\in\mathbb{Z}_{\geq 1}} \coprod_{\gamma\in\Gamma^*_{(m,p)}} \mathcal{E}_\gamma,$$

namely $\mathcal{E}$ is the set of systems of linear differential equations on $\mathbb{P}^1$ which have regular singularity at infinity, and $\mathcal{F}$ is the set of Fuchsian systems on $\mathbb{P}^1$.

Definition of $\pi : \mathcal{GO} \to \mathcal{E}$

Let $[T, R, G]$ be an arbitrary element of $\mathcal{GO}$, that is, a system of the form

$$(xI - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = -GRG^{-1}\Psi. \tag{3}$$

Here T is an $n \times n$ Jordan matrix (not necessarily diagonal).

We put $R = \mathrm{diag}(\rho_1, \ldots, \rho_m, 0, \ldots, 0)$.

By changing the unknown function of (3) as $\Psi = G\tilde{\Psi}$, we have

$$\frac{\mathrm{d}\tilde{\Psi}}{\mathrm{d}x} = -G^{-1}(xI - T)^{-1}GR\tilde{\Psi}.$$

The coefficient of the right-hand side is rewritten into the following form:

$$-G^{-1}(xI - T)^{-1}GR = \sum_{\nu=1}^{p} \sum_{k=0}^{r_\nu} \frac{B_\nu^{(-k)}}{(x - t_\nu)^{k+1}}.$$

Here

$$B_\nu^{(-k)} := -G^{-1} J_\nu^{(-k)} GR.$$

$J_\nu^{(-k)}$ denotes the coefficient matrix of $1/(x - t_\nu)^{k+1}$ in $(xI - T)^{-1}$.

Since the last $n-m$ columns of R are zero, the matrix $B_\nu^{(-k)}$ is of the form

$$B_\nu^{(-k)} = \begin{pmatrix} A_\nu^{(-k)} & O_{m,n-m} \\ X_\nu^{(-k)} & O_{n-m,n-m} \end{pmatrix},$$

$A_\nu^{(-k)}$ being some $m \times m$ matrix and $X_\nu^{(-k)}$ some $(n-m) \times m$ matrix. Starting from $[T, R, G]$, we obtain

$$\sum_{\nu=1}^{p} \sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x-t_\nu)^{k+1}} \in \mathcal{E}.$$

The definition of $\pi : \mathcal{GO} \to \mathcal{E}$ is summarized as follows:

For $[T, R, G] \in \mathcal{GO}$,

$$\pi(T, R, G) := \text{the principal } m \times m \text{ part of}(-G^{-1}(xI - T)^{-1}GR).$$

This is well-defined.

Relation to the middle convolution

We investigate the relation between the map $\pi|_{\mathcal{O}} : \mathcal{O} \to \mathcal{F}$ and the middle convolution.

Let $F = \left[\sum_{\nu=1}^{p} \frac{A_\nu^{(0)}}{x - t_\nu}\right]$ be an element of $\mathcal{F}$ whose matrix size is m. Put $\text{rank} A_\nu^{(0)} = l_\nu$. Then $A_\nu^{(0)}$ is factorized into

$$A_\nu^{(0)} = B_\nu C_\nu,$$

where B_ν is $m \times l_\nu$ matrix, C_ν is $l_\nu \times m$ matrix, and

$$\text{rank} B_\nu = \text{rank} C_\nu = l_\nu.$$

We put $n = l_1 + \cdots + l_p$.

We define the $n \times n$ matrices T_{min}, A_{min} as follows:

$$T_{\mathsf{min}} = \begin{pmatrix} t_1 I_{l_1} & & \\ & \ddots & \\ & & t_p I_{l_p} \end{pmatrix},$$

$$A_{\mathsf{min}} = \begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} (B_1 \dots B_p).$$

Proposition 1. *The minimal size Okubo system in $\pi^{-1}(F)$ uniquely exists up to conjugate action of* $\mathsf{Stab}(T_{\mathsf{min}})$ *and is given as follows:*

$$(xI - T_{\mathsf{min}}) \frac{\mathsf{d}\Psi}{\mathsf{d}x} = A_{\mathsf{min}} \Psi.$$

In particular, $\pi|_{\mathcal{O}} : \mathcal{O} \to \mathcal{F}$ *is surjective.*

Example. $m = 2$, $p = 3$

Eigenvalues of $A_\nu^{(0)}$ are $0, \theta_\nu$ $(\nu = 1, 2, 3)$.

$A_\nu^{(0)}$ is parametrized as

$$A_\nu^{(0)} = \frac{1}{2}\begin{pmatrix} a_\nu b_\nu + \theta_\nu & -a_\nu^2 \\ b_\nu^2 - \frac{\theta_\nu^2}{a_\nu^2} & -a_\nu b_\nu + \theta_\nu \end{pmatrix}$$
$$= \begin{pmatrix} a_\nu \\ \frac{a_\nu b_\nu - \theta_\nu}{a_\nu} \end{pmatrix} \begin{pmatrix} \frac{a_\nu b_\nu + \theta_\nu}{2a_\nu} & -\frac{a_\nu}{2} \end{pmatrix}.$$

A_{min} is given by

$$A_{\mathsf{min}} = \begin{pmatrix} \frac{a_1b_1+\theta_1}{2a_1} & -\frac{a_1}{2} \\ \frac{a_2b_2+\theta_2}{2a_2} & -\frac{a_2}{2} \\ \frac{a_3b_3+\theta_3}{2a_3} & -\frac{a_3}{2} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ \frac{a_1b_1-\theta_1}{a_1} & \frac{a_2b_2-\theta_2}{a_2} & \frac{a_3b_3-\theta_3}{a_3} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2\theta_1 & c_{12} & c_{13} \\ c_{21} & 2\theta_2 & c_{23} \\ c_{31} & c_{32} & 2\theta_3 \end{pmatrix},$$

where

$$c_{ij} := a_jb_i - a_ib_j + \theta_i\frac{a_j}{a_i} + \theta_j\frac{a_i}{a_j}.$$

The relation to the middle convolution is as follows.

Proposition 2. *For any $\lambda \in \mathbb{C}$, the middle convolution of*

$$F = \sum_{\nu=1}^{p} \frac{A_\nu^{(0)}}{x - t_\nu} \in \mathcal{F}$$

with λ coincides with the image of the following system under π:

$$(xI - T_{\mathrm{min}})\frac{\mathrm{d}\Psi}{\mathrm{d}x} = (A_{\mathrm{min}} + \lambda I)\Psi.$$

Hence, the middle convolution is obtained by the following procedure:

1. Lift a system in $\mathcal{F}$ to $\mathcal{O}$ of the minimal size.

2. Shift the right-hand side with scalar matrix:

$$T_\lambda(T, R, G) = (T, R + \lambda I, G).$$

3. Take an image of this in $\mathcal{F}$ by π.

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{T_\lambda} & \mathcal{O} \\ {\scriptstyle \pi|_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \pi|_{\mathcal{O}}} \\ \mathcal{F} & \xrightarrow[mc_\lambda]{} & \mathcal{F} \end{array}$$

Shift of the right-hand side of Okubo systems by a scalar matrix is realized by the Euler transformation:

$$\Psi(x) \mapsto \int \Psi(t)(x-t)^{\lambda}\mathrm{d}t.$$

Therefore we can say that the middle convolution is "Transform F into Okubo system + Euler transform".

By taking the above consideration into account, we can define an analogue of the middle convolution for non-Fuchsian systems by the same procedure.

$$\begin{array}{ccc} \mathcal{GO} & \xrightarrow{T_\lambda} & \mathcal{GO} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{E} & \xrightarrow[\text{``}mc_\lambda\text{''}]{} & \mathcal{E} \end{array}$$

It is necessary to show the surjectivity of π so that this procedure may work.

Surjectivity of π

Let $\displaystyle\sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu}\frac{A_\nu^{(-k)}}{(x-t_\nu)^{k+1}}$ be a size m element of $\mathcal{E}$.

We put $\tilde{r}_\nu := m(r_\nu+1), \quad n := \sum_{\nu=1}^{p}\tilde{r}_\nu$.

Let $\tilde{A}_\nu$ be the following $\tilde{r}_\nu \times n$ matrix:

$$\tilde{A}_\nu := \begin{pmatrix} & & & O_{mr_\nu,n} & & & \\ A_1^{(-r_1)} & \dots & A_1^{(0)} & \dots & A_p^{(-r_p)} & \dots & A_p^{(0)} \end{pmatrix}.$$

We put the matrices $\tilde{A}, T$, and P as follows:

$$\tilde{A} := \begin{pmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_p \end{pmatrix},$$

$$T := J_{r_1+1}(t_1)^{\oplus m} \oplus \cdots \oplus J_{r_p+1}(t_p)^{\oplus m},$$

$$P := P_{(m,r_1+1)} \oplus \cdots \oplus P_{(m,r_p+1)}.$$

Here $J_k(a)$ ($a \in \mathbb{C},\ k \in \mathbb{Z}_{\geq 1}$) is the $k \times k$ Jordan block with eigenvalue a, and $P_{(i,j)}$ is a permutation matrix

$$P_{(i,j)} = (I_i \otimes \boldsymbol{e}_1, I_i \otimes \boldsymbol{e}_2, \ldots, I_i \otimes \boldsymbol{e}_j),$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_j$ are unit vectors of $\mathbb{C}^j$.

We consider the generalized Okubo system

$$(xI_n - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = (P\tilde{A}P^{-1} + \lambda I_n)\Psi. \tag{4}$$

Definition 1. We call the generalized Okubo system (4) *convolution of E with λ* and denote it by $c_\lambda(E)$.

Theorem 3. *For any element $E \in \mathcal{E}$, $c_0(E) \in \pi^{-1}(E)$. Therefore the map $\pi : \mathcal{GO} \to \mathcal{E}$ is surjection.*

Remark 1. The assumption that E has at least one regular singular point is not essential since, by a gauge transformation $Y \to (x-a)^{\alpha} Y$, we can add the term $\frac{\alpha}{x-a}$ to E.

Remark 2. When the corank of leading terms $A_{\nu}^{(-r_{\nu})}$ $(\nu = 1, \ldots, p)$ of E are all zero, the system $c_0(E)$ is the minimal size generalized Okubo system in $\pi^{-1}(E)$.

Examples of m.c. for non-Fuchsian systems

We give three examples of the middle convolution for non-Fuchsian systems.

Example. The system satisfied by ${}_3F_1$:

$$\left(x - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \frac{\mathrm{d}\Psi}{\mathrm{d}x} = \begin{pmatrix} \lambda_1 & \frac{\rho_1\rho_2\rho_3}{\lambda_2 t} & 1 \\ t & 0 & 0 \\ u & 0 & \lambda_2 \end{pmatrix} \Psi$$

where $u = \lambda_1\lambda_2 - \rho_1\rho_2 - \rho_2\rho_3 - \rho_3\rho_1 - \frac{\rho_1\rho_2\rho_3}{\lambda_2}$.

The Riemann scheme of this system is

$$\left\{ \begin{array}{ccc} \multicolumn{2}{c}{x = 0} & x = \infty \\ \multicolumn{2}{c}{\overbrace{\hphantom{0 \quad \lambda_2}}} & \\ 0 & 0 & \rho_1 \\ 0 & \lambda_2 & \rho_2 \\ t & \lambda_1 & \rho_3 \end{array} \right\}.$$

$$\xrightarrow{mc_{\rho_3}} \frac{A_1^{(-1)}}{x^2} + \frac{A_1^{(0)}}{x}$$ (rank 2 system) where

$$A_1^{(-1)} = \frac{t}{\rho_1 - \rho_2} \begin{pmatrix} \lambda_2 + \rho_1 & \lambda_2 + \rho_2 \\ -(\lambda_2 + \rho_1) & -(\lambda_2 + \rho_2) \end{pmatrix},$$

$$A_1^{(0)} = - \begin{pmatrix} \rho_1 - \rho_3 & 0 \\ 0 & \rho_2 - \rho_3 \end{pmatrix}.$$

The Riemann scheme is

$$\left\{ \begin{array}{ccc} \overbrace{\begin{array}{cc} & \end{array}}^{x=0} & x = \infty & \\ 0 \quad \lambda_2 + \rho_3 & \rho_1 - \rho_3 & \\ t \quad \lambda_1 + 2\rho_3 & \rho_2 - \rho_3 & \end{array} \right\}.$$

$$\xrightarrow{} \atop add_{-(\lambda_2+\rho_3)}$$

$$\overrightarrow{add_{-(\lambda_2+\rho_3)}} \left\{ \begin{array}{ccc} \multicolumn{2}{c}{x=0} & x=\infty \\ \multicolumn{2}{c}{\overbrace{\hphantom{0 \quad \lambda_1-\lambda_2+\rho_3}}} & \\ 0 & 0 & \rho_1+\lambda_2 \\ t & \lambda_1-\lambda_2+\rho_3 & \rho_2+\lambda_2 \end{array} \right\}.$$

$\overrightarrow{mc_{\rho_1+\lambda_2}}$ rank 1 system.

Example. Fifth Painlevé equation

We consider the system of linear differential equations L_{V} given by the following Riemann scheme:

$$\left\{ \begin{array}{cccc} x=0 & \multicolumn{2}{c}{x=1} & x=\infty \\ 0 & \overbrace{0 \qquad\qquad 0} & & \alpha_0 \\ \alpha_3 & t & \alpha_2-\alpha_0 & \alpha_0+\alpha_1-1 \end{array} \right\}.$$

Then the system L_{V} is written as follows:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \left(\frac{A_1^{(-1)}}{(x-1)^2} + \frac{A_1^{(0)}}{x-1} + \frac{A_0^{(0)}}{x}\right) Y,$$

$$A_1^{(-1)} = \begin{pmatrix} z_1 + t & -vz_1 \\ (z_1+t)/v & -z_1 \end{pmatrix}, \quad A_1^{(0)} = -A_0^{(0)} - \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 + \alpha_1 - 1 \end{pmatrix},$$

$$A_0^{(0)} = \begin{pmatrix} z_0 + \alpha_3 & -uz_0 \\ (z_0+\alpha_3)/u & -z_0 \end{pmatrix}, \tag{5}$$

$$
\begin{aligned}
(1-\alpha_1)z_0 = {} & \lambda^2(\lambda-1)^2\mu^2 \\
& + \{\alpha_0(\lambda-1)-\alpha_2-t\}\{(\lambda-1)\mu+\alpha_0\}\lambda \\
& + \{\alpha_0\lambda(\lambda-1)-t\}\lambda\mu+\alpha_3(\alpha_1-1), \\
(1-\alpha_1)z_1 = {} & \lambda(\lambda-1)^3\mu^2 \\
& + \{2\alpha_0\lambda^2-(2\alpha_0-\alpha_3+t)\lambda-\alpha_3\} \\
& \qquad \times \{(\lambda-1)\mu+\alpha_0\} \\
& - \alpha_0^2\lambda(\lambda-1)+(\alpha_0+\alpha_1-1)t, \\
v = {} & \frac{\lambda-1}{\lambda}\frac{z_0}{z_1}u.
\end{aligned}
$$

The parameter λ is a position of the apparent singular point. The holonomic deformation of (5) is governed by the fifth Painlevé equation P_{V}.

Proposition 4. *The minimal size generalized Okubo system in* $\pi^{-1}(L_{\mathsf{V}})$ *is uniquely given as follows:*

$$(xI_3 - T_{\mathsf{V}})\frac{\mathrm{d}\Psi}{\mathrm{d}x} = C_{\mathsf{V}}\Psi$$

where

$$T_{\mathsf{V}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_{\mathsf{V}} = \begin{pmatrix} \alpha_2 - \alpha_0 & -\frac{1}{t}\det A_1^{(0)} & (C_{\mathsf{V}})_{13} \\ t & 0 & (C_{\mathsf{V}})_{23} \\ (C_{\mathsf{V}})_{31} & (C_{\mathsf{V}})_{32} & \alpha_3 \end{pmatrix},$$

$$
\begin{aligned}
(C_{\mathsf{V}})_{23} &= ((\lambda-1)\mu+\alpha_0)\lambda+\alpha_3,\\
(C_{\mathsf{V}})_{31} &= t-\{(\lambda-1)\mu+\alpha_0\}(\lambda-1),\\
t(C_{\mathsf{V}})_{32} &= (\alpha_1-1)z_1\\
&\quad+(\alpha_0+\alpha_1-1)(t-((\lambda-1)\mu+\alpha_0)(\lambda-1)),\\
(C_{\mathsf{V}})_{13} &= \frac{1}{t-\{(\lambda-1)\mu+\alpha_0\}(\lambda-1)}\times\\
&\quad\{(\alpha_1-1)z_0-(((\lambda-1)\mu+\alpha_0)\lambda+\alpha_3)(C_{\mathsf{V}})_{32}\\
&\quad-\alpha_3(\alpha_0+\alpha_3)\}.
\end{aligned}
$$

The middle convolution of (5) with α_0 is

$$\begin{aligned}
\frac{\mathrm{d}Y}{\mathrm{d}x} &= \left(\frac{\bar{A}_1^{(-1)}}{(x-1)^2} + \frac{\bar{A}_1^{(0)}}{x-1} + \frac{\bar{A}_0^{(0)}}{x}\right) Y, \\
\bar{A}_0^{(0)} &= \begin{pmatrix} \bar{z}_0 + \alpha_3 + \alpha_0 & -\bar{u}\bar{z}_0 \\ (\bar{z}_0 + \alpha_3 + \alpha_0)/\bar{u} & -\bar{z}_0 \end{pmatrix}, \\
\bar{A}_1^{(-1)} &= \begin{pmatrix} \bar{z}_1 + t & -\bar{v}\bar{z}_1 \\ (\bar{z}_1 + t)/\bar{v} & -\bar{z}_1 \end{pmatrix}, \\
\bar{A}_1^{(0)} &= -\bar{A}_0^{(0)} - \begin{pmatrix} -\alpha_0 & 0 \\ 0 & \alpha_1 - 1 \end{pmatrix},
\end{aligned} \tag{6}$$

$$\bar{z}_0 = \frac{\alpha_1 - 1}{\alpha_0 + \alpha_1 - 1} z_0, \quad \bar{z}_1 = \frac{\alpha_1 - 1}{\alpha_0 + \alpha_1 - 1} z_1,$$
$$\bar{v} = \frac{\lambda + \alpha_0/\mu - 1}{\lambda + \alpha_0/\mu} \frac{z_0}{z_1} \bar{u}.$$

By comparing (5) and (6), we obtain the transformation

$$\alpha_0 \mapsto -\alpha_0, \quad \alpha_1 \mapsto \alpha_1 + \alpha_0, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \alpha_3 + \alpha_0,$$
$$t \mapsto t, \quad \lambda \mapsto \lambda + \frac{\alpha_0}{\mu}, \quad \mu \mapsto \mu.$$

Example. Fourth Painlevé equation

Next we consider the system of linear differential equations L_{IV} given by the following Riemann scheme:

$$\left\{ \begin{array}{cc} \overbrace{\begin{array}{ccc} & x=0 & \end{array}} & x=\infty \\ \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & t & \alpha_2-\alpha_0 \end{array} & \begin{array}{c} \alpha_0 \\ \alpha_0+\alpha_1-1 \end{array} \end{array} \right\}.$$

L_{IV} is written as follows:

$$
\begin{aligned}
\frac{\mathrm{d}Y}{\mathrm{d}x} &= \left(\frac{A_0^{(-2)}}{x^3} + \frac{A_0^{(-1)}}{x^2} + \frac{A_0^{(0)}}{x} \right) Y, \\
A_0^{(-2)} &= \begin{pmatrix} z + 1/2 & -uz \\ (z + 1/2)/u & -z \end{pmatrix}, \\
A_0^{(-1)} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \\
A_0^{(0)} &= -\begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 + \alpha_1 - 1 \end{pmatrix},
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
2(1-\alpha_1)z = \ & (2\lambda^3\mu + 2\alpha_0\lambda^2 - 2t\lambda - 1)(\lambda\mu + \alpha_0) \\
& \qquad\qquad + \alpha_0 + \alpha_1 - 1, \\
\lambda a_{11} = \ & -(z + 1/2) + \lambda^3\mu + \alpha_0\lambda^2, \\
a_{12} = \ & uz/\lambda, \\
a_{21} = \ & \frac{\lambda}{uz}\{a_{11}(t - a_{11}) - (\alpha_1 - 1)z \\
& \qquad\qquad - (\alpha_0 + \alpha_1 - 1)/2\}, \\
a_{22} = \ & t - a_{11}.
\end{aligned}
$$

Proposition 5. *The minimal size generalized Okubo system in* $\pi^{-1}(L_{\mathrm{IV}})$ *is uniquely given as follows:*

$$(xI_3 - T_{\mathrm{IV}})\frac{\mathrm{d}\Psi}{\mathrm{d}x} = C_{\mathrm{IV}}\Psi. \tag{8}$$

$$T_{\mathrm{IV}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$C_{\mathrm{IV}} = \begin{pmatrix} 2(\alpha_1 - 1)z - \alpha_0 & (C_{\mathrm{IV}})_{12} & (C_{\mathrm{IV}})_{13} \\ 0 & -2(\alpha_1 - 1)z + \alpha_2 & (C_{\mathrm{IV}})_{23} \\ \frac{1}{2} & t & 0 \end{pmatrix},$$

$$
\begin{aligned}
(C_{\mathrm{IV}})_{13} =&\ 4(\alpha_1 - 1)\lambda\{2\lambda(\lambda\mu + \alpha_0)^2 - \mu\}z, \\
(C_{\mathrm{IV}})_{23} =&\ -4(\alpha_1 - 1)\lambda(\lambda\mu + \alpha_0)z, \\
(C_{\mathrm{IV}})_{12} =&\ \frac{((-2(\alpha_1 - 1)z + \alpha_2)(C_{\mathrm{IV}})_{13}}{(C_{\mathrm{IV}})_{23}} \\
&\qquad + 2t(2(\alpha_1 - 1)z - \alpha_0).
\end{aligned}
$$

Remark 3. By means of Laplace transform

$$\Psi(x) = \int e^{-xz} \Phi(z) \mathrm{d}z,$$

(8) transforms into

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \left(T_{\mathrm{IV}} - \frac{C_{\mathrm{IV}} + I}{z} \right) \Phi.$$

This system is essentially the linear equation associated with the Noumi-Yamada system of type $A_2^{(1)}$.

The middle convolution of (7) with α_0 is

$$
\begin{aligned}
\frac{\mathrm{d}Y}{\mathrm{d}x} &= \left(\frac{\bar{A}_0^{(-2)}}{x^3} + \frac{\bar{A}_0^{(-1)}}{x^2} + \frac{\bar{A}_0^{(0)}}{x}\right) Y, \\
\bar{A}_0^{(-2)} &= \begin{pmatrix} \bar{z} + 1/2 & -\bar{u}\bar{z} \\ (\bar{z} + 1/2)/\bar{u} & -\bar{z} \end{pmatrix}, \\
\bar{A}_0^{(-1)} &= \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix}, \\
\bar{A}_0^{(0)} &= -\begin{pmatrix} -\alpha_0 & 0 \\ 0 & \alpha_1 - 1 \end{pmatrix},
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\bar{z} &= \frac{\alpha_0 + \alpha_2}{\alpha_2} z, \\
\bar{a}_{11} &= \frac{1}{\alpha_2} \left\{ (\alpha_0 + \alpha_2) a_{11} - \alpha_0 t \right\}, \ \bar{a}_{12} = \frac{\bar{u}\bar{z}}{\lambda + \alpha_0/\mu}, \\
\bar{a}_{21} &= \frac{\lambda + \alpha_0/\mu}{\bar{u}\bar{z}} \left\{ \bar{a}_{11}(t - \bar{a}_{11}) + \alpha_2 \bar{z} + \frac{\alpha_0 + \alpha_2}{2} \right\}, \\
\bar{a}_{22} &= t - \bar{a}_{11}.
\end{aligned}
$$

By comparing (7) and (9), we have

$$
\begin{gathered}
\alpha_0 \mapsto -\alpha_0, \quad \alpha_1 \mapsto \alpha_1 + \alpha_0, \quad \alpha_2 \mapsto \alpha_2, \\
t \mapsto t, \quad \lambda \mapsto \lambda + \frac{\alpha_0}{\mu}, \quad \mu \mapsto \mu.
\end{gathered}
$$