Kavli Asian Winter School 2023

Asymptotic Symmetries & Celestial Holography

Plan : I) Infrared aspects of gravity (& gauge thy.) in 4d - AFS

a) Bondi gauge & asymptotic symm.

b) Matching condition & charge conserv.

c) Scattering & tower of soft theorems

d) Observables / memory effects

II) a) Conformal primary basis

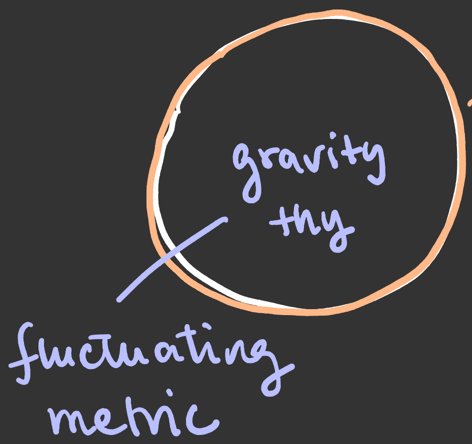
b) Celestial amplitudes

c) Comments on AdS / CFT in flat space limit

III) a) Celestial OPE / symmetries

b) Infinite symmetry algebras

Gauge-gravity correspondence



QFT / CFT
fixed background

Protohypical example : AdS/CFT

* match "symmetries"
* —"— observables
* geometry / entanglement
* BH / thermal physics

...

$S_{BH} \propto \frac{A}{4G}$ $\Rightarrow$ "holographic" principle / beyond AdS ($\Lambda < 0$)

(Susskind, 't Hooft '9)

$\boxed{Q}$: Which aspects of gravity are captured by "CFT"?

This course : $\Lambda = 0$

should care about it bc.

- textbook GR
- (3+1)-dimensional
- gravitational waves & astrophys. BH.

4D leading soft thm $\longrightarrow$ subleading $\longrightarrow$ ∞ tower — OUR UNIV.

'13 ; '14 ; '22 matching

4D Asymptotic symm. $\rightarrow$ Virasoro^2 $\longrightarrow$ W_∞ (Part I) ×2

BBMS '62

$\cup$

(C)CFT$_2$ $\longrightarrow$ W_∞ '21 (Part II) ×2

I) IR aspects of gravity in AFS

a) Bondi, v.d. Burg, Metzner & Sachs '60 - '62

4d AFS ($\Lambda = 0$ GR)

* framework for quantifying radiation from isolated sources in spacetime

gravitational waves

Neutron stars, Black Holes, ...

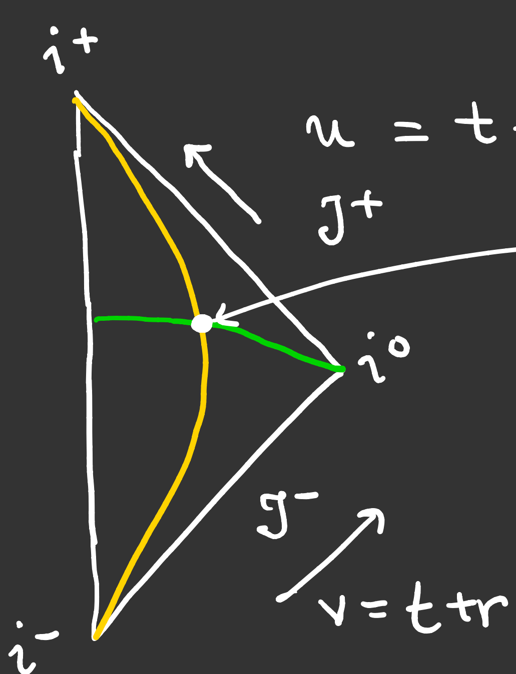
* observer "far away" from source

$\Rightarrow$ perturbation around Mink. background.

[can also do similar analysis in BH (near-horizon region), see eg. REFS.]

In retarded coords (outgoing radiation):

$$ds^2 = -du^2 - 2du\,dr + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z}$$



each pt on S^2 with metric $\gamma_{z\bar{z}} = \dfrac{2}{(1+z\bar{z})^2}$

- $(z, \bar{z})$ related to (θ, ϕ) by stereographic projection (AB)

* fix "residual" diff invariance by choosing a coord. system in which:

1. waves propagate radially along family of null geodesics $u = ct$. $(u = t - r)$

$$\Rightarrow g^{\mu\nu} \partial_\mu u \, \partial_\nu u = 0 \Rightarrow \boxed{g^{uu} = 0} \quad (\times 1)$$

2. Angular coordinates x^A $(A, B = 1, 2)$ are constant along null rays:

$$g^{\mu\nu} \partial_\mu u \, \partial_\nu x^A = 0 \Rightarrow \boxed{g^{uA} = 0} \quad (\times 2)$$

3. Wave fronts are spherical:

$$\boxed{\partial_r \det \left(r^{-2} g_{AB} \right) = 0} \quad (\times 1)$$

Total 4 conditions $(\xi^\mu, \mu = 0, \dots 3$ in 4d$)$

Most general Bondi metric that $\to$ Mink. as $r \to \infty$

$$(*) \quad ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} du dr + g_{AB} (dx^A - U^A du)(dx^B - U^B du)$$

where β, V, g_{AB}, U^A are functions of r, u, x^A.

Let $g_{AB} = r^2 \gamma_{AB} + r C_{AB}(u, z, \bar{z}) + \frac{J_{AB}}{r} + \mathcal{O}(r^{-2})$

- Solving the VACUUM radial E.E $[G_{ur}, G_{rz}, G_{rr} = 0]$ in a large-r expansion yields further constraints:

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ (a lot of structure always pres. in vacuum)

curvature w.r.t $\bar{\gamma}_{AB}$

$$\frac{V}{r} = -\frac{\bar{R}}{2} + \frac{2M}{r} + \mathcal{O}(r^{-2})$$

$$\beta = \frac{1}{r^2}\left(-\frac{1}{32} C_{AB} C^{AB}\right) + \mathcal{O}(r^{-3})$$

$$U^A = -\frac{1}{2r^2} D_B C^{BA} - \frac{2}{3}\frac{1}{r^3}\left[N^A - \frac{1}{2} C^{AB} D^C C_{BC}\right] + \mathcal{O}(r^{-4})$$

[depends on convention]

- Solving G_{uu}, G_{uA}, G_{AB} at $1/r^2$ provides "constraints" / "evolution eq^{n}s" / "flux-balance laws" for M, N^A, J_{AB}

eg for $G_{uu} = 0$: Bondi mass-loss formula

$$\partial_u M = -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{8} \bar{\Box} \bar{R} + \frac{1}{4} D_A D_B N^{AB}$$

← Laplacian w.r.t $\bar{\gamma}_{AB}$

* convenient to choose $\bar{\gamma}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (cel. plane)

$N_{AB} \equiv \partial_u C_{AB}$ where $C_{AB}(u, z, \bar{z})$ is free data [undetermined by the eom] → $N_{AB} \neq 0 \Leftrightarrow$ flux

Comment: $\partial_u N_A$, $\partial_u J_{AB}$ a mess, but will see later how to simplify these eq^{n}s upon organizing the asy. expansion in terms of data that carry definite "spacetime" weights under the action of asymptotic symmetries [in particular superrotations]

- Metrics of the form $(*)$ enjoy a large degree of symmetry

asymptotic symm. = diffeos that preserve the boundary conditions (or fall-offs at large r) and that survive as $r \to \infty$. (conservative def)

Ex: Look for v.f. ξ that obey

$$\mathcal{L}_\xi g_{uu} = \mathcal{O}(r^{-1}), \quad \mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_\xi g_{uz} = \mathcal{O}(1)$$

$$\mathcal{L}_\xi g_{zz} = \mathcal{O}(r), \quad \boxed{\mathcal{L}_\xi g_{z\bar{z}} = \mathcal{O}(1)} \quad (**)$$

$\mathcal{L}$ is the Lie derivative $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$.

$$\xi(\mathcal{T}, Y, W) = \mathcal{T}(z,\bar{z})\,\partial_u + Y^A(z,\bar{z})\,\partial_A + \frac{1}{2} D\cdot Y\,(u\partial_u - r\partial_r) + \ldots$$

$\mathcal{T}(z,\bar{z})$ is an arbitrary fct^n on the sphere

8 param. supertranslations - original BMS extension of Poincaré (Y^A 6 Lorentz transf. $\mathcal{W} = \frac{1}{2} D_A Y^A$)

Y^A enlarge this symm. group by allowing for violation of (2D) conformal Killing eq^{n}s at isolated points on sphere

i) extended BMS: $\partial_z Y^{\bar{z}} = \partial_{\bar{z}} Y^z = 0$ from (**)

$\Rightarrow$ superrotations (2 copies of Witt alg.)

ii) generalized BMS: relax (**) $\Rightarrow$ Diff(S^2)

[motivated by bijection between subleading soft thm. & cons. law] - see Campiglia - Laddha

iii) Allow for Weyl rescalings of S^2

$\frac{1}{2} D \cdot Y \to \mathcal{W}$ is now arbitrary.

$\boxed{\text{Will focus on i)}}$ - classical symm. of gravity include conformal symm. of S^2.

MINIMAL EXT. OF BMS

MOTIVATION for celestial holography

"Codimension-2 holography":

thy. of gravity in 4d AFS ~ CFT on 2d cut of $\mathcal{I}$.

Review Lecture I

* Asy. flat metrics in Bondi gauge + radial EE

$$\Rightarrow ds^2 = ds^2_{Mink} + \left(r\, C_{zz}(u,z,\bar z)\, dz^2 + cc\right)$$
$$+ \frac{2M}{r} du^2 + \dots \equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

↖ subleading in $1/r$

$G_{uu}, G_{uz}, G_{zz} = 0 \Rightarrow$ MESS

(*) $\partial_u M = \dots$, $\partial_u N_A = \dots$, $\partial_u T_{AB} = \dots$

free data : $C_{zz}(u,z,\bar z)$ & fct^{n}s on sphere
(unconstrained) (integration cts of (*))

* | large gauge transformations

Large-r falloffs preserved under $\mathcal{L}_\xi$

* note asy data, in part. C_{AB} still changes as we only demand $\mathcal{L}_\xi g_{zz} = O(r)$

$$\nabla_{(\mu}\xi_{\nu)} = O(r^{-\#})$$

IN GENERAL wrt. full g *, but (if you did exercise) to leading orders in $1/r$ amounts to solving $\nabla^{(0)}_{(\mu}\xi_{\nu)} = O(r^{-\#})$ $\boxed{\text{NOTE NOT } 0}$

$$\Rightarrow \xi = \underset{\text{ST}}{f}\,\partial_u + \underset{\text{SR}}{Y^A(z,\bar z)}\,\partial_A + \frac{1}{2} D\cdot Y\,(u\partial_u - r\partial_r) + O(1/r)$$

Subl. comp. determined from req. Bondi gauge $\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{rA} = 0$

$\xi(J,Y)$ form an algebra (ext BMS4)

$$[\xi(J_1,Y_1), \xi(J_2,Y_2)] = \xi(J_{12}, Y_{12})$$

where $J_{12} = Y_1^A \partial_A J_2 - \frac{1}{2} \partial_A Y^A_1 J_2 - (1 \leftrightarrow 2)$

$$Y_{12}^A = Y_1^A \partial_B Y_2^B - (1 \leftrightarrow 2)$$

[field-dependent bracket for subleading orders in 1/r]

Definition: $\Phi_{h,\bar{h}}(z,\bar{z})$ is a conformal primary field of weights $(h,\bar{h})$ if it obeys

f ST. $(J=0)$
$$\delta_Y \Phi_{h,\bar{h}} = (Y^A \partial_A + h\,\partial_z Y^z + \bar{h}\,\partial_{\bar{z}} Y^{\bar{z}})\Phi_{h\bar{h}} \quad (*)$$

[cf. conformal primary field in CFT_2]

* Explicit computation of $\delta_\xi M$, $\delta_\xi N_A$, $\delta_\xi C_{AB}$, $\delta_\xi N_{AB}$ reveals that only C_{AB} obeys $(*)$. @ $u=0$
However, one can construct $\hat{M}$, $\hat{N}_A$, $\hat{J}_{AB}$ that do ! [Donnay, Ruzziconi; Freidel, Pranzetti
Barnich, Ruzziconi ...]

Exercise : Show that :

Need $\mathcal{L}_Y g_{uu}$ @ $\frac{1}{r}$ hence need $1/r$ comp. of ξ (ξ^r too); imp. g_{uu} & not $g^{(0)}_{uu}$!

$$\hat{M} = M + \frac{1}{8} C_{AB} N^{AB} \quad \text{obeys } (*) \text{ at } u=0$$

[* COINCIDES W. REAL PART OF $\Psi_2^{(0)}$ Weyl t*]

Similar analysis allows one to identify the following spacetime primaries (at a cut)

		$(h,\bar{h})$
$M \to \hat{M}_{\mathbb{C}} = \hat{M} + i\tilde{M} \equiv \Psi_2^{(0)}$	not to confuse with N_{AB}	$(3/2, 3/2)$
$N_A \to \hat{J}_A \supset \Psi_1^{(0)}$		$(2, 1)$
$\tilde{J}_{AB} \to \hat{\tilde{J}}_{AB} \supset \Psi_0^{(0)}$		$(\frac{5}{2}, \frac{1}{2})$

Covariant quantities identifeed by Newman, Penrose a long time ago. (70s)

Ψ_i defined from Weyl tensor by contraction w. $l, n, m, \bar{m}$ (null frame), eg. $\Psi_2 = -C_{lm\bar{m}n}$

TRACELESS component of $R_{\mu\nu\rho\sigma}$

$\Psi_0 = C_{lmlm}$

$\Psi_1 = C_{lnlm}$

$l \cdot n = -1,\ m \cdot \bar{m} = 1$

$$\left\{ \begin{array}{l} l = \partial_r,\ n = e^{-2\beta}\left(\partial_u + \frac{V}{2r}\partial_r + r^{-2}U^A\partial_A\right) + \ldots \\ m = m^A \partial_A \quad ;\ m^A \bar{m}_A = 1 \end{array} \right\}$$ null vectors

cf polarization tensors

* $g_{ab} = -l_a n_b - l_b n_a + m_a \bar{m}_b + m_b \bar{m}_a$?

NP. variables: $\Psi_i = \sum_{n \geq 0} \Psi_i^{(n)} r^{-n-5+i}$

$s \equiv h - \bar{h}$, $\Delta \equiv h + \bar{h}$, note all have $\underline{\Delta = 3}$
but $\underline{s = 0, 1, 2}$.

Separating the $s = 1$ and $s = 2$ into positive (z) & neg. $(\bar{z})$ helicity components

$$\mathcal{J} \equiv m^A \hat{\mathcal{J}}_A \,, \quad \mathcal{J}_- \equiv \bar{m}^A \hat{\mathcal{J}}_A$$

[think $z\bar{z}$ vs $\bar{z}\bar{z}$ comp.]

$$J \equiv m^A m^B \hat{J}_{AB} \,, \quad J_- \equiv \bar{m}^A \bar{m}^B \hat{J}_{AB}$$

$$D \equiv m^A D_A \,, \quad \bar{D} \equiv \bar{m}^A D_A \,, \quad N \equiv \bar{m}^A \bar{m}^B N_{AB}$$

$$M \sim \Psi_2^{(0)} \,, \quad \mathcal{J} \sim \Psi_1^{(0)} \,, \quad J \sim \Psi_0^{(0)}$$

(and c.c. for $-$)

the G_{uu}, G_{uA}, G_{AB} constraints take a particularily simple form:

2112.15573 + refs

$$\boxed{\partial_u Q_s = D\, Q_{s-1} + \frac{s+1}{2} C \cdot Q_{s-2}} \quad (**)$$

for $s = 0, 1, 2$; $\left\{ \begin{array}{l} Q_{-1} \equiv \frac{1}{2} D N \\ Q_{-2} \equiv \frac{\partial_u N}{2} \end{array} \right\}$ bdry cond.

$Q_0 \equiv M_c$, $Q_1 \equiv \mathcal{J}$, $Q_2 \equiv J$

similar eq. for $-$ variables

END 1

b) $\boxed{\text{Comment}}$: In the following will consider (**) with $s \in \mathbb{N}$. For $s > 2$ these can be shown to be truncations of the evolution eq^{n}s for $\mathcal{I}_0^{(n)}$ with $n \geq 1$.

[or equiv. - evolution eq. for $g_{AB}^{(n)}$]

(**) can be solved order by order in # of fields

$$Q_s = Q_s^{(1)} + Q_s^{(2)} + \dots + Q_s^{(s+1)} \quad \text{(polynomial)}$$

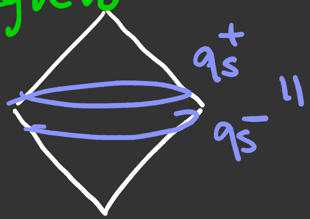
- $\lim_{u \to -\infty} Q_s(u, z, \bar{z})$ should yield a conserved quantity, but instead diverges. (for $s \geq 1$)
- regularize $\Rightarrow q_s(z, \bar{z})$ [w/ Pranzetti, Freidel]

$\boxed{\text{Define}}$ pairing $\int_{S^2} \mathcal{F}(z, \bar{z})\, q_s^{\pm}(z, \bar{z}) \equiv Q_s^{\pm}$

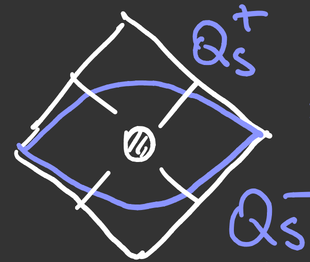
* infinity of charges for all s.
* $s = 0, 1 \Rightarrow$ ST & SR charges
* $s \geq 2 \Rightarrow$ higher multipole moments of grav. field

Similar analysis on $\mathcal{I}^-$ (indep BMS4 = C) no flux through i^0

$\boxed{\text{match}}$ $\Rightarrow$ spacetime conservation law



$\parallel$ $=$



$=$

[after using Stokes' thm. to extend to integrals over $\mathcal{J}^\pm$]

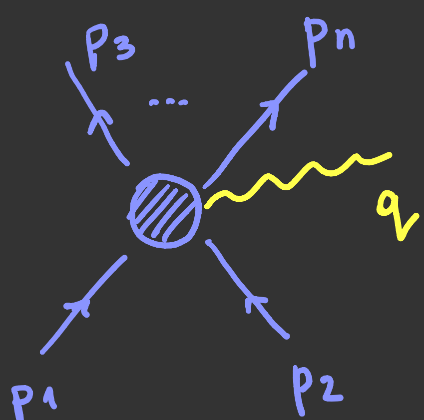
matching $\Rightarrow$ $\langle \text{out} | Q_s^+ S - S Q_s^- | \text{in} \rangle = 0 \quad \forall s.$

$Q^\pm$ truncated to quadratic order.

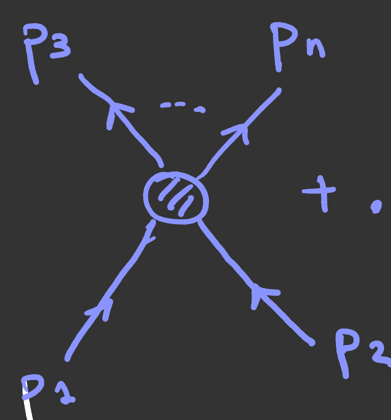
$$(=) \lim_{q \to 0} \langle \text{out} | a(q) S | \text{in} \rangle = \underbrace{\sum_{n=0}^{\infty} \omega^{n-1} S^{(n)}}_{\text{tower of soft thms.}} \langle \text{out} | S | \text{in} \rangle$$

$\hookrightarrow$ fix $s = n$ $\Rightarrow$ $(\text{sub})^s$- leading soft thm.

DEF: SOFT THM./EXPANSION



$$= \sum_{n=0}^{\infty} \omega^{n-1} S^{(n)} \times$$



$+ \ldots$

$n=0$:
$$S_\pm^{(0)} = \sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu \varepsilon_{\mu\nu}^\pm}{p_i \cdot \hat{q}}$$
universal (leading) soft factor

c) Example: Leading soft graviton as conservation law of supertranslation charge

$(S=0)$

<u>Goal</u>: show that $\langle \text{out} | Q^{+}_{S=0} S - S Q^{-}_{S=0} | \text{in} \rangle = 0$

implies $\lim_{\omega \to 0} \omega \langle \text{out} | a_{\pm}(\omega \hat{q}) S | \text{in} \rangle = S^{(0)}_{\pm} \langle \text{out} | S | \text{in} \rangle$

Start with the definition:

$$Q^{\pm}_{S=0} = \int_{S^2} F(z,\bar{z}) \, M_C(z,\bar{z}) \Big|_{\mathcal{J}^{\pm}_{\mp}}$$

$$= \int_{\mathcal{J}^{\pm}} F(z,\bar{z}) \, \partial_u M_C(u,z,\bar{z})$$

b.c. $Q_{S=0}\big|_{\mathcal{J}^{\pm}_{\pm}} = 0$

$$= \int_{\mathcal{J}^{\pm}} F(z,\bar{z}) \Big[\underbrace{\tfrac{1}{2} D^2 N}_{\equiv Q_S = Q_0^{(1)}} + \underbrace{\tfrac{1}{4} C \cdot \partial_u N}_{\equiv Q_H = Q_0^{(2)}} \Big] \quad (**)$$

$(**)$

→ How do these act on asymptotic scattering states?

Recall: $h_{AB} \equiv r C_{AB}$ ~ free graviton

transv. traceless metric part.

$$r\,C_{AB} = \frac{\partial x^\mu}{\partial x^A}\frac{\partial x^\nu}{\partial x^B}\underbrace{h_{\mu\nu}}\qquad (*)\qquad \boxed{(-+++)}$$

$$\int \widetilde{d^3q}\,\left[a_{\mu\nu}(q)\,e^{iq\cdot x} + a^\dagger_{\mu\nu}(q)\,e^{-iq\cdot x}\right]$$

where $a_{\mu\nu}(q) = \sum_{\alpha,\beta=\pm} \varepsilon^{\alpha\beta *}_{\mu\nu}\, a_{\alpha\beta}(q)$ [check $*$]

$\equiv$ polarization tensors

Exercise : Use the stationary phase approx. to take $r \to \infty$ limit of $(*)$ & show that

$$C_{zz} \propto \int_0^\infty d\omega\left(a_+(\vec{q})\,e^{-i\omega u} - a_-^\dagger(\vec{q})\,e^{i\omega u}\right)$$

$N_{zz} \equiv \partial_u C_{zz}$ $\rightarrow$ substitute these mode expansions into $(***)$:

$$Q_S \propto \int_{-\infty}^{\infty} du\, N_{zz} = \lim_{\omega\to 0}\int_{-\infty}^{\infty} du\, e^{i\omega u} N_{zz}(u)$$

$\propto$ SOFT GRAVITON

$Q_H \propto$ quadratic in graviton modes

use $\left[a_\pm(\omega), a_\pm^\dagger(\omega')\right] = 2\omega\,\delta(\omega-\omega')\,\delta^2(z,z')$

to compute action on asy. particle states $\Rightarrow$

$$[Q_H, a^{\dagger}_{\pm}(p_i)] \propto S^{(0)}_{\pm}(p_i)\, a^{\dagger}_{\pm}(p_i)$$

Finally, inserting this into conservation law:

$$\underbrace{\langle out | [Q_S, S] | in \rangle}_{\substack{\text{III} + \text{crossing} \\ \propto \lim_{\omega \to 0} \omega \langle out | a(\omega) S | in \rangle}} = - \underbrace{\langle out | [Q_H, S] | in \rangle}_{\sum_{i=1}^{n} S^{(0)}(p_i) \langle out | S | in \rangle}$$

- repeat same steps to deduce subleading & whole tower of soft thms from Ward id.

<u>KEY OBSERVATION</u>: $Q_S = Q_S[N, C]$

and $\{N(u,z), C(u',z')\} \propto \delta(u-u')\, \delta^2(z,z')$

CHARGE ALGEBRA

$$\{q_s(z,\bar{z}), q_{s'}(z',\bar{z}')\}^1 = \{q^2_s, q^1_{s'}\} + (s \leftrightarrow s')$$

$$= \frac{\kappa^2}{8} \left[-(s'+1)\, q^1_{s'+s-1}(z')\, D_z \delta(z,z') + (s+1)\, q^1_{s'+s-1}(z)\, D_{z'} \delta(z,z') \right]$$

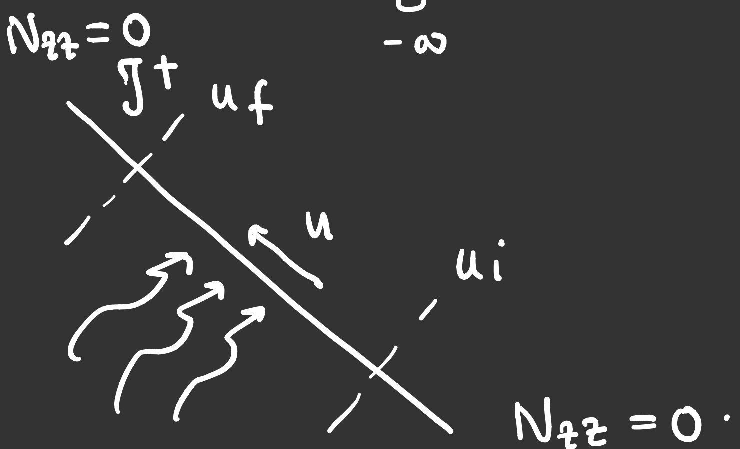
$\Rightarrow$ <u>w_∞ algebra</u> on gravitational phase space!!

$s = 1 \longrightarrow$ Virasoro algebra

Brief comment on memory

$$Q_S = \int du\, N_{zz} \propto \lim_{\omega\to 0} \omega(a(\omega) + a^\dagger(\omega))$$

$$= \int_{-\infty}^{\infty} du\, \partial_u C_{zz} = C_{zz}(u=+\infty) - C_{zz}(u=-\infty)$$



- vacuum before u_i & after u_f
- $N_{zz} \neq 0$ (+ zero mode)

$\Rightarrow \Delta C_{zz} \neq 0$

eg. $C_{zz} = D_z^2 N\, \Theta(u-u_i) + \dots$

Gravitational memory effect.
is the FT of the leading soft pole
[conversely, scattering amplit allow us to extract an <u>infrared</u> <u>classical</u> obs. in the soft limit]
$\shortparallel$
G. MEMORY.

* $Q_S^{(1)}$, $s \in \mathbb{N}_+$: tower of $(\text{sub})^s$-lead. memories

('23 w/ Freidel & Pranzetti)

d) Aside on vacuum structure & memory

First notice that vacuum metrics ($N_{AB}=0$) are param. by $C^{vac}_{AB} = -2\partial_A\partial_B C \neq 0$ where $C = C(z,\bar z)$. Under ST: $\delta_f C = f$ / shift (cf. Goldstone).

Vacuum near $i^0, i^+ (u\to\mp\infty)$ parameterized by

$$C_{zz} \overset{u\to\pm\infty}{=} -2\partial_z^2 C_\pm(z,\bar z) + (u+C_\pm) N^{vac}_{zz} + \mathcal{O}(u^{-1})$$

$$N_{zz} \overset{u\to\pm\infty}{=} N^{vac}_{zz} + \mathcal{O}(u^{-2})$$

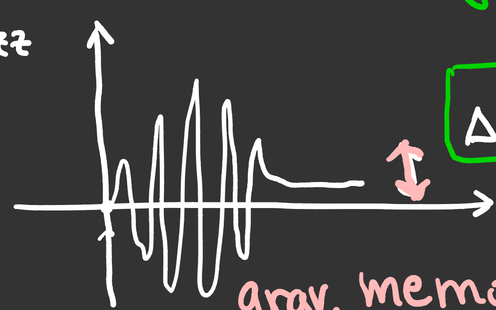
* where $C_\pm$ transf. as primaries (*) of $(-\frac{1}{2},-\frac{1}{2})$
* $C_\pm$ enter in def. of Goldstone & Mem. modes par. soft sector of grav. phase space

$$\mathcal{C} = \frac{1}{2}(C_+ + C_-), \quad \mathcal{N}^{(0)} = \frac{1}{2}(C_+ - C_-)$$

canonically paired:

$$\{\partial_z^2 \mathcal{N}^{(0)}, \partial_{z'}^2 \mathcal{C}\} \propto \delta^{(2)}(z,z')$$

obs. memory effect



$$\boxed{\Delta C_{zz} \propto \partial_z^2 \Delta C} + \boxed{\propto \partial_z^2 \mathcal{N}^{(0)}}$$

grav. memory effect (leading)

$N_{zz}^{vac} = \frac{1}{2}(\partial_z \varphi)^2 - \partial_z^2 \varphi$ where $\varphi(z)$ is a Liouville field transforming as $\delta\varphi = Y^A \partial_A \varphi + \underbrace{\partial_A Y^A}_{\text{shift}}$

where $Y^A = (Y^z, 0)$ (otherwise weight 0)

$$\delta N_{zz}^{vac} = (Y^z \partial_z + 2\partial_z Y^z) N_{zz}^{vac} - \underline{\underline{\partial_z^3}} Y^z$$

recall this is holomorphic

[can also write
$N_{zz}^{vac} = -\{G(z), z\}$
$G'(z) = e^{\varphi}$]

$\{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ Schw. der.

N_{zz}^{vac} param. the superrotation vacua and it will shift under superrotations.

Can use these to construct:

$$\hat{C}_{AB} \equiv C_{AB} \left[- C_{AB}^{vac} - u N_{AB}^{vac} \right]$$
$$\hat{N}_{AB} \equiv N_{AB} - N_{AB}^{vac}$$

not needed for primary @ $u=0$ to hold.

that transform like primaries of weights

$(\hat{C}_{zz})$ $\left(+\frac{3}{2}, -\frac{1}{2}\right)$ and $(2,0)$ $(\hat{N}_{zz})$
$(\hat{C}_{\bar{z}\bar{z}})$ $\left(-\frac{1}{2}, \frac{3}{2}\right)$ $(0,2)$ $(\hat{N}_{\bar{z}\bar{z}})$

under superrot at $u = 0$.

Review lecture 2

vacuum Einstein equations @ large r $\rightarrow$

- $\partial_u Q_s = D Q_{s-1} + \frac{s+1}{2} C \cdot Q_{s-2}$, $s \in \mathbb{N}$

  Q_s "conformal primaries" of $\Delta = 3$, $J = s$ at $\underline{u=0}$. [$\bar{Q}_s$ — another tower]

- $\left\{\begin{matrix} Q_{s=0} \\ Q_{s=1} \\ Q_{s=2} \end{matrix}\right\}$ $O(\frac{1}{r})$ components of g_{uu}, g_{uA}, g_{AB}

  $Q_{s\geq 3}$ from $O(\frac{1}{r^{s-1}})$ components of g_{AB}

- $\lim_{u\to -\infty} Q_s = \infty$ for $s \geq 1$; regularize $\rightarrow$

  $q_s(z,\bar{z})$; towers at $\mathcal{I}^+$ and $\mathcal{I}^-$ matched across i^0 $\rightsquigarrow$

- conservation law:

  $$\langle out \,|\, q_s^+(z,\bar{z})\, S - S\, q_s^-(z,\bar{z}) \,|\, in\rangle = 0 \Big|_{quadr}$$

  $$\Updownarrow$$

$$\lim_{\omega\to 0} \underbrace{\partial_\omega^s \left(\omega \langle out \,|\, a_{\pm}(\omega) S \,|\, in\rangle\right)}_{(sub)^s\text{-leading soft insertion}} = \underbrace{S_{\pm}^{(s)}}_{\text{"tree-level" sub-s leading soft factor}} \langle out \,|\, S \,|\, in\rangle$$

$$-\{q_s(z), q_{s'}(z')\}^{(1)} = (s+1)\underbrace{D_{z'}}_{q_{s+s'-1}(z)}\delta(z,z') - (s'+1)\underbrace{D_z}_{q_{s+s'-1}(z')}\delta(z,z')$$

II) a) Conformal primary basis

Observables in 4D AFS constructed from S-matrix elements / amplitudes for a collection of particles in the far past to evolve into one in the far future.

Interactions assumed to be localized in space and time $\Rightarrow$ particles freely moving as $t \to \pm\infty$.

$(-+++)$

Free scalar states $\Leftrightarrow$ solutions to KG eom: $(\Box + m^2)\Phi = 0$ $\quad (*)$

$-\partial_\mu \partial^\mu$

More generally, for spinning particles

$$\mathcal{D} \cdot \Phi = 0 \qquad [\text{eg. } s = 1/2 \quad \mathcal{D} = \gamma^\mu \partial_\mu + m\,\mathbb{I}]$$

Time translation invariance $\subset$ Poincaré $\Rightarrow$

$$S = S_p \oplus \bar{S}_p$$

where S_p, $\bar{S}_p$ are positive & neg. freq. subspaces.

space of solutions to $(*)$

$\Updownarrow$

completely specified by $(\Phi, \partial_t \Phi)$ on any equal time / Cauchy slice Σ_t and the split into $\omega \gtrless 0$ follows from the

time-independent "inner" product on Σ_t
(conserved)

$$(\alpha,\beta) \equiv \langle \alpha,\beta \rangle_{KG} = \int_{\Sigma_t} d^3x \, \underline{\underline{n^a}} \, j_a(\alpha,\beta) \qquad (*)$$

normal to Σ_t

$$j_a(\phi_1,\phi_2) = -i\,(\phi_1^* \partial_a \phi_2 - \phi_2 \partial_a \phi_1^*)$$

$(\alpha,\beta) = -(\beta^*,\alpha^*) \Rightarrow (*)$ is not positive definite ; S_p, $\bar{S}_p$ are the definite frequency subspaces : $\Phi_+ \in S_p$, $\Phi_- \in \bar{S}_p$

$$(*) \quad \boxed{\partial_t \phi_\pm = \mp i \omega \phi_\pm \ , \ \omega > 0}$$

Solutions to KG eq are superpositions of pos/negative freq. modes :

$$\phi_\pm(x;p) = e^{\pm i p \cdot x} \qquad \text{and}$$

$$\Phi(x) = \int \widetilde{d^3p}\,(a_p^\dagger \phi_p + a_p \phi_p^*)$$

The choice of $(*)$ is motivated by global translation invariance $\Rightarrow$ asy. states = reps. of Poincaré

We learned that asy. symm. group $\gg$ Poincaré so we may want to organize in reps of asy. symm. group.

Reps. of ext. BMS4 not fully classified yet ...

Virasoro^2 $\subset$ eBMS$_4$ $\Rightarrow$ organize asy. states in reps. of Virasoro^2 ! [cf. conf. primary @ cut...]

$\Leftrightarrow$ symmetry group of CFT$_2$ so may be able to exploit 2D CFT methods to understand 4D physical observables ...

Replace plane wave basis above by conformal primary basis [Pasterski, Shao, Strominger '16]

Def. Scalar conf. prim. wave functions are solutions to the wave equation:

$$(\Box + m^2)\,\Phi = 0$$

which are "heighest weight" w.r.t the Lorentz group.

$SO(1,3) \simeq SL(2,\mathbb{C})$: $M_{\mu\nu} \equiv -(x_\mu \partial_\nu - x_\nu \partial_\mu)$ (+)

Lorentz generators organize into

- $K_i \equiv M_{0i}$ (boosts), $J_i \equiv \varepsilon_{ijk} M_{jk}$ (rot) obeying the Lorentz algebra

$$[J_i, J_j] = \epsilon_{ijk} J_k \ , \quad [K_i, K_j] = -\varepsilon_{ijk} J_k$$

$$[J_i, K_j] = \epsilon_{ijk} K_k$$

↳ reorganize into $SL(2,\mathbb{C})$ algebra by taking linear combinations

$$L_0 + \bar{L}_0 \equiv K_3$$

$$L_0 - \bar{L}_0 \equiv J_3$$

$$L_1 = J_1 + iK_1 + i(J_2 + iK_2)$$

$$L_{-1} = J_1 + iK_1 - i(J_2 + iK_2)$$

$$\bar{L}_1 = L_{+1}^{\dagger}$$

$$\bar{L}_{-1} = L_{-1}^{\dagger}$$

Then $[L_m, L_n] = (m-n) L_{m+n}$ & similarily for $[\bar{L}_m, \bar{L}_n]$ CHECK

Def: Highest weight states of $SL(2,\mathbb{C})$ are defined by

$$(L_0 + \bar{L}_0)\Psi_\Delta = \Delta \Psi_\Delta \quad \text{(boost eigenstate)}$$

$$(L_0 - \bar{L}_0)\Psi_\Delta = 0 \quad \text{(for scalars)}$$

$$L_1 \Psi_\Delta = \bar{L}_1 \Psi_\Delta = 0$$

Using the rep. (*) $\Rightarrow$

$$\Psi_\Delta \propto \frac{1}{(x^0+x^3)^\Delta}$$

(diagonalize boost towards $(1,0,0,-1)$)

Can generalize to solutions that diagonalize boosts towards an arbitrary point on the sphere

$$\hat{q} = (1+z\bar{z},\ z+\bar{z},\ -i(z-\bar{z}),\ 1-z\bar{z})$$

with associated Lorentz gens. obtained from • via a rotation

$$J_i' = R_{ij}(\hat{q}) J_j \quad , \quad K_i' = R_{ij} K_j \ \bullet$$

Exercise : Show that highest weight wavefunctions wrt. • take the form

$$\Psi_\Delta(\hat{q}; x) = \frac{f(x^2)}{(-\hat{q}\cdot x)^\Delta} \qquad (*)$$

Note : $f(x^2)$ is Lorentz invariant & does not affect the highest weight conditions. It is fixed by requiring that (*) obeys the eom :

$$4x^2 f''(x^2) - 4(\Delta-2) f'(x^2) - m^2 f(x^2) = 0$$

(exercise: derive this eq. by substituting (*) into the KG eq^n)

Solutions are Bessel fct^{n}s + bdy. conditions ($\Psi \to 0$ as $x^2 \to \infty$) $\Rightarrow$

$$f(x^2) \propto \left(\sqrt{-x^2}\right)^{\Delta-1} K_{\Delta-1}\left(im\sqrt{-x^2}\right)$$

CHECK.

Can check that $\Psi_\Delta(x;\hat{q})$ transforms like a 2D conformal primary under Lor. transf.:

$$\Psi_\Delta\left(\Lambda^\mu{}_\nu x^\nu ; \vec{z}'(\vec{z})\right) = \left|\frac{\partial \vec{z}'}{\partial \vec{z}}\right|^{-\Delta/2} \Psi_\Delta(x;\vec{z})$$

fix. in note

* Massless wavefunctions obtained by taking the limit $m \to 0$ of CPW.

$$\Rightarrow \varphi_\Delta(\hat{q};x) \propto \frac{1}{(-\hat{q}\cdot x)^\Delta} \equiv \int_0^\infty d\omega\, \omega^{\Delta-1} e^{i\omega\hat{q}\cdot x}$$

* Spinning (massless) wave functions obtained by dressing φ_Δ with polarization tensors :

eg. $A_{\Delta, J=+1} = m\,\varphi_\Delta$, $A_{\Delta, J=-1} = \bar{m}\,\varphi_\Delta$

$h_{\Delta, J=+2} = m\,m\,\varphi_\Delta$, $h_{\Delta, J=-2} = \bar{m}\bar{m}\,\varphi_\Delta$

where $m, \bar{m}$ were introduced before :

$$m_\mu = \varepsilon^+_\mu + \# \, \hat{q}_\mu . \qquad \frac{\epsilon \cdot x}{(-\hat{q}\cdot x)}$$

Think of $\Phi^\pm_\Delta(\hat{q}\,; x)$ as replacing $e^{\pm i\omega \hat{q}\cdot x}$

$\qquad ///$ needs regulator

$\Phi_\Delta(\hat{q}\,; x_\pm)$ for branch cut at

$x_\pm = x \mp i\epsilon n$ $\qquad \hat{q}\cdot x = 0$.

Basis for $\Delta = 1 + i\lambda$ [Pasterski, Shao '17]

Bulk scalar field admits expansion in cp. modes

$$\phi(x) = \int_{-\infty}^{\infty} d\lambda \int d^2z \, [\, \mathcal{O}_\lambda \, \varphi_{1+i\lambda}(\hat{q}\,; x) + \mathcal{O}_\lambda^{+\,*} \, \varphi^*_{1-i\lambda}(\hat{q}\,; x) \,]$$

celestial operator

$$\mathcal{O}_\lambda(\hat{q}) \equiv \langle \Phi(x) , \varphi_{1+i\lambda}(\hat{q}\,; x) \rangle_{\Sigma_t}$$

same KG ip

Note : $\langle \varphi_{1+i\lambda_1} , \varphi_{1+i\lambda_2} \rangle = 8\pi^4 \, \delta(\lambda_1 - \lambda_2) \, \delta^2(z_1, z_2)$.

b) Celestial amplitudes

Massless scattering:

$$\tilde{A}(\Delta_i, z_i) \equiv \langle \widetilde{out} | S | \widetilde{in} \rangle = \prod_{i=1}^{n} \left(\int_0^\infty d\omega_i \, \omega_i^{\Delta_i - 1} \right) \langle out | S | in \rangle$$

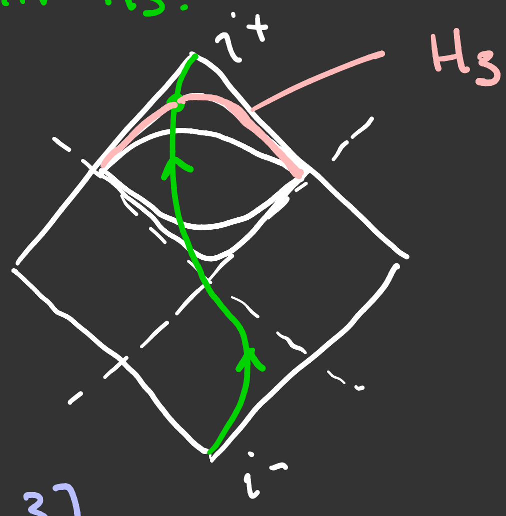
where $|\widetilde{in/out}\rangle$ are boost eigenstates

Massive scattering:

$$\tilde{A}_m(\Delta_i, z_i) = \prod_{i=1}^{n} \int_{H_3} d^3\hat{p}_i \, \underbrace{G_{\Delta_i}(\hat{p}_i ; \hat{q})} \langle p_{out} | S | p_{in} \rangle$$

- Fourier transform of massive CPW
- Bulk to bdry. propagator on Euclidean AdS$_3$ (H_3) (de Boer, Solodukhin 2002)

can be understood by resolving timelike infinity w. H_3 slices recalling that $\hat{p}_i^2 = -1 \leftrightarrow$ point in in H_3.



Exercise: Compute the 3-pt cel. amplitude with 2 massless & 1 massive particles.

[END OF L.3]

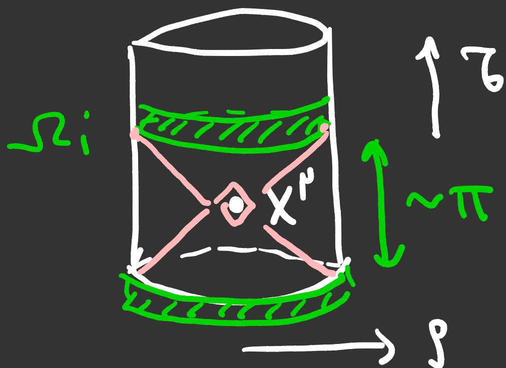
c) Celestial amplitudes from limit of AdS-Witten diagrams [skip?]

Lorentzian AdS_{d+1} defined as a max. symm. space of rad. ℓ inside $Mink_{d+2}$ with $(--++\ldots+)$

$$-(X^0)^2 + \sum_{i=1}^{d}(X^i)^2 - (X^{d+1})^2 = -\ell^2$$ (check)

Parameterize points in AdS_{d+1} as

$$X^0 = \ell \sin\tau/\cos\rho$$
$$X^{d+1} = \ell \cos\tau/\cos\rho$$
$$X^i = \ell \tan\rho\, \Omega_i \,, \quad \Omega_i^2 = 1.$$

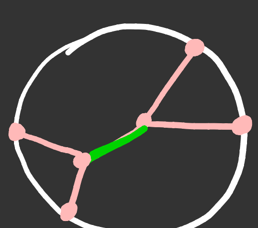


points on boundary:

$$P = \lim_{\rho\to\infty} \frac{\cos\rho}{\ell} X(\tau,\rho,\Omega_i)$$

cf. CFT_d in embedding space

Witten diagrams: $\prod_{\substack{\text{bulk pts.}\\ \alpha}}\left(\int_{AdS_{d+1}} d^{d+1}x_\alpha\right) \prod_{\substack{\text{bdry pts}\\ i}} K_{\Delta_i}(P_i\,;x_\alpha)$



$\times \; B(x_1,\ldots)$

$K_\Delta(P, x)$ is a bulk-to-bdry propagator

$$= \frac{C_\Delta}{(-P\cdot x)^\Delta}$$

while B is a product of bulk-to-bulk propagators (each solves sourced wave eq. in AdS_{d+1})

Observations: 1) for boundary points $\tau_i = \pm \frac{\pi}{2}$

blk pts $\begin{Bmatrix} r = \ell\cdot\rho \\ t = \ell\cdot\tau \end{Bmatrix}$, $\begin{Bmatrix} \ell\to\infty \\ \text{fixed}(r,t) \end{Bmatrix}$ the bulk-to-bdry prop becomes a massless CPW in $(d+1)$-flat space with $\Delta = \Delta_i$ inherited from CFT_d operator.

2) for $\tau_i = \pm\frac{\pi}{2} + \frac{u_i}{\ell}$, $\ell\to\infty$

$$\int_{-\infty}^{\infty} du_i \; u_i^{-\lambda_i} \, K_{\Delta_i}(P_i; x_i) \underset{\ell\to\infty}{=} \varphi_{\Delta_i+\lambda_i-1}$$

EXERCISE

Suggests that for this kinematic configuration AdS_{d+1} Witten diagrams $\to$ celestial amplit. in $(d-1)$-dim.

* infinitesimal time bands around $\pm\frac{\pi}{2}$

$\Leftrightarrow$ $\mathcal{I}^{\pm}$; compactification $\Rightarrow$ celestial amplitudes ; otherwise Carrollian correlators *

Refs : w/ de Gioia

III) Holographic aspects of gauge & gravity thys. in 4dim. (massless scattering)

a) Celestial symmetries :

- Lorentz $SL(2,\mathbb{C})$ symmetries

$$\sum_{n=1}^{N} \mathcal{L}_i^{(n)} \quad \tilde{A}(\Delta_i, z_i) = 0 \quad , \qquad i = -1, 0, 1$$

and similarily for $\bar{\mathcal{L}}_i$. $\mathcal{L}_i, \bar{\mathcal{L}}_i$ admit a 2D representation (cf. global conformal generators in 2D CFT)

- Poincaré symmetries

$$\sum_{n=1}^{N} \hat{P}^{(n)} \tilde{A}(\Delta_i, z_i) = 0 \quad ,$$

$$\hat{P}^{(n)} = \hat{q}(z_n)\, \underbrace{e^{\partial_{\Delta_n}}}$$

- weight shifting operator
- conformal primary basis rep.

In momentum basis $p = \omega \hat{q}$ and

$$p^{(j)} \tilde{A}(\Delta_i, z_i) = \ldots \int_0^\infty d\omega \; \omega^{\Delta_j - 1} \ldots \; \omega \hat{q}_j \; A(\omega, z_i)$$

acts on j^{th} external leg

$$= \hat{q}_j \, \tilde{A}(\ldots, \Delta_j + 1, z_j, \ldots) \text{ with all other } \Delta \text{ fixed.}$$

Example : 4-point functions

$$\tilde{A}_4 = K_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\,\delta(r-\bar{r})\,f^{h_i,\bar{h}_i}(r,\bar{r})$$

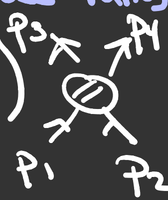
conformally covariant / transl. invar.

cross-ratio

$$K_{h_i,\bar{h}_i} = \prod_{i<j=1}^{4} z_{ij}^{h/3-h_i-h_j}\,\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \quad , \quad h \equiv \sum_{i=1}^{4} h_i$$

$r = \dfrac{z_{13}z_{24}}{z_{12}z_{34}}$, $\bar{r} = r^*$ are conf. invariant cross ratios

$(r = -t/s)$



$\delta(r-\bar{r})$ due to momentum conservation

$f^{h_i,\bar{h}_i}(r,\bar{r})$ in 2D CFT is not fixed by symm, but instead other constraints (eg. crossing).

Here, translation invariance imposes an additional constraint on f:

Exercise

Since $\sum_{j=1}^{4} K_{h_j+\frac{1}{2},\bar{h}_j+\frac{1}{2}} = 0$, Poincaré invar $\Rightarrow$

$$f^{h_i+\frac{1}{2},\bar{h}_i+\frac{1}{2}} = f^{h_j+\frac{1}{2},\bar{h}_j+\frac{1}{2}} \quad , \quad \forall i,j$$

By induction $\Rightarrow$ $f^{h_i,\bar{h}_i} = f^{\beta,J_i}$ where

$$\beta \equiv \sum_{i=1}^{4}(h_i+\bar{h}_i) = \sum_{i=1}^{4}\Delta_i$$

Poincaré invariance can be used to constrain the form of 3-point cel. amplitudes.

Ex: 2 massless, 1 massive obey

$$\left(P_1 + P_2 + P_3^{(m)}\right) \widetilde{A}_3(1,2,3^{(m)}) = 0$$

Lorentz: $$\widetilde{A}(1,2,3^{(m)}) = \frac{C(\Delta_1,\Delta_2,\Delta_3)}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}\,|z_{13}|^{\Delta_1+\Delta_3-\Delta_2}}$$

and $C(\Delta_1,\Delta_2,\Delta_3)$ is subject to recursion relations that are solved by

$$C(\Delta_1,\Delta_2,\Delta_3) = B\left(\frac{\Delta_{12}+\Delta_3}{2}, \frac{\Delta_{21}+\Delta_3}{2}\right) \times \text{const.}$$

- Conformally soft symmetries

  are 2D repres. of 4D asy. symmetries discussed in the first lecture.

<u>Recall soft thms</u>:

Soft photon thm. in 4D

$$\langle J_z\, \mathcal{O}_1(\omega_1,z_1,\bar{z}_1) \ldots \mathcal{O}_n(\omega_n,z_n,\bar{z}_n)\rangle$$

$$= \lim_{\omega\to 0} \omega \langle \mathcal{O}^+(\omega,z,\bar{z})\, \mathcal{O}_1 \ldots \mathcal{O}_n\rangle \qquad (*)$$

$$= \underbrace{\sum_{k=1}^{n} \frac{Q_k}{z-z_k}}\langle \mathcal{O}_1 \ldots \mathcal{O}_n\rangle \qquad \text{EXERCISE}$$

$$S^{(0)+}_{QED} = \sum_{k=1}^{n} \frac{p_k\cdot\varepsilon^+}{p_k\cdot q} \quad \text{where } p_k, q$$

are null momenta & $\varepsilon^{+} \equiv \partial_z q$.

(*) Ward identity of U(1) current in 2D CFT

$(h, \bar{h}) = (1,0)$ or $(0,1)$.

expect the dim. of a positive-helicity & spin

conf. soft gluon are $\Delta = 1$, $S = 1$. Can see that indeed, insertions of $\Delta = 1$ ops in 2D

$\Rightarrow$ leading soft ops. in 4D :

$$\mathcal{O}^{+}_{\Delta}(z,\bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1}\, \mathcal{O}^{+}(\omega, z, \bar{z})$$

$$\lim_{\Delta\to 1} (\Delta-1)\, \mathcal{O}^{+}_{\Delta}(z,\bar{z}) = \lim_{\Delta\to 1} \int_0^\infty d\omega\, (\Delta-1)\, \omega^{\Delta-1}\, \mathcal{O}^{+}(\omega, z)$$

$$= 2\int_0^\infty d\omega\, \delta(\omega)\, \omega\, \mathcal{O}^{+}(\omega, z, \bar{z})$$

$$= \lim_{\omega\to 0} \omega\, \mathcal{O}^{+}(\omega, z, \bar{z}).$$

We used the identity

$$\lim_{\epsilon\to 0} \frac{\epsilon}{2} |x|^{\epsilon-1} = \delta(x)$$

split into low & high en.

More generally :

$$\lim_{\Delta\to -n} (\Delta+n)\, \mathcal{O}^{+}_{\Delta}(z,\bar{z}) = \lim_{\Delta\to -n} (\Delta+n) \int_0^{\omega_*} d\omega\, \omega^{\Delta-1}\, \mathcal{O}^{+}$$

$$= \lim_{\Delta \to -n} (\Delta+n) \sum_k \int_0^{\omega_*} d\omega\, \omega^{\Delta+k-1} O_k^+(z,\bar z)$$

$$= O_n^+(z,\bar z)$$

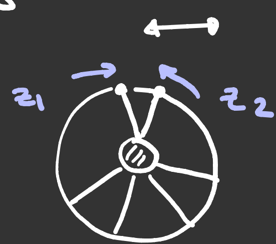
where $O^+(\omega, z,\bar z) = \sum_k \omega^k O_k^+(z,\bar z)$

[need to choose ω_* small enough & note that $\int_{\omega_*}^{\infty} d\omega$ will not have poles at negative integer Δ].

$(\text{Sub})^n$-subleading soft photons correspond to Residues at $\Delta = 1-n$, $n \in \mathbb{N}$.

b) Celestial operator products & symm. algebras

4d collinear limits of amplitudes



2d operator product expansions

$z_{12} \equiv z_1 - z_2 \to 0$

Example : Positive helicity Gluon OPE (fix sign in notes)

$$(*) \quad \underbrace{O_{\Delta_1}^{+,a}(z_1)\, O_{\Delta_2}^{+,b}(z_2)}_{\Delta_1+\Delta,\ J=2} \sim \underbrace{\frac{-if^{abc}}{z_{12}}}_{\substack{\text{from Lorentz invariance} \\ \Delta=1,\ J=1}} C(\Delta_1,\Delta_2)\, \underbrace{O_{\Delta_1+\Delta_2-1}^{+c}(z_2)}_{\checkmark}$$

* useful to work in bulk 2-2 signature, in which case $z, \bar{z}$ real independent variables

$SL(2,\mathbb{C}) \to SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

* can take $z_{12} \to 0$ while keeping $\bar{z}_{12}$ fixed *

* use subleading soft gluon thm. to fix the $C(\Delta_1, \Delta_2)$ OPE coefficient:

$$\bar{\delta}_b \mathcal{O}^{\pm a}_{\Delta} = -(\Delta - 1 \mp 1 + \bar{z}\partial_{\bar{z}}) i f^{abc} \mathcal{O}^{\pm c}_{\Delta-1}$$

negative helicity

$\Rightarrow$ recursion relation (EXERCISE)

$$(\Delta_1 - 2) C(\Delta_1 - 1, \Delta_2) = (\Delta_1 + \Delta_2 - 3) C(\Delta_1, \Delta_2)$$

[Ref: Pate, AR, Strominger, Yuan '19]

with the unique solution

$$C(\Delta_1, \Delta_2) = B(\Delta_1 - 1, \Delta_2 - 1).\ ; \quad B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1}$$

b') Holographic symmetry algebras

- include $SL(2,\mathbb{R})$ descendants in (*)

$$O^{+a}_{\Delta_1}(z_1)\, O^{+b}_{\Delta_2}(z_2) \sim \frac{-i f^{abc}}{z_{12}} \sum_{n=0}^{\infty} B(\Delta_1 - 1 + n, \Delta_2 - 1) \times \frac{\bar{z}_{12}^{\,n}}{n!} \bar{\partial}^n \mathcal{O}^{+c}_{\Delta_1 + \Delta_2 - 1}(z_2)$$

and study the limit when $\Delta_1, \Delta_2 \in \{1, 0, -1, \dots\}$

(conf. soft limit discussed before) $\leadsto$ algebra of (sub)^s- leading soft modes (from 4d pt. of view)

* note that the algebra closes because $\Delta_1+\Delta_2-1 \in \{1,0,-1,\dots\}$ as well.

* note that taking the residue at $\Delta_1 = 1-k$ only finite # of terms in ope survive since $B(x,y)$ only has poles at $x,y \in \mathbb{Z}_-$ (∞ upper limit in sum replaced by k)

* defining $R^{k,a} = \lim_{\epsilon \to 0} \mathcal{O}^{+a}_{k+\epsilon}$

$$\Rightarrow \bar{\partial}^{k+1} R^{k,a}(z,\bar{z}) = 0 \quad \text{so} \quad R^{k,a}(z,\bar{z})$$

are polynomials in $\bar{z}$ $\hookrightarrow$ finite dim. reps. of $SL(2,\mathbb{R})_R$.

* further taking residue at $\Delta_2 = 1-l$ $l \in \mathbb{N}$ $\Rightarrow$

$$R^{k,a}(z_1,\bar{z}_1)\, R^{l,b}(z_2,\bar{z}_2) \sim \frac{-if^{abc}}{z_{12}} \times \sum_{n=0}^{k} \binom{1+k-l-n}{l} \frac{\bar{z}_{12}^{\,n}}{n!} \bar{\partial}^n R^{k+l-1}(z_2,\bar{z}_2)$$

from which one can compute algebra

of soft modes (wrt $\bar{z}$ expansion):

$$[R^{k,a}_n(z), R^{l,b}_{n'}(z')] = -i f^{abc} \begin{pmatrix} \frac{k}{2} - n + \frac{l}{2} - n' \\ \frac{k}{2} - n \end{pmatrix} \begin{pmatrix} \frac{k}{2} + n + \frac{l}{2} + n' \\ \frac{k}{2} + n \end{pmatrix} R^{k+l-1,c}_{n+n'}$$

or rescaling $\hat{R}^{k,a}_n = \left(\frac{k}{2} - n\right)! \left(\frac{k}{2} + n\right)! R^{k,a}_n$

$$[\hat{R}^{k,a}_n, \hat{R}^{l,b}_{n'}] = -i f^{abc} \hat{R}^{k+l-1,c}_{n+n'}$$

Same analysis in gravity $\Rightarrow$ w_∞ algebra that we saw before from $\mathcal{E}\mathcal{E}$. Relation can be made precise

[Ref. /AR, Freidel, Pranzetti '21]